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Detecting induced subgraphs

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ABSTRACT

An *s-graph* is a graph with two kinds of edges: *subdivisible* edges and *real* edges. A *realisation* of an s-graph B is any graph obtained by subdividing subdivisible edges of B into paths of arbitrary length (at least one). Given an s-graph B, we study the decision problem Π_B whose instance is a graph G and question is "Does G contain a realisation of B as an induced subgraph?". For several B's, the complexity of Π_B is known and here we give the complexity for several more.

Our NP-completeness proofs for Π_B's rely on the NP-completeness proof of the following problem. Let $\mathcal{S}$ be a set of graphs and d be an integer. Let $\Gamma^d_{\mathcal{S}}$ be the problem whose instance is (G, x, y) where G is a graph whose maximum degree is at most d, with no induced subgraph in $\mathcal{S}$ and $x, y \in V(G)$ are two non-adjacent vertices of degree 2. The question is "Does G contain an induced cycle passing through x, y?". Among several results, we prove that $\Gamma^3_{\emptyset}$ is NP-complete. We give a simple criterion on a connected graph H to decide whether $\Gamma^{+\infty}_{\{H\}}$ is polynomial or NP-complete. The polynomial cases rely on the algorithm three-in-a-tree, due to Chudnovsky and Seymour.

1. Introduction

In this paper graphs are simple and finite. A *subdivisible graph* (*s-graph* for short) is a triple $B = (V, D, F)$ such that $(V, D \cup F)$ is a graph and $D \cap F = \emptyset$. The edges in D are said to be *real edges of* B while the edges in F are said to be *subdivisible edges of* B. A *realisation* of B is a graph obtained from B by subdividing edges of F into paths of arbitrary length (at least one). The problem Π_B is the decision problem whose input is a graph G and whose question is "Does G contain a realisation of B as an induced subgraph?". On figures, we depict real edges of an s-graph with straight lines, and subdivisible edges with dashed lines.

Several interesting instances of Π_B are studied in the literature. For some of them, the existence of a polynomial time algorithm is trivial, but efforts are devoted toward optimized algorithms. For example, Alon, Yuster and Zwick [2] solve Π_T in time $O(m^{1.41})$ (instead of the obvious $O(n^3)$ algorithm), where T is the s-graph depicted on Fig. 1. This problem is known as *triangle detection*. Rose, Tarjan and Lueker [10] solve Π_H in time $O(n + m)$ where H is the s-graph depicted on Fig. 1.

For some Π_B's, the existence of a polynomial time algorithm is non-trivial. A *pyramid* (resp. *prism*, *theta*) is any realisation of the s-graph B_1 (resp. B_2, B_3) depicted on Fig. 2. Chudnovsky and Seymour [4] gave an $O(n^9)$-time algorithm for Π_{B_1} (or equivalently, for detecting a pyramid). As far as we know, that is the first example of a solution to a Π_B whose complexity is non-trivial to settle. In contrast, Maffray and Trotignon [8] proved that Π_{B_2} (or detecting a prism) is NP-complete.

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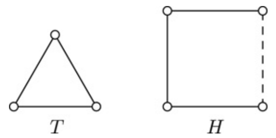


Fig. 1. s-graphs yielding trivially polynomial problems.

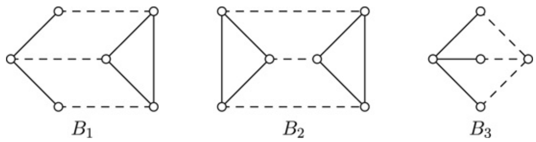


Fig. 2. Pyramids, prisms and thetas.

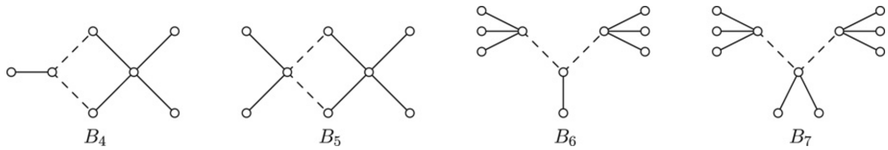


Fig. 3. Some s-graphs with pending edges.

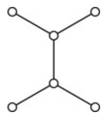


Fig. 4. I_1.

Chudnovsky and Seymour [5] gave an $O(n^{11})$-time algorithm for P_{B_3} (or detecting a theta). Their algorithm relies on the solution of a problem called "three-in-a-tree", that we will define precisely and use in Section 2. The three-in-tree algorithm is quite general since it can be used to solve a lot of Π_B problems, including the detection of pyramids.

These facts are a motivation for a systematic study of Π_B. A further motivation is that very similar s-graphs can lead to a drastically different complexity. The following example may be more striking than pyramid/prism/theta: Π_{B_4}, Π_{B_6} are polynomial and Π_{B_5}, Π_{B_7} are NP-complete, where $B_4, \dots, B_7$ are the s-graphs depicted on Fig. 3. This will be proved in Section 3.1.

1.1. Notation and remarks

By C_k ($k \geq 3$) we denote the cycle on k vertices, by K_l ($l \geq 1$) the clique on l vertices. A *hole* in a graph is an induced cycle on at least four vertices. We denote by I_l ($l \geq 1$) the tree on $l+5$ vertices obtained by taking a path of length l with ends a, b, and adding four vertices, two of them adjacent to a, the other two to b; see Fig. 4. When a graph G contains a graph isomorphic to H as an induced subgraph, we will often say "G contains an H".

Let (V, D, F) be an s-graph. Suppose that $(V, D \cup F)$ has a vertex of degree one incident to an edge e. Then $\Pi_{(V, D\cup\{e\}, F\setminus\{e\})}$ and $\Pi_{(V, D\setminus\{e\}, F\cup\{e\})}$ have the same complexity, because a graph G contains a realisation of $(V, D \cup \{e\}, F \setminus \{e\})$ if and only if it contains a realisation of $(V, D \setminus \{e\}, F \cup \{e\})$. For the same reason, if $(V, D \cup F)$ has a vertex of degree two incident to the edges $e \neq f$ then $\Pi_{(V, D\setminus\{e\}\cup\{f\}, F\setminus\{f\}\cup\{e\})}$, $\Pi_{(V, D\setminus\{f\}\cup\{e\}, F\setminus\{e\}\cup\{f\})}$ and $\Pi_{(V, D\setminus\{e,f\}, F\cup\{e,f\})}$ have the same complexity. If $|F| \leq 1$ then $\Pi_{(V,D,F)}$ is clearly polynomial. Thus, in the rest of the paper, we will consider only s-graphs (V, D, F) such that:

- $|F| \geq 2$;
- no vertex of degree one is incident to an edge of F;
- every induced path of $(V, D \cup F)$ with all interior vertices of degree 2 and whose ends have degree $\neq 2$ has at most one edge in F. Moreover, this edge is incident to an end of the path;
- every induced cycle with at most one vertex v of degree at least 3 in $(V, D \cup F)$ has at most one edge in F and this edge is incident to v if v exists (if it does not then the cycle is a component of $(V, D \cup F)$).

2. Detection of holes with prescribed vertices

Let $\Delta(G)$ be the maximum degree of G. Let $\mathcal{S}$ be a set of graphs and d be an integer. Let $\Gamma_{\mathcal{S}}^d$ be the problem whose instance is (G, x, y) where G is a graph such that $\Delta(G) \leq d$, with no induced subgraph in $\mathcal{S}$ and $x, y \in V(G)$ are two non-adjacent vertices of degree 2. The question is "Does G contain a hole passing through x, y?". For simplicity, we write $\Gamma_{\mathcal{S}}$ instead of

$\Gamma_{\mathcal{S}}^{+\infty}$ (so, the graph in the instance of $\Gamma_{\mathcal{S}}$ has unbounded degree). Also we write Γ^d instead of $\Gamma_{\emptyset}^d$ (so the graph in the instance of Γ^d has no restriction on its induced subgraphs). Bienstock [3] proved that $\Gamma = \Gamma_{\emptyset}$ is NP-complete. For $\mathcal{S} = \{K_3\}$ and $\mathcal{S} = \{K_{1,4}\}$, $\Gamma_{\mathcal{S}}$ can be shown to be NP-complete, and a consequence is the NP-completeness of several problems of interest: see [8,9].

In this section, we try to settle $\Gamma_{\mathcal{S}}^d$ for as many $\mathcal{S}$'s and d's as we can. In particular, we give the complexity of $\Gamma_{\mathcal{S}}$ when $\mathcal{S}$ contains only one connected graph and of Γ^d for all d. We also settle $\Gamma_{\mathcal{S}}^d$ for some cases when $\mathcal{S}$ is a set of cycles. The polynomial cases are either trivial, or are a direct consequence of an algorithm of Chudnovsky and Seymour. The NP-complete cases follow from several extensions of Bienstock's construction.

2.1. Polynomial cases

Chudnovsky and Seymour [5] proved that the problem whose instance is a graph G together with three vertices a, b, c and whose question is "Does G contain a tree passing through a, b, c as an induced subgraph?" can be solved in time $O(n^4)$. We call this algorithm "three-in-a-tree". Three-in-a-tree can be used directly to solve $\Gamma_{\mathcal{S}}$ for several $\mathcal{S}$'s. Let us call *subdivided claw* any tree with one vertex u of degree 3, three vertices v_1, v_2, v_3 of degree 1 and all the other vertices of degree 2.

Theorem 2.1. *Let H be a graph on k vertices that is either a path or a subdivided claw. There is an $O(n^k)$-time algorithm for $\Gamma_{\{H\}}$.*

Proof. Here is an algorithm for $\Gamma_{\{H\}}$. Let (G, x, y) be an instance of Γ_H. If H is a path on k vertices then every hole in G is on at most k vertices. Hence, by a brute-force search on every k-tuple, we will find a hole through x, y if there is any. Now we suppose that H is a subdivided claw. So $k \geq 4$. For convenience, we put $x_1 = x$, $y_1 = y$. Let x_0, x_2 (resp. y_0, y_2) be the two neighbours of x_1 (resp. y_1).

First check whether there is in G a hole C through x_1, y_1 such that the distance between x_1 and y_1 in C is at most $k-2$. If $k = 4$ or $k = 5$ then $\{x_0, x_1, x_2, y_0, y_1, y_2\}$ either induces a hole (that we output) or a path P that is contained in every hole through x, y. In this last case, the existence of a hole through x, y can be decided in linear time by deleting the interior of P, deleting the neighbours in $G \setminus P$ of the interior vertices of P and by checking the connectivity of the resulting graph. Now suppose $k \geq 6$. For every l-tuple $(x_3, \dots, x_{l+2})$ of vertices of G, with $l \leq k-5$, test whether $P = x_0 - x_1 - \dots - x_{l+2} - y_2 - y_1 - y_0$ is an induced path, and if so delete the interior vertices of P and their neighbours except x_0, y_0, and look for a shortest path from x_0 to y_0. This will find the desired hole if there is one, after possibly swapping x_0, x_2 and doing the work again. This takes time $O(n^{k-3})$.

Now we may assume that in every hole through x_1, y_1, the distance between x_1, y_1 is at least $k-1$.

Let k_i be the length of the unique path of H from u to v_i, $i = 1, 2, 3$. Note that $k = k_1 + k_2 + k_3 + 1$. Let us check every $(k-4)$-tuple $z = (x_3, \dots, x_{k_1+1}, y_3, \dots, y_{k_2+k_3})$ of vertices of G. For such a $(k-4)$-tuple, test whether $x_0 - x_1 - \dots - x_{k_1+1}$ and $P = y_0 - y_1 - \dots - y_{k_2+k_3}$ are induced paths of G with no edge between them except possibly $x_{k_1+1}y_{k_2+k_3}$. If not, go to the next $(k-4)$-tuple, but if yes, delete the interior vertices of P and their neighbours except $y_0, y_{k_2+k_3}$. Also delete the neighbours of $x_2, \dots, x_{k_1}$, except $x_1, x_2, \dots, x_{k_1}, x_{k_1+1}$. Call G_z the resulting graph and run three-in-a-tree in G_z for the vertices $x_1, y_{k_2+k_3}, y_0$. We claim that the answer to three-in-a-tree is YES for some $(k-4)$-tuple if and only if G contains a hole through x_1, y_1 (after possibly swapping x_0, x_2 and doing the work again).

To prove this, first assume that G contains a hole C through x_1, y_1 then up to a symmetry this hole visits $x_0, x_1, x_2, y_2, y_1, y_0$ in this order. Let us name $x_3, \dots, x_{k_1+1}$ the vertices of C that follow after x_1, x_2 (in this order), and let us name $y_3, \dots, y_{k_2+k_3}$ those that follow after y_1, y_2 (in reverse order). Note that all these vertices exist and are pairwise distinct since in every hole through x_1, y_1 the distance between x_1, y_1 is at least $k-1$. So the path from y_0 to $y_{k_2+k_3}$ in $C \setminus y_1$ is a tree of G_z passing through $x_1, y_{k_2+k_3}, y_0$, where z is the $(k-4)$-tuple $(x_3, \dots, x_{k_1+1}, y_3, \dots, y_{k_2+k_3})$.

Conversely, suppose that G_z contains a tree T passing through $x_1, y_{k_2+k_3}, y_0$, for some $(k-4)$-tuple z. We suppose that T is vertex-inclusion-wise minimal. If T is a path visiting $y_0, x_1, y_{k_2+k_3}$ in this order, then we obtain the desired hole of G by adding $y_1, y_2, \dots, y_{k_2+k_3-1}$ to T. If T is a path visiting $x_1, y_0, y_{k_2+k_3}$ in this order, then we denote by $y_{k_2+k_3+1}$ the neighbour of $y_{k_2+k_3}$ along T. Note that T contains either x_0 or x_2. If T contains x_0, then there are three paths in G: $y_0 - T - x_0 - x_1 - \dots - x_{k_1}$, $y_0 - T - y_{k_2+k_3+1} - \dots - y_{k_3+2}$ and $y_0 - y_1 - \dots - y_{k_3}$. These three paths form a subdivided claw centered at y_0 that is long enough to contain an induced subgraph isomorphic to H, a contradiction. If T contains x_2 then the proof works similarly with $y_0 - T - x_{k_1+1} - x_{k_1} - \dots - x_1$ instead of $y_0 - T - x_0 - x_1 - \dots - x_{k_1}$. If T is a path visiting $x_1, y_{k_2+k_3}, y_0$ in this order, the proof is similar, except that we find a subdivided claw centered at $y_{k_2+k_3}$. If T is not a path, then it is a subdivided claw centered at a vertex u of G. We obtain again an induced subgraph of G isomorphic to H by adding to T sufficiently many vertices of $\{x_0, \dots, x_{k_1+1}, y_0, \dots, y_{k_2+k_3}\}$. $\square$

2.2. NP-complete cases (unbounded degree)

Many NP-completeness results can be proved by adapting Bienstock's construction. We give here several polynomial reductions from the problem 3-Satisfiability of Boolean functions. These results are given in a framework that involves a few parameters, so that our result can possibly be used for different problems of the same type. Recall that a Boolean function with n variables is a mapping f from $\{0, 1\}^n$ to $\{0, 1\}$. A Boolean vector $\xi \in \{0, 1\}^n$ is a *truth assignment satisfying* f if $f(\xi) = 1$. For any Boolean variable z on $\{0, 1\}$, we write $\bar{z} := 1 - z$, and each of $z, \bar{z}$ is called a *literal*. An instance of

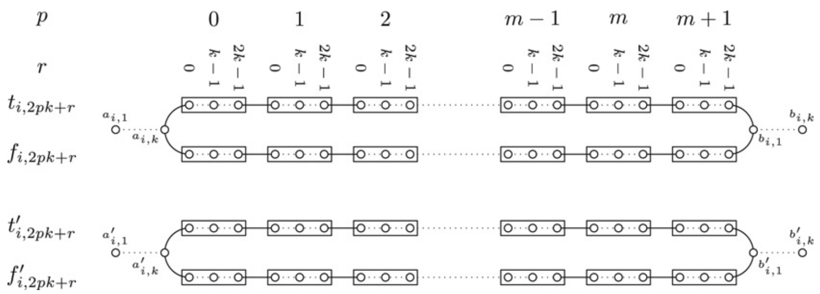


Fig. 5. The graph $G(z_i)$ (only blue edges are depicted).

3-Satisfiability is a Boolean function f given as a product of clauses, each clause being the Boolean sum $\vee$ of three literals; the question is whether f is satisfied by a truth assignment. The NP-completeness of 3-Satisfiability is a fundamental result in complexity theory, see [6].

Let f be an instance of 3-Satisfiability, consisting of m clauses $C_1, \ldots, C_m$ on n variables $z_1, \ldots, z_n$. For every integer $k \geq 3$ and parameters $\alpha \in \{1, 2\}$, $\beta \in \{0, 1\}$, $\gamma \in \{0, 1\}$, $\delta \in \{0, 1, 2, 3\}$, $\varepsilon \in \{0, 1\}$, $\zeta \in \{0, 1\}$ such that if $\alpha = 2$ then $\varepsilon = \beta = \gamma$, let us build a graph $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ with two specified vertices x, y of degree 2. There will be a hole containing x and y in $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ if and only if there exists a truth assignment satisfying f. In $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ (we will sometimes write G_f for short), there will be two kinds of edges: blue and red. The reason for this distinction will appear later. Let us now describe G_f.

2.2.1. *Pieces of G_f arising from variables*

For each variable z_i ($i = 1, \ldots, n$), prepare a graph $G(z_i)$ with $4k$ vertices $a_{i,r}, b_{i,r}, a'_{i,r}, b'_{i,r}$, $r \in \{1, \ldots, k\}$ and $4(m + 2)2k$ vertices $t_{i,2pk+r}, f_{i,2pk+r}, t'_{i,2pk+r}, f'_{i,2pk+r}$, $p \in \{0, \ldots, m + 1\}$, $r \in \{0, \ldots, 2k - 1\}$. Add blue edges so that the four sets $\{a_{i,1}, \ldots a_{i,k}, t_{i,0}, \ldots, t_{i,2k(m+2)-1}, b_{i,1}, \ldots, b_{i,k}\}$, $\{a_{i,1}, \ldots a_{i,k}, f_{i,0}, \ldots, f_{i,2k(m+2)-1}, b_{i,1}, \ldots, b_{i,k}\}$, $\{a'_{i,1}, \ldots a'_{i,k}, t'_{i,0}, \ldots, t'_{i,2k(m+2)-1}, b'_{i,1}, \ldots, b'_{i,k}\}$, $\{a'_{i,1}, \ldots a'_{i,k}, f'_{i,0}, \ldots, f'_{i,2k(m+2)-1}, b'_{i,1}, \ldots, b'_{i,k}\}$ all induce paths (and the vertices appear in this order along these paths). See Fig. 5.
Add red edges according to the value of α, β, γ, as follows:

- If $\alpha = 1$ then, for every $p = 1, \ldots, m + 1$, add all edges between $\{t_{i,2kp}, t_{i,2kp+\beta}\}$ and $\{f_{i,2kp}, f_{i,2kp+\gamma}\}$, between $\{f_{i,2kp}, f_{i,2kp+\gamma}\}$ and $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$, between $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$ and $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$, between $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$ and $\{t_{i,2kp}, t_{i,2kp+\beta}\}$.
- If $\alpha = 2$ then, for every $p = 1, \ldots, m$, add all edges between $\{t_{i,2kp+k-1}, t_{i,2kp+k-1+\beta}\}$ and $\{f_{i,2kp+k-1}, f_{i,2kp+k-1+\gamma}\}$; for every $p = 1, \ldots, m + 1$, add all edges between $\{f_{i,2kp+k-1}, f_{i,2kp+k-1+\gamma}\}$ and $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$, between $\{t'_{i,2kp}, t'_{i,2kp+\beta}\}$ and $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$, between $\{f'_{i,2kp}, f'_{i,2kp+\gamma}\}$ and $\{t_{i,2k(p-1)+k-1}, t_{i,2k(p-1)+k-1+\beta}\}$.

  See Figs. 6 and 7.

2.2.2. *Pieces of G_f arising from clauses*

For each clause C_j ($j = 1, \ldots, m$), with $C_j = y_j^1 \vee y_j^2 \vee y_j^3$, where each y_j^q ($q = 1, 2, 3$) is a literal from $\{z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n\}$, prepare a graph $G(C_j)$ with $2k$ vertices $c_{j,p}, d_{j,p}$, $p \in \{1, \ldots, k\}$ and $6k$ vertices $u^q_{j,p}$, $q \in \{1, 2, 3\}$, $p \in \{1, \ldots, 2k\}$. Add blue edges so that the three sets $\{c_{j,1}, \ldots, c_{j,k}, u^q_{j,1}, \ldots, u^q_{j,2k}, d_{j,1}, \ldots d_{j,k}\}$, $q \in \{1, 2, 3\}$ all induce paths (and the vertices appear in this order along these paths).
Add red edges according to the value of δ:

- If $\delta = 0$, add no edge.
- If $\delta = 1$, add $u^1_{j,1}u^2_{j,1}$, $u^1_{j,2k}u^2_{j,2k}$.
- If $\delta = 2$, add $u^1_{j,1}u^2_{j,1}$, $u^1_{j,2k}u^2_{j,2k}$, $u^1_{j,1}u^3_{j,1}$, $u^1_{j,2k}u^3_{j,2k}$.
- If $\delta = 3$, add $u^1_{j,1}u^2_{j,1}$, $u^1_{j,2k}u^2_{j,2k}$, $u^1_{j,1}u^3_{j,1}$, $u^1_{j,2k}u^3_{j,2k}$, $u^2_{j,1}u^3_{j,1}$, $u^2_{j,2k}u^3_{j,2k}$.

  See Fig. 8.

2.2.3. *Gluing the pieces of G_f*

The graph $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ is obtained from the disjoint union of the $G(z_i)$'s and the $G(C_j)$'s as follows. For $i = 1, \ldots, n - 1$, add blue edges $b_{i,k}a_{i+1,1}$ and $b'_{i,k}a'_{i+1,1}$. Add a blue edge $b'_{n,k}c_{1,1}$. For $j = 1, \ldots, m - 1$, add a blue edge $d_{j,k}c_{j+1,1}$. Introduce the two special vertices x, y and add blue edges $xa_{1,1}$, $xa'_{1,1}$ and $yd_{m,k}$, $yb_{n,k}$. See Fig. 9.

Add red edges according to f, ε, ζ. For $q = 1, 2, 3$, if $y^q_j = z_i$, then add all possible edges between $\{f_{i,2kj+k-1}, f_{i,2kj+k-1+\varepsilon}\}$ and $\{u^q_{j,k}, u^q_{j,k+\zeta}\}$ and between $\{f'_{i,2kj+k-1}, f'_{i,2kj+k-1+\varepsilon}\}$ and $\{u^q_{j,k}, u^q_{j,k+\zeta}\}$; if $y^q_j = \bar{z}_i$ then add all possible edges between $\{t_{i,2kj+k-1}, t_{i,2kj+k-1+\varepsilon}\}$ and $\{u^q_{j,k}, u^q_{j,k+\zeta}\}$ and between $\{t'_{i,2kj+k-1}, t'_{i,2kj+k-1+\varepsilon}\}$ and $\{u^q_{j,k}, u^q_{j,k+\zeta}\}$. See Fig. 10.

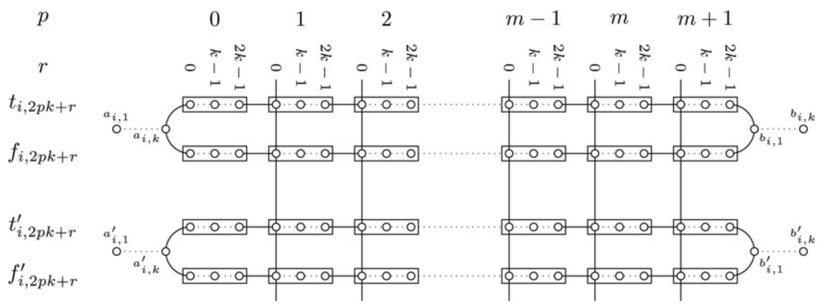


Fig. 6. The graph $G(z_i)$ when $\alpha = 1, \beta = 0, \gamma = 0$.

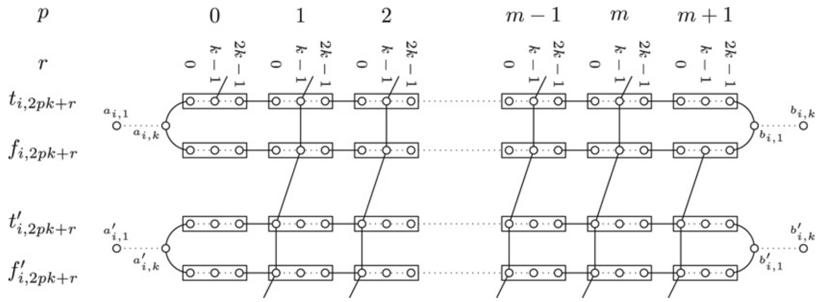


Fig. 7. The graph $G(z_i)$ when $\alpha = 2, \beta = 0, \gamma = 0$.

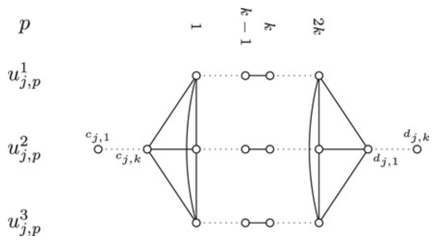


Fig. 8. The graph $G(C_j)$ when $\delta = 3$.

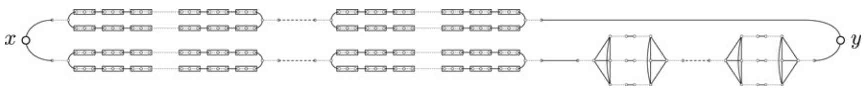


Fig. 9. The whole graph G_f.

Clearly the size of $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ is polynomial (actually quadratic) in the size $n+m$ of f, and x, y are non-adjacent and both have degree two.

Lemma 2.2. *f is satisfied by a truth assignment if and only if $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ contains a hole passing through x, y.*

Proof. Recall that if $\alpha = 2$ then $\varepsilon = \beta = \gamma$. We will prove the lemma for $\beta = 0, \gamma = 0, \varepsilon = 0, \zeta = 0$ because the proof is essentially the same for the other possible values.

Suppose that f is satisfied by a truth assignment $\xi \in \{0, 1\}^n$. We can build a hole in G by selecting vertices as follows. Select x, y. For $i = 1, \dots, n$, select $a_{i,p}, b_{i,p}, a'_{i,p}, b'_{i,p}$ for all $p \in \{1, \dots, k\}$. For $j = 1, \dots, m$, select $c_{j,p}, d_{j,p}$ for all $p \in \{1, \dots, k\}$. If $\xi_i = 1$ select $t_{i,p}, t'_{i,p}$ for all $p \in \{0, \dots, 2k(m+2) - 1\}$. If $\xi_i = 0$ select $f_{i,p}, f'_{i,p}$ for all $p \in \{0, \dots, 2k(m+2) - 1\}$. For $j = 1, \dots, m$, since ξ is a truth assignment satisfying f, at least one of the three literals of C_j is equal to 1, say $y_j^q = 1$ for some $q \in \{1, 2, 3\}$. Then select $u^q_{j,p}$ for all $p \in \{1, \dots, 2k\}$. Now it is a routine matter to check that the selected vertices induce a cycle Z that contains x, y, and that Z is chordless, so it is a hole. The main point is that there is no chord in Z between some subgraph $G(C_j)$ and some subgraph $G(z_i)$, for that would be either an edge $t_{i,p}u^q_{j,r}$ with $y^q_j = z_i$ and $\xi_i = 1$, or, symmetrically, an edge $f_{i,p}u^q_{j,r}$ with $y^q_j = \bar{z}_i$ and $\xi_i = 0$, and in either case this would contradict the way the vertices of Z were selected.

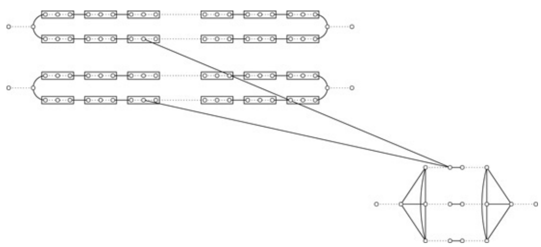


Fig. 10. Red edges between $G(z_i)$ and $G(C_j)$ when $\varepsilon = \zeta = 0$.

Conversely, suppose that $G_f(k, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ admits a hole Z that contains x, y.

(1) *For* $i = 1, \ldots, n$, *Z contains at least* $4k + 4k(m+2)$ *vertices of* $G(z_i)$: $4k$ *of these are* $a_{i,p}, a'_{i,p}, b_{i,p}, b'_{i,p}$ *where* $p \in \{1, \ldots, k\}$, *and the others are either the* $t_{i,p}, t'_{i,p}$*'s or the* $f_{i,p}, f'_{i,p}$*'s where* $p \in \{0, \ldots, 2k(m+2)-1\}$.

Let us first deal with the case $i = 1$. Since $x \in Z$ has degree 2, Z contains $a_{1,1}, \ldots, a_{1,k}$ and $a'_{1,1}, \ldots, a'_{1,k}$. Hence exactly one of $t_{1,0}, f_{1,0}$ is in Z. Likewise exactly one of $t'_{1,0}, f'_{1,0}$ is in Z. If $t_{1,0}, f'_{1,0}$ are both in Z then there is a contradiction: indeed, if $\alpha = 1$ then, $t_{1,0}, \ldots, t_{1,2k}$ and $f'_{1,0}, \ldots, f'_{1,2k}$ must all be in Z, and since $t_{1,2k}$ sees $f'_{1,2k}$, Z cannot go through y; and if $\alpha = 2$ the proof is similar. Similarly, $t'_{1,0}, f_{1,0}$ cannot both be in Z. So, there exists a largest integer $p \le 2k(m+2)-1$ such that either $t_{1,0}, \ldots, t_{1,p}$ and $t'_{1,0}, \ldots, t'_{1,p}$ are all in Z or $f_{1,0}, \ldots, f_{1,p}$ and $f'_{1,0}, \ldots, f'_{1,p}$ are all in Z.

We claim that $p = 2k(m+2)-1$. For otherwise, some vertex w in $\{t_{1,p}, t'_{1,p}, f_{1,p}, f'_{1,p}\}$ is incident to a red edge e of Z. If $\alpha = 1$ then, up to a symmetry, we assume that $t_{1,0}, \ldots, t_{1,p}$ and $t'_{1,0}, \ldots, t'_{1,p}$ are all in Z. Let w' be the vertex of e that is not w. Then w' (which is either an $f_{1,\cdot}$, an $f'_{1,\cdot}$ or a $u^{\cdot}_{j,\cdot}$) is a neighbour of both $t_{1,p}, t'_{1,p}$. Hence, Z cannot go through y, a contradiction. This proves our claim when $\alpha = 1$. If $\alpha = 2$, we distinguish between the following six cases.

Case 1: $p = k-1$. Then $e = t_{1,k-1}f'_{1,2k}$. Clearly $t_{1,0}, \ldots, t_{1,k-1}$ must all be in Z. If $t'_{1,0}, \ldots, t'_{1,2k}$ are in Z, there is a contradiction because of $t'_{1,2k}f'_{1,2k}$, and if $f'_{1,0}, \ldots, f'_{1,2k}$ are in Z, there is a contradiction because of e.

Case 2: $p = 2kl$ *where* $1 \le l \le m+1$ *and* $w = t'_{1,2kl}$. Then e is $t'_{1,2kl}f_{1,2kl+k-1}$ or $t'_{1,2kl}f'_{1,2kl}$. In either case, $t_{1,2kl}, \ldots, t_{1,2kl+k-1}$ are all in Z, and there is a contradiction because of the red edge $f_{1,2kl+k-1}t_{1,2kl+k-1}$ or $t_{1,2(l-1)k+k-1}f'_{1,2kl}$, or when $l = m+1$ because of $b_{1,1}$.

Case 3: $p = 2kl$ *where* $1 \le l \le m+1$ *and* $w = f'_{1,2kl}$. Then e is $f'_{1,2kl}t_{1,2(l-1)k+k-1}$ or $t'_{1,2kl}f'_{1,2kl}$. In either case, $f_{1,2kl}, \ldots, f_{1,2kl+k-1}$ are all in Z, and there is a contradiction because of the red edge $t_{1,2(l-1)k+k-1}f_{1,2(l-1)k+k-1}$ or $t'_{1,2kl}f_{1,2kl+k-1}$, or when $l = 1$ because of $a_{1,k}$.

Case 4: $p = 2kl + k - 1$ *where* $1 \le l \le m$ *and* $w = t_{1,2kl+k-1}$. Then e is $t_{1,2kl+k-1}f_{1,2kl+k-1}$, $t_{1,2kl+k-1}f'_{1,2(l+1)k}$, or $t_{1,2kl+k-1}u^q_{j,k}$ for some j, q. In the last case, there is a contradiction since $t'_{1,2kl+k-1} \in Z$ also sees $u^q_{j,k}$. For the same reason, $t'_{1,2kl+k-1}u^q_{j,k}$ is not an edge of Z and $t'_{1,2kl+k-1}, \ldots, t'_{1,2(l+1)k}$ are all in Z. So there is a contradiction because of the red edge $t'_{1,2kl}f_{1,2kl+k-1}$ or $t'_{1,2(l+1)k}f'_{1,2(l+1)k}$.

Case 5: $p = 2kl+k-1$ *where* $2 \le l \le m$ *and* $w = f_{1,2kl+k-1}$. Then e is either $f_{1,2kl+k-1}t_{1,2kl+k-1}$ or $f_{1,2kl+k-1}t'_{1,2kl}$ or $f_{1,2kl+k-1}u^q_{j,k}$ for some j, q. In the last case, there is a contradiction since $f'_{1,2kl+k-1} \in Z$ also sees $u^q_{j,k}$. For the same reason, $f'_{1,2kl+k-1}u^q_{j,k}$ is not an edge of Z and $f'_{1,2kl+k-1}, \ldots, f'_{1,2(l+1)k}$ are all in Z. So there is a contradiction because of the red edge $t'_{1,2kl}f'_{1,2kl}$ or $t_{1,2kl+k-1}f'_{1,2(l+1)}$.

Case 6: $p = 2k(m+1)+k-1$ *and* $w = f_{1,2k(m+1)+k-1}$. Then there is a contradiction because of the red edge $t'_{1,2k(m+1)}f'_{1,2k(m+1)}$. This proves our claim.

Since $p = 2k(m+2)-1$, $b_{1,1}$ is in Z. We claim that $b_{1,2}$ is in Z. For otherwise, the two neighbours of $b_{1,1}$ in Z are $t_{1,2k(m+2)-1}$ and $f_{1,2k(m+2)-1}$. This is a contradiction because of the red edges $t_{1,2km+k-1}f'_{1,2k(m+1)}$, $t'_{1,2k(m+1)}f_{1,2k(m+1)+k-1}$ (if $\alpha = 2$) or $t_{1,2k(m+1)}f'_{1,2k(m+1)}$, $t'_{1,2k(m+1)}f_{1,2k(m+1)}$ (if $\alpha = 1$). Similarly, $b'_{1,1}, b'_{1,2}$ are in Z. So $b_{1,1}, \ldots, b_{1,k}$ and $b'_{1,1}, \ldots, b'_{1,k}$ are all in Z.

This proves (1) for $i = 1$. The proof for $i = 2, \ldots, n$ is essentially the same as for $i = 1$. This proves (1).

(2) *For* $j = 1, \ldots, m$, *Z contains* $c_{j,1}, \ldots, c_{j,k}, d_{j,1}, \ldots, d_{j,k}$ *and exactly one of* $\{u^1_{j,1}, \ldots, u^1_{j,2k}\}, \{u^2_{j,1}, \ldots, u^2_{j,2k}\}, \{u^3_{j,1}, \ldots, u^3_{j,2k}\}$.

Let us first deal with the case $j = 1$. By (1), $b'_{n,k}$ is in Z and so $c_{1,1}, \ldots, c_{1,k}$ are all in Z. Consequently exactly one of $u^1_{1,1}, u^2_{1,1}, u^3_{1,1}$ is in Z, say $u^1_{1,1}$ up to a symmetry. Note that the neighbour of u^1_1 in $Z \setminus c_{1,k}$ cannot be a vertex among $u^2_{1,1}, u^3_{1,1}$ for this would imply that Z contains a triangle. Hence $u^1_{1,2}, \ldots, u^1_{1,k}$ are all in Z. The neighbour of $u^1_{1,k}$ in $Z \setminus u^1_{1,k-1}$ cannot be in some $G(z_i)$ $(1 \le i \le n)$. Else, up to a symmetry we assume that this neighbour is $t_{1,p}$, $p \in \{0, \ldots, 2k(m+2)-1\}$. If $t_{1,p} \in Z$, there is a contradiction because then $t'_{1,p}$ is also in Z by (1) and $t'_{1,p}$ would be a third neighbour of $u^1_{1,k}$ in Z. If $t_{1,p} \notin Z$, there is a contradiction because then the neighbour of $t_{1,p}$ in $Z \setminus u^1_{1,k}$ must be $t_{1,p+1}$(or symmetrically $t_{1,p-1}$) for otherwise Z contains a triangle. So, $t_{1,p+1}, t_{1,p+2}, \dots$ must be in Z, till reaching a vertex having a neighbour $f_{1,p'}$ or $f'_{1,p'}$ in Z (whatever α). Thus the neighbour of $u^1_{1,k}$ in $Z \setminus u^1_{1,k-1}$ is $u^1_{1,k+1}$. Similarly, we prove that $u_{1,k+2}, \dots, u_{1,2k}$ are in Z, that $d_{1,1}, \dots, d_{1,k}$ are in Z, and so the claim holds for $j = 1$. The proof of the claim for $j = 2, \dots, m$ is essentially the same. This proves (2).

Together with x, y, the vertices of Z found in (1) and (2) actually induce a cycle. So, since Z is a hole, they are the members of Z and we can replace "at least" by "exactly" in (1). We can now make a Boolean vector ξ as follows. For $i = 1, \dots, n$, if Z contains $t_{i,0}, t'_{i,0}$ set $\xi_i = 1$; if Z contains $f_{i,0}, f'_{i,0}$ set $\xi_i = 0$. By (1) this is consistent. Consider any clause C_j $(1 \le j \le m)$. By (2) and up to symmetry we may assume that $u^1_{j,k}$ is in Z. If $y^1_j = z_i$ for some $i \in \{1, \dots, n\}$, then the construction of G implies that $f_{i,2kj+k-1}, f'_{i,2j+k-1}$ are not in Z, so $t_{i,2kj+k-1}, t'_{i,2kj+k-1}$ are in Z, so $\xi_i = 1$, so clause C_j is satisfied by x_i. If $y^1_j = \bar{z}_i$ for some $i \in \{1, \dots, n\}$, then the construction of G_f implies that $t_{i,2kj+k-1}, t'_{i,2kj+k-1}$ are not in Z, so $f_{i,2kj+k-1}, f'_{i,2kj+k-1}$ are in Z, so $\xi_i = 0$, so clause C_j is satisfied by $\bar{z}_i$. Thus ξ is a truth assignment satisfying f. □

Theorem 2.3. *Let $k \ge 5$ be an integer. Then $\Gamma_{\{C_3,\dots,C_k,K_{1,6}\}}$ and $\Gamma_{\{I_1,\dots,I_k,C_5,\dots,C_k,K_{1,4}\}}$ are NP-complete.*

Proof. It is a routine matter to check that the graph $G_f(k, 2, 0, 0, 0, 0, 0)$ contains no C_l $(3 \le l \le k)$ and no $K_{1,6}$ (in fact it has no vertex of degree at least 6). So Lemma 2.2 implies that $\Gamma_{\{C_3,\dots,C_k,K_{1,6}\}}$ is NP-complete.

It is a routine matter to check that the graph $G_f(k, 1, 1, 1, 3, 1, 1)$ contains no $K_{1,4}$, no C_l $(5 \le l \le k)$ and no $I_{l'}$ $(1 \le l' \le k)$. So Lemma 2.2 implies that $\Gamma_{\{K_{1,4},C_5,\dots,C_k,I_1,\dots,I_k\}}$ is NP-complete. □

2.3. Complexity of $\Gamma_{\{H\}}$ when H is a connected graph

Theorem 2.4. *Let H be a connected graph. Then one of the following holds:*

- *H is a path or a subdivided claw and $\Gamma_{\{H\}}$ is polynomial.*
- *H contains one of $K_{1,4}$, I_k for some $k \ge 1$, or C_l for some $l \ge 3$ as an induced subgraph and $\Gamma_{\{H\}}$ is NP-complete.*

Proof. If H contains one of $K_{1,4}$, I_k for some $k \ge 1$, or C_l for some $l \ge 3$ as an induced subgraph then $\Gamma_{\{H\}}$ is NP-complete by Theorem 2.3. Otherwise, H is a tree since it contains no C_l, $l \ge 3$. If H has no vertex of degree at least 3, then H is a path and $\Gamma_{\{H\}}$ is polynomial by Theorem 2.1. If H has a single vertex of degree at least 3, then this vertex has degree 3 because H contains no $K_{1,4}$. So, H is a subdivided claw and $\Gamma_{\{H\}}$ is polynomial by Theorem 2.1. If H has at least two vertices of degree at least 3 then H contains an I_l, where l is the minimum length of a path of H joining two such vertices. This is a contradiction. □

Interestingly, the following analogous result for finding maximum stable sets in H-free graphs was proved by Alekseev:

Theorem 2.5 (*Alekseev, [1]*). *Let H be a connected graph that is not a path nor a subdivided claw. Then the problem of finding a maximum stable set in H-free graphs is NP-hard.*

But the complexity of the maximum stable set problem is not known in general for H-free graphs when H is a path or a subdivided claw. See [7] for a survey.

2.4. NP-complete cases (bounded degree)

Here, we will show that Γ^d is NP-complete when $d \ge 3$ and polynomial when $d = 2$. If $\mathcal{S}$ is any finite list of cycles $C_{k_1}, C_{k_2}, \dots, C_{k_m}$, then we will also show that $\Gamma^3_{\mathcal{S}}$ is NP-complete as long as $C_6 \notin \mathcal{S}$.

Let f be an instance of 3-Satisfiability, consisting of m clauses $C_1, \dots, C_m$ on n variables $z_1, \dots, z_n$. For each clause C_j $(j = 1, \dots, m)$, with $C_j = y_{3j-2} \vee y_{3j-1} \vee y_{3j}$, then y_i $(i = 1, \dots, 3m)$ is a literal from $\{z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n\}$.

Let us build a graph G_f with two specified vertices x and y of degree 2 such that $\Delta(G_f) = 3$. There will be a hole containing x and y in G_f if and only if there exists a truth assignment satisfying f.

For each literal y_j $(j = 1, \dots, 3m)$, prepare a graph $G(y_j)$ on 20 vertices $\alpha, \alpha', \alpha^{1+}, \dots, \alpha^{4+}, \alpha^{1-}, \dots, \alpha^{4-}, \beta, \beta', \beta^{1+}, \dots, \beta^{4+}, \beta^{1-}, \dots, \beta^{4-}$. (We drop the subscript j in the labels of the vertices for clarity.)

For $i = 1, 2, 3$ add the edges $\alpha^{i+}\alpha^{(i+1)+}, \beta^{i+}\beta^{(i+1)+}, \alpha^{i-}\alpha^{(i+1)-}, \beta^{i-}\beta^{(i+1)-}$. Also add the edges $\alpha^{1+}\beta^{1-}, \alpha^{1-}\beta^{1+}, \alpha^{4+}\beta^{4-}, \alpha^{4-}\beta^{4+}, \alpha\alpha^{1+}, \alpha\alpha^{1-}, \alpha^{4+}\alpha', \alpha^{4-}\alpha', \beta\beta^{1+}, \beta\beta^{1-}, \beta^{4+}\beta', \beta^{4-}\beta'$. See Fig. 11.

For each clause C_j $(j = 1, \dots, m)$, prepare a graph $G(C_j)$ with 10 vertices $c^{1+}, c^{2+}, c^{3+}, c^{1-}, c^{2-}, c^{3-}, c^{0+}, c^{12+}, c^{0-}, c^{12-}$. (We drop the subscript j in the labels of the vertices for clarity.)

Add the edges $c^{12+}c^{1+}, c^{12+}c^{2+}, c^{12-}c^{1-}, c^{12-}c^{2-}, c^{0+}c^{12+}, c^{0+}c^{3+}, c^{0-}c^{12-}, c^{0-}c^{3-}$. See Fig. 12.

For each variable z_i $(i = 1, \dots, n)$, prepare a graph $G(z_i)$ with $2z^-_i + 2z^+_i$ vertices, where z^-_i is the number of times $\bar{z}_i$ appears in clauses $C_1, \dots, C_m$ and z^+_i is the number of times z_i appears in clauses $C_1, \dots, C_m$.

Let $G(z_i)$ consist of two internally disjoint paths P^+_i and P^-_i with common endpoints d^+_i and d^-_i and lengths $1 + 2z^-_i$ and $1 + 2z^+_i$ respectively. Label the vertices of P^+_i as $d^+_i, p^+_{i,1}, \dots, p^+_{i,2f_i}, d^-_i$ and label the vertices of P^-_i as $d^+_i, p^-_{i,1}, \dots, p^-_{i,2g_i}, d^-_i$. See Fig. 13.

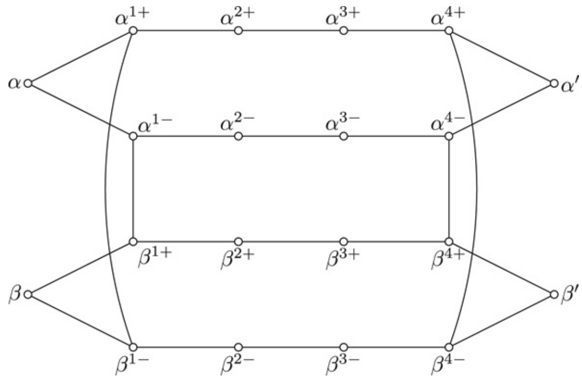


Fig. 11. The graph $G(y_j)$.

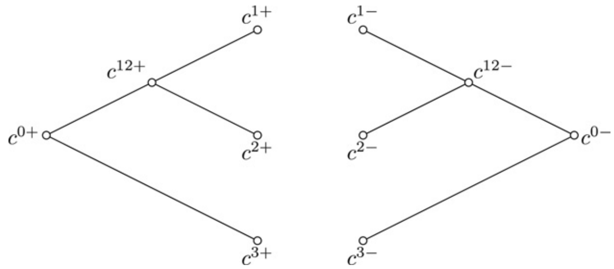


Fig. 12. The graph $G(C_j)$.

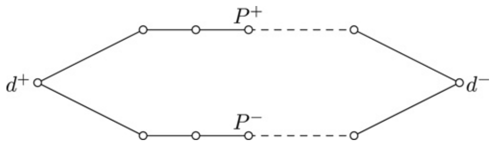


Fig. 13. The graph $G(z_i)$.

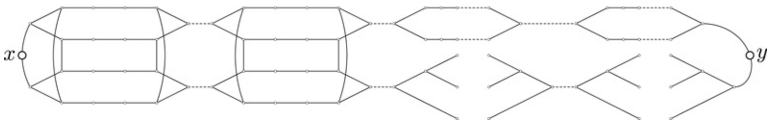


Fig. 14. The final graph G_f.

The final graph G_f (see Fig. 14) will be constructed from the disjoint union of all the graphs $G(y_j)$, $G(C_i)$, and $G(z_i)$ with the following modifications:

- For $j = 1, \ldots, 3m - 1$, add the edges $\alpha'_j\alpha_{j+1}$ and $\beta'_j\beta_{j+1}$.
- For $j = 1, \ldots, m - 1$, add the edge $c_j^{0-}c_{j+1}^{0+}$.
- For $i = 1, \ldots, n - 1$, add the edge $d_i^-d_{i+1}^+$.
- For $i = 1, \ldots, n$, let $y_{n_1}, \ldots, y_{n_{z_i^-}}$ be the occurrences of $\overline{z}_i$ over all literals. For $j = 1, \ldots, z_i^-$, delete the edge $p^+_{i,2j-1}p^+_{i,2j}$ and add the four edges $p^+_{i,2j-1}\alpha^{2+}_{n_j}$, $p^+_{i,2j-1}\beta^{2+}_{n_j}$, $p^+_{i,2j}\alpha^{3+}_{n_j}$, $p^+_{i,2j}\beta^{3+}_{n_j}$.
- For $i = 1, \ldots, n$, let $y_{n_1}, \ldots, y_{n_{z_i^+}}$ be the occurrences of z_i over all literals. For $j = 1, 2, \ldots, z_i^+$, delete the edge $p^-_{i,2j-1}p^-_{i,2j}$ and add the four edges $p^-_{i,2j-1}\alpha^{2+}_{n_j}$, $p^-_{i,2j-1}\beta^{2+}_{n_j}$, $p^-_{i,2j}\alpha^{3+}_{n_j}$, $p^-_{i,2j}\beta^{3+}_{n_j}$.
- For $i = 1, \ldots, m$ and $j = 1, 2, 3$, add the edges $\alpha^{2-}_{3(i-1)+j}c_i^{j+}$, $\alpha^{3-}_{3(i-1)+j}c_i^{j-}$, $\beta^{2-}_{3(i-1)+j}c_i^{j+}$, $\beta^{3-}_{3(i-1)+j}c_i^{j-}$.
- Add the edges $\alpha'_{3m}d_1^+$ and $\beta'_{3m}c_1^{0+}$
- Add the vertex x and add the edges $x\alpha_1$ and $x\beta_1$.
- Add the vertex y and add the edges yc_m^{0-} and yd_n^-.

It is easy to verify that $\Delta(G_f) = 3$, that the size of G_f is polynomial (actually linear) in the size $n + m$ of f, and that x, y are non-adjacent and both have degree two.

Lemma 2.6. *f is satisfied by a truth assignment if and only if G_f contains a hole passing through x and y.*

Proof. First assume that f is satisfied by a truth assignment $\xi \in \{0, 1\}^n$. We will pick a set of vertices that induce a hole containing x and y.

1. Pick vertices x and y.
2. For $i = 1, \ldots, 3m$, pick the vertices $\alpha_i, \alpha'_i, \beta_i, \beta'_i$.
3. For $i = 1, \ldots, 3m$, if y_i is satisfied by ξ, then pick the vertices $\alpha_i^{1+}, \alpha_i^{2+}, \alpha_i^{3+}, \alpha_i^{4+}, \beta_i^{1+}, \beta_i^{2+}, \beta_i^{3+}$, and β_i^{4+}. Otherwise, pick the vertices $\alpha_i^{1-}, \alpha_i^{2-}, \alpha_i^{3-}, \alpha_i^{4-}, \beta_i^{1-}, \beta_i^{2-}, \beta_i^{3-}$, and β_i^{4-}.
4. For $i = 1, \ldots, n$, if $\xi_i = 1$, then pick all the vertices of the path P_i^+ and all the neighbours of the vertices in P_i^+ of the form α_k^{2+} or α_k^{3+} for any k.
5. For $i = 1, \ldots, n$, if $\xi_i = 0$, then pick all the vertices of the path P_i^- and all the neighbours of the vertices in P_i^- of the form α_k^{2+} or α_k^{3+} for any k.
6. For $i = 1, \ldots, m$, pick the vertices c_i^{0+} and c_i^{0-}. Choose any $j \in \{3i-2, 3i-1, 3i\}$ such that ξ satisfies y_j. Pick vertices α_j^{2-}, and α_j^{3-}. If $j = 3i-2$, then pick the vertices $c_i^{12+}, c_i^{1+}, c_i^{1-}, c_i^{12-}$. If $j = 3i-1$, then pick the vertices $c_i^{12+}, c_i^{2+}, c_i^{2-}, c_i^{12-}$. If $j = 3i$, then pick the vertices c_i^{3+} and c_i^{3-}.

It suffices to show that the chosen vertices induce a hole containing x and y. The only potential problem is that for some k, one of the vertices $\alpha_k^{2+}, \alpha_k^{3+}, \alpha_k^{2-}$, or α_k^{3-} was chosen more than once. If α_k^{2+} and α_k^{3+} were picked in Step 3, then y_k is satisfied by ξ. Therefore, α_k^{2+} and α_k^{3+} were not chosen in Step 4 or Step 5. Similarly, if α_k^{2-} and α_k^{3-} were picked in Step 6, then y_k is satisfied by ξ and α_k^{2-} and α_k^{3-} were not picked in Step 3. Thus, the chosen vertices induce a hole in G containing vertices x and y.

Now assume G_f contains a hole H passing through x and y. The hole H must contain α_1 and β_1 since they are the only two neighbours of x. Next, either both α_1^{1+} and β_1^{1+} are in H, or both α_1^{1-} and β_1^{1-} are in H.

Without loss of generality, let α_1^{1+} and β_1^{1+} be in H (the same reasoning that follows will hold true for the other case). Since β_1^{1-} and α_1^{1-} are both neighbours of two members in H, they cannot be in H. Thus, α_1^{2+} and β_1^{2+} must be in H. Since α_1^{2+} and β_1^{2+} have the same neighbour outside $G(y_1)$, it follows that H must contain α_1^{3+} and β_1^{3+}. Also, H must contain α_1^{4+} and β_1^{4+}. Suppose that α_1^{4-} and β_1^{4-} are in H. Because α_1^{i-} has the same neighbour as β_1^{i-} outside $G(y_1)$ for $i = 2, 3$, it follows that H must contain $\alpha_1^{3-}, \alpha_1^{2-}$, and α_1^{1-}. But then H is not a hole containing b, a contradiction. Therefore, α_1^{4-} and β_1^{4-} cannot both be in H, so H must contain $\alpha'_1, \beta'_1, \alpha_2$, and β_2.

By induction, we see for $i = 1, 2, \ldots, 3m$ that H must contain $\alpha_i, \alpha'_i, \beta_i, \beta'_i$. Also, for each i, either H contains $\alpha_i^{1+}, \alpha_i^{2+}, \alpha_i^{3+}, \alpha_i^{4+}, \beta_i^{1+}, \beta_i^{2+}, \beta_i^{3+}, \beta_i^{4+}$ or H contains $\alpha_i^{1-}, \alpha_i^{2-}, \alpha_i^{3-}, \alpha_i^{4-}, \beta_i^{1-}, \beta_i^{2-}, \beta_i^{3-}, \beta_i^{4-}$.

As a result, H must also contain d_1^+ and c_1^{0+}. By symmetry, we may assume H contains $p_{1,1}^+$ and α_k^{2+} for some k. Since α_k^{1+} is adjacent to two vertices in H, H must contain α_k^{3+}. Similarly, H cannot contain α_k^{4+}, so H contains $p_{1,2}^+$ and $p_{1,3}^+$. By induction, we see that H contains $p_{1,i}^+$ for $i = 1, 2, \ldots, z_i^+$ and d_1^-. If H contains $p_{1,z_i^-}^-$, then H must contain $p_{1,i}^-$ for $i = z_i^-, \ldots, 1$, a contradiction. Thus, H must contain d_2^+. By induction, for $i = 1, 2, \ldots, n$, we see that H contains all the vertices of the path P_i^+ or P_i^- and by symmetry, we may assume H contains all the neighbours of the vertices in P_i^+ or P_i^- of the form α_k^{2+} or α_k^{3+} for any k.

Similarly, for $i = 1, 2, \ldots, m$, it follows that H must contain c_i^{0+} and c_i^{0-}. Also, H contains one of the following:

- $c_i^{12+}, c_i^{1+}, c_i^{1-}, c_i^{12-}$ and either α_j^{2-} and α_j^{3-} or β_j^{2-} and β_j^{3-} (where α_j^{2-} is adjacent to c_i^{1+}).
- $c_i^{12+}, c_i^{2+}, c_i^{2-}, c_i^{12-}$ and either α_j^{2-} and α_j^{3-} or β_j^{2-} and β_j^{3-} (where α_j^{2-} is adjacent to c_i^{2+}).
- c_i^{3+} and c_i^{3-} and either α_j^{2-} and α_j^{3-} or β_j^{2-} and β_j^{3-} (where α_j^{2-} is adjacent to c_i^{3+}).

We can recover the satisfying assignment ξ as follows. For $i = 1, 2, \ldots, n$, set $\xi_i = 1$ if the vertices of P_i^+ are in H and set $\xi_i = 0$ if the vertices of P_i^- are in H. By construction, it is easy to verify that at least one literal in every clause is satisfied, so ξ is indeed a satisfying assignment. □

Note that the graph G_f used above contains several C_6's that we could not eliminate, induced for instance by $\alpha, \alpha^{1+}, \beta^{1-}, \beta, \beta^{1+}, \alpha^{1-}$.

Theorem 2.7. *The following statements hold:*

- *For any $d \in \mathbb{Z}$ with $d \geq 2$, the problem Γ^d is NP-complete when $d \geq 3$ and polynomial when $d = 2$.*
- *If $\mathcal{H}$ is any finite list of cycles $C_{k_1}, C_{k_2}, \ldots, C_{k_m}$ such that $C_6 \notin \mathcal{H}$, then $\Gamma_{\mathcal{H}}^3$ is NP-complete.*

Proof. In the above reduction, $\Delta(G_f) = 3$ so Γ^d is NP-complete for $d \geq 3$. When $d = 2$, there is a simple $O(n)$ algorithm. Any hole containing x and y must be a component of G so pick the vertex x and consider the component C of G that contains x. It takes $O(n)$ time to verify whether C is a hole containing x and y or not.

To show the second statement, let K be the length of the longest cycle in $\mathcal{H}$. In the above reduction, do the following modifications.

- For $i = 1, 2, 3$ and $j = 1, 2, \dots, 3m$, replace the edges $\alpha_j^{i+}\alpha_j^{(i+1)+}$, $\alpha_j^{i-}\alpha_j^{(i+1)-}$, $\beta_j^{i+}\beta_j^{(i+1)+}$, and $\beta_j^{i-}\beta_j^{(i+1)-}$ by paths of length K.
- For $j = 1, 2, \dots, 3m - 1$, replace the edges $\alpha'_j\alpha_{j+1}$ and $\beta'_j\beta_{j+1}$ by paths of length K.
- Replace the edges $x\alpha_1$ and $x\beta_1$ by paths of length K.

This new reduction is polynomial in n, m and contains no graph of the list $\mathcal{H}$. The proof of Lemma 2.6 still holds for this new reduction, therefore $\Gamma^3_{\mathcal{H}}$ is NP-complete. □

3. Π_B for some special s-graphs

3.1. Holes with pending edges and trees

Here, we study $\Pi_{B_4}, \dots, \Pi_{B_7}$ where $B_4, \dots, B_7$ are the s-graphs depicted on Fig. 3. Our motivation is simply to give a striking example and to point out that, surprisingly, pending edges of s-graphs matter and that even an s-graph with no cycle can lead to NP-complete problems.

Theorem 3.1. *There is an $O(n^{13})$-time algorithm for Π_{B_4} but Π_{B_5} is NP-complete.*

Proof. A realisation of B_4 has exactly one vertex of degree 3 and one vertex of degree 4. Let us say that the realisation H is *short* if the distance between these two vertices in H is at most 3. Detecting short realisations of B_4 can be done in time n^9 as follows: for every 6-tuple $F = (a, b, x_1, x_2, x_3, x_4)$ such that $G[F]$ has edge-set $\{x_1a, ax_2, x_2b, bx_3, bx_4\}$ and for every 7-tuple $F = (a, b, x_1, x_2, x_3, x_4, x_5)$ such that $G[F]$ has edge-set $\{x_1a, ax_2, x_2x_3, x_3b, bx_4, bx_5\}$, delete $x_1, \dots, x_5$ and their neighbours except a, b. In the resulting graph, check whether a and b are in the same component. The answer is YES for at least one 7-or-6-tuple if and only if G contains at least one short realisation of B_4.

Here is an algorithm for Π_{B_4}, assuming that the entry graph G has no short realisation of B_4. For every 9-tuple $F = (a, b, c, x_1, \dots, x_6)$ such that $G[F]$ has edge-set $\{x_1a, bx_2, x_2x_3, x_3x_4, cx_5, x_5x_3, x_3x_6\}$ delete $x_1, \dots, x_6$ and their neighbours except a, b, c. In the resulting graph, run three-in-a-tree for a, b, c. It is easily checked that the answer is YES for some 9-tuple if and only if G contains a realisation of B_4.

Let us prove that Π_{B_5} is NP-complete by a reduction of Γ^3 to Π_{B_5}. Since by Theorem 2.7, Γ^3 is NP-complete, this will complete the proof. Let (G, x, y) be an instance of Γ^3. Prepare a new graph G': add four vertices x', x'', y', y'' to G and add four edges xx', xx'', yy', yy''. Since $\Delta(G) \leq 3$, it is easily seen that G contains a hole passing through x, y if and only if G' contains a realisation of B_5. □

The proof of the theorem below is omitted since it is similar to the proof of Theorem 3.1.

Theorem 3.2. *There is an $O(n^{14})$-time algorithm for Π_{B_6} but Π_{B_7} is NP-complete.*

3.2. Induced subdivisions of K_5

Here, we study the problem of deciding whether a graph contains an induced subdivision of K_5. More precisely, we put: $sK_5 = \left(\{a, b, c, d, e\}, \emptyset, \binom{\{a,b,c,d,e\}}{2}\right)$.

Theorem 3.3. *Π_{sK_5} is NP-complete.*

Proof. We consider an instance (G, x, y) of Γ^3. Let us denote by x', x'' the two neighbours of x and by y', y'' the two neighbours of y. Let us build a graph G' by adding five vertices a, b, c, d, e. We add the edges $ab, bd, dc, ca, ea, eb, ec, ed, ax', bx'', cy'', dy'$. We delete the edges xx', xx'', yy', yy''. We define a very similar graph G'', the only change being that we do not add edges cy'', dy' but edges cy', dy'' instead. See Fig. 15.

Now in G' (and similarly G'') every vertex has degree at most 3, except for a, b, c, d, e. We claim that G contains a hole going through x and y if and only if at least one of G', G'' contains an induced subdivision of K_5. Indeed, if G contains a hole passing through x, x', y', y, y'', x'' in that order then G' obviously contains an induced subdivision of K_5, and if the hole passes in order through x, x', y'', y, y', x'' then G'' contains such a subgraph. Conversely, if G' (or symmetrically G'') contains an induced subdivision of K_5 then a, b, c, d, e must be the vertices of the underlying K_5, because they are the only vertices with degree at least 4. Hence there is a path from x' to y' in $G \setminus \{x, y\}$ and a path from x'' to y'' in $G \setminus \{x, y\}$, and consequently a hole going through x, y in G. □

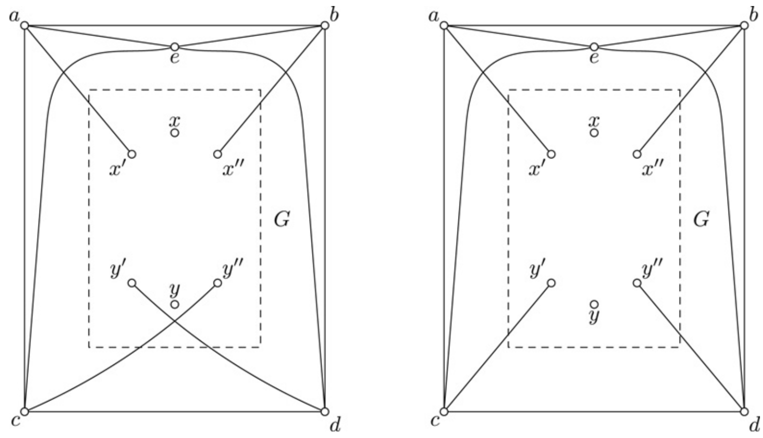
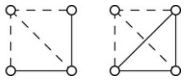


Fig. 15. Graphs G' and G''.

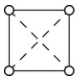
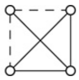
3.3. Π_B for small B's

Here, we survey the complexity Π_B when B has at most four vertices. By the remarks in the introduction, if $|V| \leq 3$ then $\Pi_{(V,D,F)}$ is polynomial. Up to symmetries, we are left with twelve s-graphs on four vertices as shown below.

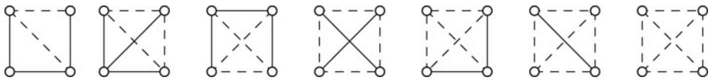
For the following two s-graphs, there is a polynomial algorithm using three-in-a-tree. The two algorithms are essentially similar to those for thetas and pyramids (see Fig. 2). See [5] for details.



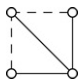
The next two s-graphs yield an NP-complete problem:

 (by $\Gamma_{\{C_4\}}$)  (by $\Gamma_{\{K_3\}}$)

For the next seven graphs on four vertices, we could not get an answer:



For the last graph represented below, it was proved recently by Trotignon and Vušković [11] that the problem can be solved in time $O(nm)$, using a method based on decompositions.



In conclusion we would like to point out that, except for the problem solved in [11], every detection problem associated with an s-graph for which a polynomial time algorithm is known can be solved either by using three-in-a-tree or by some easy brute-force enumeration.

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References

[1] V.E. Alekseev, On the local restrictions effect on the complexity of finding the graph independence number, Combinatorial-Algebraic Methods in Applied Mathematics 132 (1983) 3–13. Gorky University Press, Gorky (in Russian).
[2] N. Alon, R. Yuster, U. Zwick, Finding and counting given length cycles, in: Proceedings of the 2nd European Symposium on Algorithms, Utrecht, The Netherlands, 1994, pp. 354–364.
[3] D. Bienstock, On the complexity of testing for odd holes and induced odd paths, Discrete Mathematics 90 (1991) 85–92. See also Corrigendum by B. Reed, Discrete Math. 102 (1992) 109.
[4] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, K. Vušković, Recognizing Berge graphs, Combinatorica 25 (2005) 143–186.
[5] M. Chudnovsky, P. Seymour, The three-in-a-tree problem, Manuscript.
[6] M.R. Garey, D.S. Johnson, Computer and Intractability: A Guide to the Theory of NP-completeness, W.H. Freeman, San Fransisco, 1979.

[7] A. Hertz, V.V. Lozin, The maximum independent set problem and augmenting graphs, in: D. Avis, A. Hertz, O. Marcotte (Eds.), Graph Theory and Combinatorial Optimization, Springer, 2005, pp. 69–99.
[8] F. Maffray, N. Trotignon, Algorithms for perfectly contractile graphs, SIAM Journal on Discrete Mathematics 19 (3) (2005) 553–574.
[9] F. Maffray, N. Trotignon, K. Vušković, Algorithms for square-3*PC*(·, ·)-free Berge graphs, SIAM Journal on Discrete Mathematics 22((1) (2008) 51–71.
[10] D.J. Rose, R.E. Tarjan, G.S. Lueker, Algorithmic aspects of vertex elimination of graphs, SIAM Journal on Computing 5 (1976) 266–283.
[11] N. Trotignon, K. Vušković, A structure theorem for graphs with no cycle with a unique chord and its consequences, Journal of Graph Theory (in press).