Homework 1
Due December 8 in class or by email at 13h00 [1]

- Exercise 1 - Distribution of Minimum Cuts

True or False: For a graph $G = (V, E)$, let $S_1 \subset V$ and $S_2 \subset V$ denote two distinct, nonempty subsets of vertices, each corresponding to a minimum cut of G. Then each of these cuts is equally likely to be output by the routine RANDOM-CONTRACT(G). Justify your answer.

- Exercise 2 - Dominating Set: LP Formulations and Approximation

Let $G = (V, A)$ denote a directed graph. We use (i, j) to denote an edge directed from vertex i to vertex j. A *dominating set* in G is a subset of vertices, $S \subseteq V$, such that for every vertex $j \in V$, either $j \in S$ or there is a directed edge $(i, j) \in A$ such that $i \in S$. The *minimum dominating set problem* is to find a dominating set S of minimum cardinality (i.e. $|S|$ is minimum).

a. Write an integer program for the minimum dominating set problem.

b. Write a linear programming relaxation for the minimum dominating set problem.

c. A *tournament* is a directed graph $T = (V, A)$ such that for every pair of vertices $i, j \in V$, either edge (i, j) or edge (j, i) belongs to A. Give a $\log n$-approximation algorithm for the minimum dominating set problem in a tournament.

- Exercise 3 - Correlation Clustering with One Hyperplane

In Lecture 5 (Sections 1.3 and 1.4), we presented a rounding algorithm for the correlation clustering problem. What approximation ratio can be obtained with using just one hyperplane? (i.e. use g rather than both g and h in Section 1.4.)

- Exercise 4 - Asymmetric TSP: Separation Oracles

Given a directed graph $G = (V, A)$, with edge weights $w : A \to \mathbb{R}^+$ where $w_{ij} + w_{jk} \geq w_{ik}$ for all $\{i, j, k\} \in V$, our goal is to find a spanning, Eulerian, multigraph (i.e. edges may be used multiple times) in G of minimum weight. By *spanning*, we mean that the multigraph includes each vertex. Note that the edge weights may be *asymmetric*, i.e. $w_{ij} \neq w_{ji}$.

a. Write an integer program for this problem.

b. Write a linear programming relaxation for this problem.

c. Give an efficient separation oracle for the LP in Part b.

[1] Homework submitted by email must be written using LaTex and attached in .pdf format.

- Exercise 5 - Rounding Linear Program for s-t-Min-Cut

Recall the following linear programming relaxation for the s-t-min-cut problem. The following variables correspond to edges and vertices:

- $\ell : E \rightarrow \mathbb{R}$, where $\ell(i,j)$ is the *length* of an edge ij,
- $p : V \rightarrow \mathbb{R}$, where $p(i)$ is the *potential* of a vertex i.

$$
\begin{aligned}
& \min \sum_{ij \in E} \ell(ij) \\
\text{subject to:} \quad & \ell(ij) \geq p(i) - p(j), \quad ij \in E, \\
& \ell(ij) \geq p(j) - p(i), \quad ij \in E, \\
& p(s) = 0, \\
& p(t) = 1 \\
& p(i) \geq 0, \quad i \in V, \\
& \ell(ij) \geq 0, \quad ij \in E. \qquad (P_{min-cut})
\end{aligned}
$$

For a fixed graph $G = (V, E)$, let $\{p^*(i), \ell^*(ij)\}$ denote an optimal solution for $(P_{min-cut})$ on G. Consider the following algorithm.

s-t-Random-Cut(G)

1. $S \leftarrow \emptyset$.
2. Choose $\beta \in (0, 1)$ uniformly at random.
3. For each edge $ij \in E$:
   (a) Without loss of generality, assume $p^*(i) \leq p^*(j)$.
   (b) If $\beta \in [p^*(i), p^*(j)]$, add ij to solution set S.
4. Output the set of edges in S.

Prove or disprove: Algorithm s-t-Random-Cut outputs a minimum s-t-cut of G with high probability.