**Homework 2 Solutions**

- Exercise 1 - Eigenvalues and Max-Cut

Let $G = (V, E)$ be an undirected graph and let $A$ denote the adjacency matrix of $G$. (Recall that $A = \{a_{ij}\}$ is a symmetric matrix, and $a_{ii} = 0$ for all $i \in V$, and $a_{ij} = 1$ for all $ij \in E$.) Let $MC(G)$ denote the optimal value of a maximum cut of $G$. Suppose we compute the following:

$$\tau(G) = \min_{|x|=n} x^T Ax$$

a. Derive an upper bound on the value of $MC(G)$ in terms of $\tau(G)$.

**Solution:**

$$\tau(G) = \min_{|x|=n} x^T Ax \leq \min_{x \in \{-1,1\}^n} x^T Ax$$

$$= 2 \sum_{uv \in E} x_u \cdot x_v$$

$$= 2(|E| - MC(G)) - 2MC(G) = 2|E| - 4 \cdot MC(G)$$

So, we have:

$$MC(G) \leq \frac{|E|}{2} - \frac{\tau(G)}{4}.$$  

b. Find a graph $G$ for which the upper bound on $MC(G)$ given in Part a. is much larger than the actual maximum cut in $G$. (Find as large a gap as you can.)

**Solution:**

Several students gave the following solution. Consider the (disconnected) graph on $n$ vertices with only a single edge. Let $x = \{\sqrt{\frac{n}{2}}, -\sqrt{\frac{n}{2}}, 0, \ldots, 0\}$. Then the bound from Part a. is:

$$MC(G) \leq \frac{|E|}{2} - \frac{\tau(G)}{4}$$

$$= \frac{1}{2} + \frac{n}{4},$$

which is much larger than the maximum cut, which consists of one edge.
- Exercise 2 - The Fractional Eulerian Property

Let \( G = (V, A) \) be a directed graph. For a nonempty subset \( S \subset V \), let \( \delta^+(S) \subset A \) denote all edges \((i, j) \in A\) with \( i \in S \) and \( j \notin S \). Similarly, let \( \delta^-(S) \subset E \) denote all edges \((i, j) \in A\) with \( j \in S \) and \( i \notin S \). Let \( x : A \to \mathbb{R}^+ \).

Suppose that for each vertex \( i \in V \), we have:

\[
x(\delta^+(i)) = x(\delta^-(i)).
\]  

(1)

Show that for every \( S \subset V, S \neq \emptyset \), the following holds:

\[
x(\delta^+(S)) = x(\delta^-(S)).
\]  

(2)

**Solution:**

Consider a subset \( S \subset V \) such that \(|S| > 1\). Let \( A(S) \) denote edges with both endpoints in \( S \). Then:

\[
x(\delta^+(S)) = \sum_{i \in S} \delta^+(i) - \sum_{ij \in A(S)} x(ij),
\]

\[
x(\delta^-(S)) = \sum_{i \in S} \delta^-(i) - \sum_{ij \in A(S)} x(ij).
\]

Assumption (1) implies:

\[
\sum_{i \in S} x(\delta^+(i)) = \sum_{i \in S} x(\delta^+(i)),
\]

which implies statement (2).
- Exercise 3 - Tight Examples for Combinatorial TSP Algorithms

In Lecture 7, we presented the Double-Spanning-Tree Algorithm and Christofides’ Algorithm for the TSP problem on an undirected graph.

a. Give an unweighted graph for which the Double-Spanning-Tree Algorithm yields a solution whose weight is (arbitrarily close to) twice the weight of an optimal solution.

![Figure 1](Image)

Figure 1: All edges shown in the first figure have unit cost. All edges not shown have cost determined by shortest path. The bold edges in the second figure denote an MST.

Consider the graph in Figure 1 with \( n \) blue vertices and consider the MST shown in the second figure. Suppose the vertices are labeled such that edge \((1, n)\) and \((i, i+1)\) for all \(i \in \{1, n-1\}\) have unit cost. The tour produced by doubling the spanning tree alternates between blue and red vertices. Suppose that when we shortcut this tour (to obtain a Hamilton cycle in the metric completion), adjacent blue vertices in the Hamilton cycle do not have consecutive labels. Then, the cost of the resulting shortcut tour remains \(2n\), while the optimal tour has cost \(n+1\).

b. Give an unweighted graph for which Christofides’ Algorithm yields a solution whose weight is (arbitrarily close to) \(\frac{3}{2}\) times the weight of an optimal solution.

Example 1:

![Figure 2](Image)

Figure 2: The bold magenta edges form a perfect matching for the vertices with odd degree in the MST shown in Figure 1.

Consider the tour consisting of the MST from Figure 1 and the perfect matching on its odd-degree nodes shown in Figure 2. When we shortcut this tour to obtain a Hamilton cycle in the metric completion, it can be the (worst) case that the only consecutively labeled blue vertices correspond to matching edges. In this case, the cost of the Hamilton cycle resulting from the shortcutting remains \(\frac{3n}{2}\).
Example 2:

\[ \begin{align*}
1 & \quad 2 \quad 3 \quad 4 \quad \ldots \quad n-1 \quad n \\
\end{align*} \]

Figure 3: All solid edges have unit cost. If the Eulerian tour constructed is 1, 2, 3, \ldots n − 1, n, 1, then even after shortcircuiting, the cost of the tour produced will be \( \approx \frac{3n^2}{2} \).

- Exercise 4 - Intersection of a Spanning Tree and a Perfect Matching

Let \( G = (V, E) \) be a bridgeless, cubic graph. Suppose that \( G \) contains a spanning tree \( T \) and a perfect matching \( M \) such that \( T \) and \( M \) intersect in \( k \) edges. In other words, there are \( k \) edges in \( E \) that belong both to the spanning tree, \( T \), and to the perfect matching, \( M \).

1. Show that \( G \) contains a TSP tour of length at most \( n + 2k \).

**Solution:** Let \( C = E \setminus M \) denote the cycle cover of \( G \) obtained by removing the perfect matching \( M \) from \( G \). Since there exists a spanning tree that intersects \( M \) in \( k \) edges, the cycle cover \( C \) consists of at most \( k \) cycles. Thus we consider the Eulerian tour consisting of the cycle cover \( C \) plus a double spanning tree on the graph constructed by contracting the cycles in \( C \). This tour contains at most \( n + 2k \) edges.

2. Do the following problems have polynomial-time algorithms? Justify your answers.

   (a) For a fixed spanning tree \( T \), find a perfect matching \( M \) that minimizes the number of edges belonging to both \( T \) and \( M \).

   Assign each edge in \( T \) cost 1 and each edge in \( E \setminus T \) cost 0. Since we can find a minimum cost perfect matching in polynomial time, this problem can be solved in polynomial time.

   (b) For a fixed perfect matching \( M \), find a spanning tree \( T \) that minimizes the number of edges belonging to both \( T \) and \( M \).

   Assign each edge in \( M \) cost 1 and each edge in \( E \setminus M \) cost 0. Since we can find a minimum cost spanning tree in polynomial time, this problem can be solved in polynomial time.

   (c) Find a spanning tree \( T \) and a perfect matching \( M \) that minimize the number of edges belonging to both \( T \) and \( M \).

   This problem is NP-complete. Note that a cubic graph \( G \) contains a spanning tree \( T \) and perfect matching \( M \) intersecting in zero edges iff the graph contains a Hamilton cycle. Thus, suppose that we have an efficient algorithm for this problem. Then we can determine whether or not a given cubic graph is Hamiltonian, which is an NP-complete problem.
- Exercise 5 - Red and Blue Tournaments

Let $T = (V, A)$ be a tournament, i.e. an oriented complete graph $K_n$ on $n$ vertices. Suppose that each edge is colored either red or blue. For two vertices $u$ and $v$, we say that there is a red path from $u$ to $v$ if: (i) $u = v$, or (ii) edge $(u, v) \in A$ and $(u, v)$ is colored red, or (iii) there is a vertex $i \neq u, v$ such that edges $(u, i)$ and $(i, v)$ belong to $A$ and both these edges are colored red. We define a blue path from $u$ to $v$ analogously.

a. True or False: There is a vertex $u$ in $V$ such that for each vertex $v$ in $V$, there is either a red path or a blue path from $u$ to $v$.

Solution: The statement is False as shown by the tournament on four vertices in Figure 4. (This is the smallest counterexample.)

Figure 4:

b. Here are some related problems to think about:

1. Let $S \subset V$ be a subset of vertices such that for all $v \notin S$, there is some $u \in S$ such that there is either a red path or blue path from $u$ to $v$. What is the minimum cardinality of $S$ (in the worst case)? (The solution to Part a. shows that $|S| \geq 2$.)

2. We defined a red (blue) path from $u$ to $v$ as a directed path of red (blue) edges from $u$ to $v$ of distance zero, one or two. Let us now change this definition to be a directed path of red (blue) edges from $u$ to $v$ of distance between zero and $n - 1$.

True or False: There is a vertex $u$ in $V$ such that for each vertex $v$ in $V$, there is either a red path or a blue path from $u$ to $v$. 
