Advanced Heuristic and Approximation Algorithms Grenoble-INP (ORCO)

Lecture 7

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1 Traveling Salesman Problem

Let $G = (V, E)$ be an undirected, unweighted graph. The *traveling salesman problem* (TSP) on G is to find a spanning, connected, Eulerian multisubset of edges, $F \subseteq E$, that has minimum cardinality, i.e. $|F|$ is minimized. The graph formed by the edge set F might be a *multigraph*, i.e. edges can be included more than once. (However, note that the input graph G is not a multigraph.) We say that F is *spanning* if every vertex in V is adjacent to some edge in F. We say that F is *connected* if the edges in F form a single connected component. And we say that F is *Eulerian* if every vertex in V has even degree in F. (Note that a graph with vertices but no edges is Eulerian, but not spanning.) Sometimes, we refer to a feasible solution as a *tour*. We consider some examples.

- Let $G = K_n$, the complete graph on n vertices. Then since G contains a Hamilton cycle, which is (by definition) spanning, connected and Eulerian, there is a solution consisting of n edges.
- The graph shown in Figure 1, which consists of three paths on k edges. (In Figure 1, $k = 6$.) For such a graph, the number of edges is $\approx 3k$, while the number of edges in a minimum tour is $\approx 4k$.

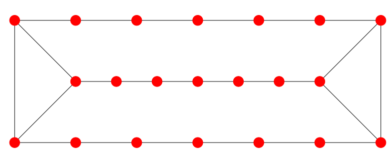

Figure 1: The "envelope graph" consists of three paths on k edges. To make this graph Eulerian, at least k edges must be doubled.

Some observations:

- We can always assume that the input graph G is 2-vertex connected. Otherwise, we can decompose G into 2-vertex connected components and solve the problem on each component separately.
- No edge needs to be included in the multigraph more than twice. Suppose an edge were included, say, three times. Then two copies can be removed and the remaining (smaller) multigraph remains spanning, connected and Eulerian. Thus each edge in E is included in F once, twice or not at all.

When the input graph G is unweighted (as described above), the problem is often referred to as *graph-TSP*.

1.1 Metric TSP

Suppose we are given a connected graph $G = (V, E)$ with edge weights that obey the triangle inequality, i.e. $w_{ij} + w_{jk} \geq w_{ik}$ for all $ijk \in V$. For all $i, j \in V$ such that $ij \notin E$, we can let w_{ij} denote the weight of the shortest path between i and j in G. If we do this for every pair of vertices in V, we obtain the *metric completion* of G, which we denote by G_{met}. Observe that G_{met} is a complete, weighted graph. Now the traveling salesman problem (TSP) on G is equivalent to finding a Hamilton cycle in G_{met} of minimum weight. Such an instance of TSP is often referred to as *metric TSP* or *symmetric TSP*, since the edge weights are symmetric, i.e. $w_{ij} = w_{ji}$. Note that graph-TSP is a special case of metric TSP with $w_{ij} = 1$ for all $ij \in E$.

2 Combinatorial Algorithms

Double-Spanning-Tree Algorithm: Given a complete, weighted graph whose edge weights obey triangle inequality, find a minimum spanning tree (MST). Doubling all edges in the MST yields an Eulerian multigraph. Shortcut the tour so that no vertex is visited more than once. The total cost of this tour is at most $2 \cdot w(MST) \leq 2 \cdot OPT$.

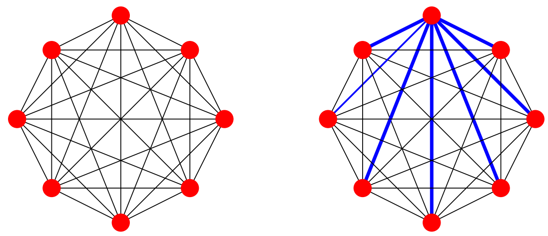

Figure 2: Doubling the minimum spanning tree of the complete graph K_n results in an Eulerian multigraph of weight $2n-2$. However, shortcutting leads to an optimal tour.

Exercise 1. *Find a graph for which the* Double-Spanning-Tree Algorithm *produces a tour close to* $2 \cdot OPT$*.*

Christofides' Algorithm: An improvement upon the previous algorithm is due to Christofides [Chr76]. Find a minimum spanning tree MST of G. Let J be the set of vertices with odd degree in MST. Find a minimum cost matching M on the induced graph $G' = (J, E(J))$. It can be shown that $w(M) \leq OPT/2$. Since $MST + T$ is Eulerian, this results in a tour of weight at most $w(MST) + w(M) \leq OPT + OPT/2$. So this is a $\frac{3}{2}$-approximation algorithm.

Our next goal is to give an alternative analysis of Christofides' Algorithm using linear programming. First, we present some necessary polyhedral tools.

3 Matching Polyhedra

For a nonempty subset $S \subset V$, let $\delta(S) \subset E$ denote all edges with exactly one endpoint in S. ($P_{BIP\text{-}PM}$) is a linear programming relaxation for the perfect matching problem on a bipartite graph.

$$\begin{aligned} \min \sum_{e \in E} & w_e x_e \\ x(\delta(i)) = 1, &\quad \text{for all } i \in V, \\ x_e \geq 0, &\quad \text{for each } e \in E. \end{aligned} \tag{$P_{BIP\text{-}PM}$}$$

An solution for a linear program can always be achieved by a point that is a *vertex* of the polyhedron. For example, the simplex algorithm always returns an optimal solution that is also a vertex.

Definition 1. *Let $P \in \mathbb{R}^n$ be a polyhedron. Then $\mathbf{x} \in P$ is an* vertex *of P if there exists a $\mathbf{c} \in \mathbb{R}^n$ such that $c^\intercal \mathbf{x} < \mathbf{c}^\intercal \mathbf{y}$ for all $\mathbf{y} \in P$ where $\mathbf{y} \neq \mathbf{x}$.*

A vertex of a polyhedron is in fact equivalent to an *extreme point* of a polyhedron.

Definition 2. *Let $P \in \mathbb{R}^n$ be a polyhedron. Then $\mathbf{x} \in P$ is an* extreme point *of P if $\nexists \mathbf{y}, \mathbf{z} \in P$ where $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{z} \neq \mathbf{x}$ such that $\mathbf{x} = \lambda \mathbf{y} + (1-\lambda)\mathbf{z}$ for some $\lambda \in (0,1)$.*

Exercise 2. *Show that an extreme point of* $(P_{BIP\text{-}PM})$ *is integral.*

Any point in a polyhedron $P \in \mathbb{R}^n$ can be written as a convex combination of its extreme points. In other words, let $\mathbf{x}$ be a point in P and let $\{\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots, \mathbf{x}_d^*\}$ denote extreme points of P. Then we have the following:

$$\sum_{i=1}^{d} \lambda_i = 1, \quad \mathbf{x} = \sum_{i=1}^{d} \lambda_i \cdot \mathbf{x}_i^*.$$

By Carathéodory's Theorem, we have $d \leq n+1$.

Now consider the matching polytope in a general graph.

$$\begin{aligned} \min \sum_{e \in E} w_e x_e & \\ x(\delta(i)) = 1, &\quad \text{for all } i \in V, \\ x(\delta(U)) \geq 1, &\quad \text{for all } U \in V \text{ such that } |U| \text{ is odd}, \\ x_e \geq 0, &\quad \text{for each } e \in E. \end{aligned} \tag{P_{PM}}$$

A famous theorem of Edmonds states that the extreme points of (P_{PM}) are integral [Edm65]. Moreover, (P_{PM}) has an efficient separation oracle. Thus, we can find integral solutions for (P_{PM}) in polynomial time.

4 Polyhedral Analysis of Christofides' Algorithm

Recall the following linear programming relaxation for TSP (P_{TSP}).

$$\begin{aligned} \min \sum_{e \in E} x_e & \\ x(\delta(i)) = 2, &\quad \text{for all } i \in V, \\ x(\delta(S)) \geq 2, &\quad \text{for all } S \subset V \text{ such that } S \neq \emptyset, \\ x_e \geq 0, &\quad \text{for each } e \in E. \end{aligned} \tag{P_{TSP}}$$

For a complete, weighted graph $G = (V, E)$, let $OPT_f(G)$ denote the optimal value of (P_{TSP}) on G. Let $V' \subset V$ be a subset of vertices. Define $G' = (V', E')$, where G' is the graph induced on the vertex set V'. The following Theorem is due to Williamson and Shmoys [SW90].

Theorem 3. $OPT_f(G') \leq OPT_f(G)$.

Exercise 3. *Find a simpler proof of Theorem 3.*

Consider an MST of G and let $J \subset V$ denote the vertices that have odd degree in MST. Now let $V' = J$ and let $\mathbf{x}^*$ denote an optimal solution for (P_{TSP}) on G'. Observe that

$$\mathbf{y}^* = \frac{\mathbf{x}^*}{2}$$

is a solution for (P_{PM}) on G'. Then by Theorem 3 and the fact that the extreme points of (P_{PM}) are integral, we see that there is a perfect matching in G with weight at most $OPT_f(G)/2$.

References

[Chr76] Nicos Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report Technical Report 388, Graduate School of Industrial Administration, Carnegie-Mellon University, 1976.

[Edm65] Jack Edmonds. Maximum matching and a polyhedron with 0,l-vertices. *Journal of Research of the National Bureau of Standards*, 69B(1965):125–130, 1965.

[SW90] David B. Shmoys and David P. Williamson. Analyzing the Held-Karp TSP bound: A monotonicity property with application. *Information Processing Letters*, 35(6):281–285, 1990.