# Topology of Combinatorial Surfaces

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# 1 What is a Map?

Topologically, a *map* corresponds to a **cellular embedding** of a graph in a 2-dimensional manifold: This is a drawing of a graph in a topological surface without crossing of the edges such that the embedded graph dissects the surface into topological open discs. Figure 1 shows a cellular embedding in a genus two surface. Up to homeomorphism,

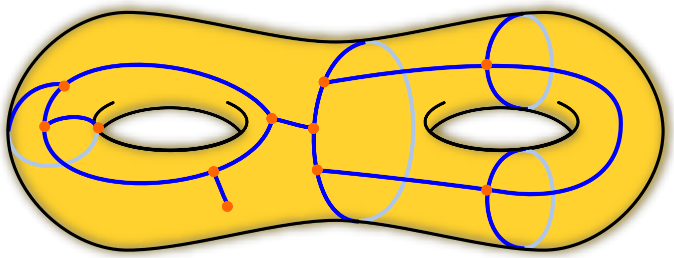

Figure 1: The complement of the graph in the surface is a disjoint union of open discs.

such a cellular embedding can be described by the graph together with the circular ordering of the edges incident to each vertex. These are purely combinatorial data referred to as a **combinatorial map**, a **combinatorial surface**, a **cellular embedding of a graph**, or just a **map**.

The theory of combinatorial maps was developed from the early 1970's in two parallel and independent directions. Both developments acknowledge the original works of Heffter [Hef91, Hef98] and Edmonds [Edm60] for the notion of combinatorial description of a graph embedded on a surface. On the more abstract side, mathematicians have succeeded to make beautiful connections between analysis, topology and algebra, going from Riemann surfaces and their coverings to algebraic curves and Galois theory of field extensions. Those connections were crystallised by Grothendieck through the notion of *dessins d'enfants* thanks to Belyi's theorem (see the gentle introduction by Zvonkin [Zvo]).

On the combinatorial side, maps appeared as the adequate formalism for topological graph theory such as exposed in a dedicated volume of the Cambridge Encyclopedia of mathematics [BW09]. Applications range from colouring problems, such as the four colour theorem and its generalization to higher genus surfaces, to embedding characterizations generalizing Kuratowski's theorem, up to the modern structural graph theory of Robertson and Seymour. The monograph by Mohar and Thomassen [MT01] is another important reference representing this trend. Pushing the combinatorial aspect to its limit, Tutte [Tut73, Tut79] was among the first to develop an axiomatic theory of combinatorial surfaces. His aim was to banish any reference to topology while getting equivalent results such as the Jordan's curve theorem [Tut79, Sta83, VL89], using combinatorial properties only. This point of view lead Tutte [Tut79] "…to eschew diagrams …because of their topological flavour". This might appear as a rather extreme attitude, although necessary when it comes to implementing algorithms.

A third development appeared in the early 1990's concerning curves on surfaces with a strong algorithmic objective [VY90, DS95]. Those works were recognized as part of *Computational topology* [Veg97, DEG98], a branch of Computational geometry focusing on algorithmic problems related to the topology of discrete structures. The point of view of Tutte is especially well suited to these computational aspects.

## 1.1 The category of oriented maps

We start with the description of combinatorial orientable surfaces. Although they can be considered as special cases of general surfaces, orientable or not, they deserve their own treatment as a simpler introduction to combinatorial surfaces. Their connection with Riemann surfaces through the theory of *dessins d'enfants* also provides them with a well established status. Indeed, Riemann surfaces are naturally oriented: such a surface is defined by a complex analytic atlas whose transition maps have positive Jacobians by the Cauchy-Riemann equations.

**Definition 1.1.** An **oriented map** is a triple $M = (A, \rho, \iota)$ where

- $A$ is a set whose elements are called **arcs**,
- $\rho : A \to A$ is a permutation of $A$,
- $\iota : A \to A$ is a fixed point free involution.

The permutations $\rho$ and $\iota$ generate a subgroup of the permutation group of $A$ called the **cartographic group** or the **monodromy group** of $M$ (see below for an explanation of the terminology). The oriented map $M$ has an associated **graph** $G(M) = (A/\langle\rho\rangle, A, o, \iota)$ whose vertices are the cycles of $\rho$, i.e., the orbits of the cyclic group of permutations $\langle\rho\rangle$ generated by $\rho$. The origin of an arc $a$ is defined as the orbit $o(a) = \langle\rho\rangle a$. An **edge** is an orbit of $\iota$. We will equally refer to a **vertex** or edge of $G(M)$ as a vertex or edge of $M$. We denote the set of vertices of $M$ by $V(M)$ and its set of edges by $E(M)$. A map is **connected** if its graph is connected, or equivalently, if its monodromy group acts transitively on its arcs. ***All the surfaces will be assumed connected in this section.***

A **face** of $M$ is a cycle of the permutation $\rho \circ \iota$. The face of an arc $a$ is denoted $F(a)$ and the set of faces of $M$ is denoted by $F(M)$. The **star** of a vertex or face $x$, denoted Star($x$), is the set of arcs in the corresponding cycle. In particular, Star($x$) $= F(a)$ for $x = \langle\rho \circ \iota\rangle a$. Since vertices and faces are defined as orbits, they are formally the same as their star. We will nonetheless avoid to say that an arc belongs to a vertex $x$ and rather say that it belongs to Star($x$), or is **incident** to $x$. The size of Star($x$) is the **degree** of $x$.

*Remark* 1.2. It is sometimes useful to allow the arc involution $\iota$ to have fixed points. In this case some $\iota$ orbits may be reduced to a single arc. *We do not count those singleton orbits in $E(M)$*. An arc invariant by $\iota$ is said **self-opposite**, or self-inverse. Interpreting $M$ as a cellular embedding of its graph, a self-opposite arc corresponds to an edge folded back on itself.

The permutation $\rho$ is sometimes designated as a **rotation system** as it encodes the cyclic ordering of the arcs incident to a vertex. An oriented map can equivalently be described as a pair $(G, \rho)$ where $G$ is a graph and $\rho$ is permutation on the arcs of $G$ whose cycles are the stars of the vertices of $G$.

Every oriented map $M$ can be realized as a cellular embedding $\eta_M : |G(M)| \hookrightarrow S(M)$, where $|G(M)|$ is the topological realization of $G(M)$ and $S(M)$ is an orientable surface that is compact whenever $M$ has a finite number of edges. There are two basic ways of visualizing this cellular embedding. One way is to consider for each face of $M$ an oriented polygon with one side per arc in the corresponding face cycle. These polygons are further glued so that the sides corresponding to an arc and to its opposite are identified (see Figure 2). The graph embedding is given by the 1-skeleton of the resulting cellular decomposition. Another way consists in thickening the graph of

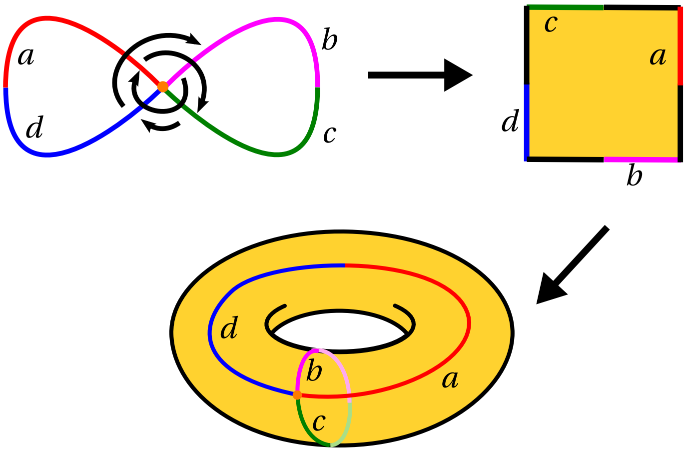


Figure 2: A cellular embedding associated to the map $(A, \rho, \iota)$ with $A = \{a, b, c, d\}$, $\rho = (a, c, d, b)$ and $\iota = (a, d)(c, b)$. The arcs are represented as colored half edges.

the map to transform it into a *ribbon graph*. We obtain a surface with boundaries that we can close with discs (see Figure 3). Here, the graph embedding is given by the inclusion of the graph into its thickening.

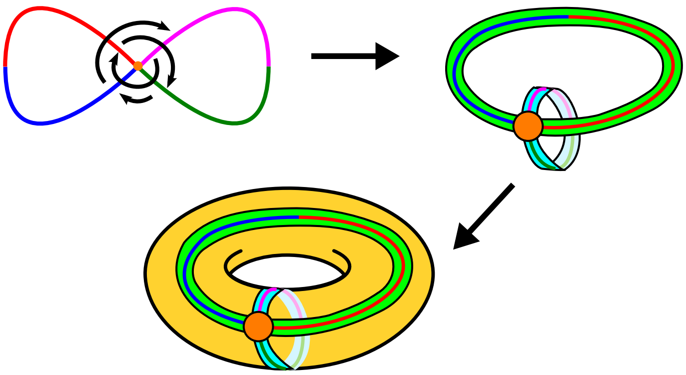

Figure 3: The same map as above. The unique vertex of the corresponding graph is replaced by a disc and each edge is replaced by a strip attached to the disc in the cyclic order of $\rho$. The resulting surface with boundary is closed with a single disc corresponding to the unique face of the map.

Conversely, every cellular embedding $\eta : |G| \hookrightarrow S$ of a graph $G$ into a topological oriented surface $S$ determines an oriented map $M(\eta) = (G, \rho)$ where $\rho$ is the rotation system corresponding to the oriented cyclic orderings of the vertex stars induced by the embedding.

**Proposition 1.3.** *The above topological realizations of $M$ through polygon gluing or graph thickening give equivalent cellular embeddings up to homeomorphism. Moreover, those topological realizations are inverse to combinatorial representations of cellular embeddings in the following sense.*

- *For a cellular embedding $\eta : |G| \hookrightarrow S$, the realization $\eta_M : |G(M(\eta))| \hookrightarrow S(M(\eta))$ is equivalent to $\eta$: There is a homeomorphism $S \simeq S(M(\eta))$ so that the diagram*
$$\begin{array}{ccc} |G| & \xrightarrow{\eta} & S \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \simeq} \\ |G(M(\eta))| & \xrightarrow{\eta_M} & S(M(\eta)) \end{array}$$
*commutes.*
- *Any oriented map $M$ is isomorphic to $M(\eta_M)$. In other words, the two maps are equal up to a renaming of the arcs.*

This proposition is essentially stated here to guide the intuition of the reader that would encounter maps for the first time. Its presence somehow contradicts the implicit credo that a purely combinatorial theory of surface can be developed without reference to topology. But, possibly in contradiction with Tutte, we strongly believe in the benefit of diagrams and topological intuition. A proof of the proposition can be found in Mohar and Thomassen's book [MT01] or Bryant and Singerman's foundational paper [BS85] for topological surfaces and in [GGD12] for the complex analytic case.

Guided by the topological realization of a map, we have

**Definition 1.4.** The **Euler characteristic** of a finite oriented map is the integer

$$\chi(M) = |V(M)| - |E(M)| + |F(M)|$$

Its **genus** is the non-negative integer $g(M) = 1 - \chi(M)/2$.

*Exercise* 1.5. Show that $g(M)$ is indeed a non-negative integer.

We now define the morphisms between oriented maps. Intuitively, a morphism of combinatorial surfaces corresponds to a branched covering of their topological realizations. This intuition is made more precise in Section 6.1.

**Definition 1.6.** A **morphism** of oriented maps $(A, \rho, \iota) \to (B, \sigma, \jmath)$ is a function $f : A \to B$ that commutes with the rotation systems and with the opposite operators, i.e., such that $f \circ \rho = \sigma \circ f$ and $f \circ \iota = \jmath \circ f$.

*Remark* 1.7. For maps with self-opposite arcs, it follows from the above definition that a morphism sends self-opposite arcs to self-opposite arcs.

## 1.2 The Riemann-Hurwitz formula

**Lemma 1.8.** *Any morphism $f:(A,\rho,\iota)\to(B,\sigma,\jmath)$ is onto and sends stars to stars surjectively. Moreover, for any vertex or face $x$ of finite degree of the map $(A,\rho,\iota)$, the restriction $f:\mathrm{Star}(x)\to f(\mathrm{Star}(x))$ of $f$ is isomorphic to the quotient*

$$\begin{array}{ccc}\mathbb{Z}/(e_x d)\mathbb{Z} & \to & \mathbb{Z}/d\mathbb{Z}\\ i \mod e_x d & \mapsto & i \mod d\end{array}$$

*where $d$ is the size of $f(\mathrm{Star}(x))$ and $e_x$ is a positive integer called the* ***ramification index*** *of $f$ at $x$.*

Proof. Let $a\in A$ be an arc. By connectedness, its orbit by the monodromy group satisfies $\langle\rho,\iota\rangle a=A$. Since $f$ commutes with the rotation systems and the opposite operators, we have $f(\langle\rho,\iota\rangle a)=\langle\sigma,\jmath\rangle f(a)=B$. Thus $f$ is onto. We also have for any integer $n$ that $f\circ\rho^n(a)=\sigma^n\circ f(a)$. It follows that the size of the orbit $\langle\rho\rangle a$, i.e., the degree of the vertex $x=o(a)$, is a multiple of the degree $d$ of the vertex $o(f(a))$. Whence $\deg(x)=e_x d$ for some positive integer $e_x$ and the lemma follows for $x$ a vertex. An analogous property holds when replacing $\rho$ by $\rho\circ\iota$ and $\sigma$ by $\sigma\circ\jmath$ proving the lemma when $x$ is a face. □

Thanks to this lemma we can define **the image by $f$ of a vertex or face** $x$ of the map $M=(A,\rho,\iota)$ as the vertex or face of $N=(B,\sigma,\jmath)$ whose star is $f(\mathrm{Star}(x))$. In particular, we can associate to $f$ a graph morphism $f:G(M)\to G(N)$. Note that this graph morphism is dimension preserving: a vertex or arc is mapped to a vertex or arc, respectively.

*Exercise* 1.9. Check that a morphism $f:(A,\rho,\iota)\to(B,\sigma,\jmath)$ induces a group epimorphism $\hat{f}:\langle\rho,\iota\rangle\twoheadrightarrow\langle\sigma,\jmath\rangle$ between the corresponding monodromy groups such that $f\circ\theta=\hat{f}(\theta)\circ f$ for all $\theta\in\langle\rho,\iota\rangle$.

**Lemma 1.10.** *All the arc fibers of a morphism $f:(A,\rho,\iota)\to(B,\sigma,\jmath)$ have the same size called the* ***degree*** *of $f$, and denoted by* $\deg(f)$.

Proof. Let $b,b'\in B$. Since $\langle\sigma,\jmath\rangle$ acts transitively, there is some $\tau\in\langle\sigma,\jmath\rangle$ such that $b'=\tau(b)$. Following Exercise 1.9 we can write $\tau=\hat{f}(\theta)$ for some $\theta\in\langle\rho,\iota\rangle$. Now, the equation $f(a)=b$ is equivalent to $\tau(f(a))=\tau(b)$, i.e., $f(\theta(a))=b'$. It follows that $\theta$ establishes a bijection from $f^{-1}(b)$ to $f^{-1}(b')$. □

**Proposition 1.11** (Index formula)**.** *Let $f:(A,\rho,\iota)\to(B,\sigma,\jmath)$ be a morphism of finite oriented maps. For any vertex or face $w$ of $(B,\sigma,\jmath)$, we have*

$$\sum_{f(v)=w} e_v=\deg(f)$$

Proof. Suppose that $w$ is a vertex and consider an arc $a\in\mathrm{Star}(w)$. We partition $f^{-1}(a)$ according to vertex stars: $f^{-1}(a)=\bigcup_{f(v)=w}(f^{-1}(a)\cap\mathrm{Star}(v))$. Because $\mathrm{Star}(v)$ wraps around $\mathrm{Star}(w)$ exactly $e_v$ times, each intersection $f^{-1}(a)\cap\mathrm{Star}(v)$ contains $e_v$ arcs (see Figure 4). The proposition then follows from Lemma 1.10. Replacing vertices by faces gives the formula when $w$ is a face. □

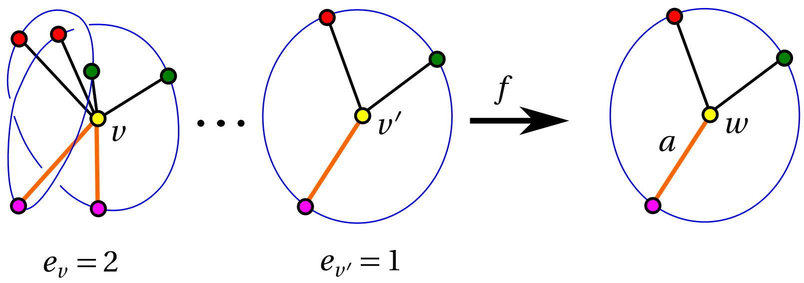


Figure 4: The preimage of the star of $w$ can be decomposed into stars.

**Theorem 1.12** (Riemann-Hurwitz Formula)**.** *For a morphism $f : M \to N$ of degree $n$ of finite oriented maps we have*

$$\chi(M) = n.\chi(N) - \sum_{v \in V(M) \cup F(M)} (e_v - 1)$$

PROOF. We know from Lemma 1.10 and Remark 1.7 that $|E(M)| = n|E(N)|$. Also, by the Index formula, we have for every vertex or face $w$ of $N$ that $n = \sum_{f(v)=w} e_v = \sum_{f(v)=w}(e_v - 1) + |f^{-1}(w)|$. So,

$$\begin{aligned}
\chi(M) &= |V(M)| - |E(M)| + |F(M)| \\
&= \sum_{w \in V(N)} |f^{-1}(w)| - n|E(N)| + \sum_{w \in F(N)} |f^{-1}(w)| \\
&= \sum_{w \in V(N) \cup F(N)} \left( n - \sum_{f(v)=w} (e_v - 1) \right) - n|E(N)| \\
&= n(|V(N)| + |F(N)|) - n|E(N)| - \sum_{v \in V(M) \cup F(M)} (e_v - 1)
\end{aligned}$$

□

# 2 Basic Operations and Classification

One advantage of the map formalism is the ability to modify the embedded graph rather easily. A notion of elementary modification gives rise to combinatorial equivalence between maps that shall replace topological homeomorphisms and allows for a classification of surfaces.

## 2.1 Modifying maps

### 2.1.1 Dual maps

Intuitively the dual of a map is obtained by inverting the roles of vertices and faces. The topological counterpart of the dual map is obtained by placing a (dual) vertex at the center of each face of the (primal) map, adding an edge between two dual vertices if their corresponding face share an edge. Figure 5 illustrates the dual of a spherical map.

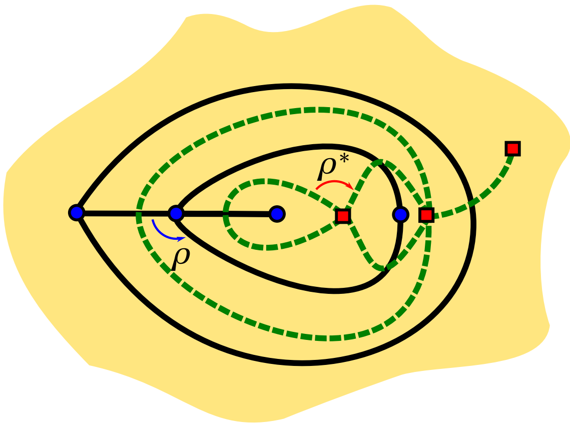


Figure 5: A map on the sphere (with plain line edges) with rotation system $\rho$ and its dual map (with dashed line edges).

**Definition 2.1.** The **dual** of the map $M = (A, \rho, \iota)$ is the map $M^* = (A, \rho \circ \iota, \iota)$. The **dual graph** of $M$ is the graph $G^*(M) = G(M^*)$ of the dual map. The vertices of the dual graph are the cycles of $\rho \circ \iota$, i.e., the faces of $M$. More precisely, $G^*(M) = (F(M), A, o^*, \iota)$ where $o^*(a) = F(a)$.

It is immediate that

**Lemma 2.2.** *$M$ and $M^*$ have the same monodromy group. In particular, $M$ is connected if and only if $M^*$ is connected.*

**Lemma 2.3.** $(M^*)^* = M$

### 2.1.2 Edge contraction

The basic operations of contraction, deletion or subdivision of an edge in a graph extend naturally to embedded graphs.

**Definition 2.4.** Let $M = (A, \rho, \iota)$ be a connected map with at least two edges. If $e = \{a, a^{-1}\}$ is a non-loop edge (*i.e.*, $o(a) \neq o(a^{-1})$) of $M$, the **contraction** of $e$ transforms $M$ to a map $M/e = (A \setminus e, \rho', \iota')$ where $\iota'$ is the restriction of $\iota$ to $A \setminus e$ and $\rho'$ is obtained by merging the cycles of $a$ and $a^{-1}$, i.e.,

$$\forall b \in A \setminus e, \rho'(b) = \begin{cases} \rho(b) & \text{if } \rho(b) \notin e, \\ \rho \circ \iota(\rho(b)) & \text{if } \rho(b) \in e \text{ and } \rho \circ \iota(\rho(b)) \notin e, \\ (\rho \circ \iota)^2(\rho(b)) & \text{otherwise.} \end{cases} \tag{1}$$

Note that the last case in (1) may only occur if $e$ has a degree one endpoint. In this case, that is when $\rho(b) \in e$ and $\rho \circ \iota(\rho(b)) \in e$, $(\rho \circ \iota)^2(\rho(b))$ reduces to $\rho^2(b)$. Figure 6 shows the effect of an edge contraction. Let $\tau$ be the transposition of $a$ and $a^{-1}$ in $A$. One may observe that the contraction of the edge $\{a, a^{-1}\}$ amounts to first replace the

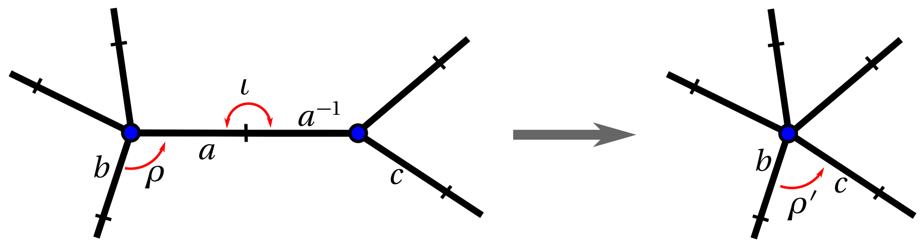


Figure 6: The contraction of a non-loop edge. $\rho(b) = a \implies \rho'(b) = \rho \circ \iota(\rho(b)) = c$.

pair of permutations $(\rho, \iota)$ by $(\rho \circ \tau, \tau \circ \iota)$ and then removing $\{a, a^{-1}\}$ from their cycle decompositions.

*Exercise* 2.5. Show that $G(M/e) = G(M)/e$. (Recall the edge contraction for graphs from the preceding lecture notes.)

**Lemma 2.6.** *If $M$ is a connected map with at least two edges and $e = \{a, a^{-1}\}$ is a non-loop edge of $M$, then $M/e$ is connected and has the same Euler characteristic as $M$.*

PROOF. Put $M = (A, \rho, \iota)$ and $M/e = (A' = A \setminus e, \rho', \iota')$. It is easily seen that the two orbits $\langle\rho\rangle a$ and $\langle\rho\rangle a^{-1}$ are merged into a single cycle of $\rho'$ after removing $a$ and $a^{-1}$ from this cycle. It follows that $|V(M/e)| = |V(M)| - 1$. (One may also argue with Exercise 2.5.) On the other hand, from (1) we get for $b \in A'$:

$$\rho' \circ \iota'(b) = \begin{cases} \rho \circ \iota(b) & \text{if } \rho \circ \iota(b) \notin e, \\ (\rho \circ \iota)^2(b) & \text{if } \rho \circ \iota(b) \in e \text{ and } (\rho \circ \iota)^2(b) \notin e, \\ (\rho \circ \iota)^3(b) = & \text{otherwise.} \end{cases}$$

The faces of $M/e$ are thus obtained by deleting $a$ and $a^{-1}$ from the faces of $M$. Since no face is reduced to the singleton $a$ or $a^{-1}$, as $e$ would be a loop edge otherwise, it follows that $|F(M/e)| = |F(M)|$. Since $|A'| = |A| - 2$, we conclude that

$$\chi(M/e) = (|V(M)| - 1) - (|E(M)| - 1) + |F(M)| = \chi(M)$$

$G(M/e)$ is moreover connected: Any path in $G(M)$ between two vertices can be transformed into a path in $G(M/e)$ between the same vertices, possibly identified, by just removing the occurrences of $a$ and $a^{-1}$ in this path. It follows that $M/e$ is indeed connected. □

### 2.1.3 Edge deletion

**Definition 2.7.** Let $M = (A, \rho, \iota)$ be a map with at least two edges. If $e = \{a, a^{-1}\}$ is an edge of $M$ without an endpoint of degree one or a free end, the **deletion** of $e$ in $M$ transforms $M$ to a map $M - e = (A \setminus e, \rho', \iota')$ where $\iota'$ is the restriction of $\iota$ to $A \setminus e$ and $\rho'$ is obtained by deleting $a$ and $a^{-1}$ in the cycles of $\rho$, i.e.,

$$\forall b \in A \setminus e, \rho'(b) = \begin{cases} \rho(b) & \text{if } \rho(b) \notin e, \\ \rho^2(b) & \text{if } \rho(b) \in e \text{ and } \rho^2(b) \notin e, \\ \rho^3(b) & \text{otherwise.} \end{cases} \quad (2)$$

Figure 7 shows the deletion of a loop edge. Observe that $G(M-e) = G(M) - e$.

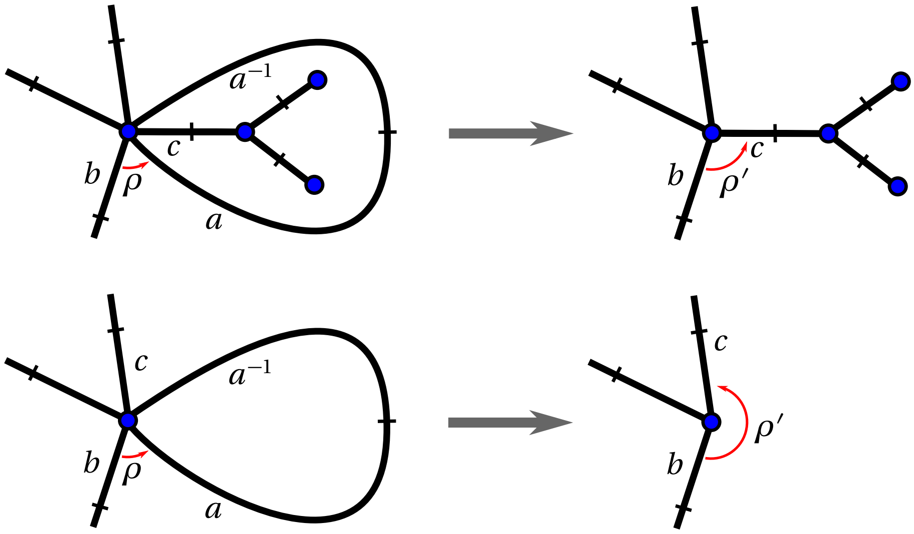


Figure 7: The deletion of a loop edge $e = \{a, a^{-1}\}$. Arc $b$ is such that $\rho(b) \in e$. Above, We have $\rho^2(b) \notin e$ implying $\rho'(b) = \rho^2(b) = c$. Below, $\rho^2(b) \in e$ so that $\rho'(b) = \rho^3(b) = c$.

An edge that is incident to two distinct faces is said **regular** and **singular** otherwise. Hence, an edge is regular if and only if $F(a) \neq F(a^{-1})$.

**Lemma 2.8.** *If $M$ is a connected map with at least two edges and $e = \{a, a^{-1}\}$ is an edge of $M$ with no endpoint of degree one and no free end, then*

$$\chi(M-e) = \begin{cases} \chi(M) & \text{if } e \text{ is regular,} \\ \chi(M)+2 & \text{otherwise.} \end{cases}$$

Note that the deletion of $e$ may disconnect the map.

PROOF. Clearly, $M'$ has the same number of vertices as $M$ and one edge less. Let $\varphi = \rho \circ \iota$ and $\varphi' = \rho' \circ \iota'$ be the facial permutations of $M$ and $M'$ respectively. We have that $e = \{a, a^{-1}\}$ is regular if and only if the $\varphi$-cycles of $a$ and $a^{-1}$ are distinct. Using the definition of $\rho'$ in (2), we see that the cycles of $\varphi'$ are the same as those of $\varphi$ except for $\langle\varphi\rangle a$ and $\langle\varphi\rangle a^{-1}$. If these cycles are distinct, then they are merged to a single cycle of $\varphi'$. Otherwise, $\langle\varphi\rangle a = \langle\varphi\rangle a^{-1}$ is split to give two distinct cycles of $\varphi'$. We infer that $M'$ has one face less in the former case and one more in the latter. We conclude that

$$\chi(M-e) = |V(M)| - (|E(M)| - 1) + (|F(M)| - 1) = \chi(M)$$

if $e$ is regular and

$$\chi(M-e) = |V(M)| - (|E(M)| - 1) + (|F(M)| + 1) = \chi(M) + 2$$

otherwise. □

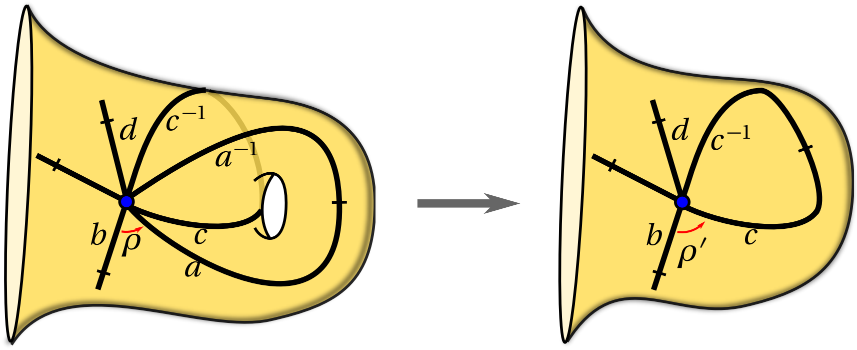


Figure 8: The deletion of the loop edge $\{a, a^{-1}\}$ replaces the facial circuit $(\ldots, b^{-1}, a, c^{-1}, a^{-1}, c, d, \ldots)$ by the two circuits $(\ldots, b^{-1}, c, d, \ldots)$ and $(c^{-1})$.

Figure 8 shows the effect of deleting a singular loop edge.

We leave as an exercise, the following link between edge contraction and deletion.

**Lemma 2.9.** *Let $e$ be an edge of a connected map $M$ with at least two edges. Then, under the condition that the following contractions or deletions are well-defined, we have $(M/e)^* = M^* - e$ and $(M - e)^* = M^*/e$.*

### 2.1.4 Edge and face subdivisions

**Definition 2.10.** Let $e = \{a, a^{-1}\}$ be an edge of a map $M = (A, \rho, \iota)$. The **subdivision** of $e$ in $M$ transforms $M$ to a map $S_e M = (A', \rho', \iota')$ where

- $A' = A \cup \{b, b'\}$, where $b, b'$ are new arcs not in $A$,
- the restriction of $\iota'$ to $A$ is equal to $\iota$ and $\iota'(b) = b'$,
- $\rho'$ is defined by

$$\forall c \in A', \rho'(c) = \begin{cases} \rho(a) & \text{if } c = b \\ a & \text{if } c = b' \\ b' & \text{if } c = a \\ b & \text{if } \rho(c) = a \\ \rho(c) & \text{otherwise.} \end{cases}$$

See Figure 9 for an illustration. We observe that $G(S_e M) = S_e G(M)$ (see the corresponding definition in the previous lecture).

**Definition 2.11.** Let $M = (A, \rho, \iota)$ be a map and let $a, b$ be two arcs, possibly equal, belonging to a same face $F(a) = F(b)$. The **subdivision** of $F(a)$ from $a$ to $b$ transforms $M$ to a map $S_{(a,b)} M = (A \cup \{c, c^{-1}\}, \rho', \iota)$ obtained by adding a new edge $\{c, c^{-1}\}$ in $F(a)$

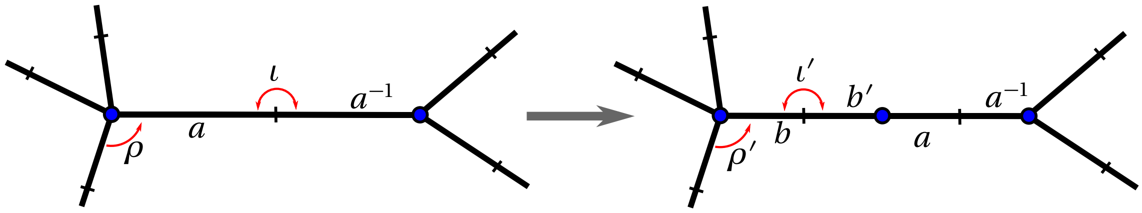


Figure 9: The subdivision of an edge splits that edge, introducing a new vertex on the edge.

between the heads of $a$ and $b$ (see Figure 10). When $a \neq b$ the new rotation system $\rho'$ is given by

$$\forall d \in A \cup \{c, c^{-1}\}, \rho'(d) = \begin{cases} c & \text{if } d = a^{-1} \\ c^{-1} & \text{if } d = b^{-1} \\ \rho(a^{-1}) & \text{if } d = c \\ \rho(b^{-1}) & \text{if } d = c^{-1} \\ \rho(d) & \text{otherwise.} \end{cases}$$

When $a = b$, $\rho'$ is given by

$$\forall d \in A \cup \{c, c^{-1}\}, \rho'(d) = \begin{cases} c & \text{if } d = a^{-1} \\ c^{-1} & \text{if } d = c \\ \rho(a^{-1}) & \text{if } d = c^{-1} \\ \rho(d) & \text{otherwise.} \end{cases}$$

We trivially check that

**Lemma 2.12.** *The edge and face subdivisions preserve the number of connected components and the Euler characteristic.*

*Remark* 2.13. The inverse of an edge subdivision amounts to contract the newly introduced edge, while the inverse of a face subdivision amounts to delete the newly introduced regular edge. Hence, face subdivision and regular edge deletion are inverse to each other. Note that the edge introduced by an edge subdivision must have a degree two endpoint. However, Zieschang et al. [ZVC80, p. 67] observe that the contraction of any non-loop edge can be obtained from a sequence of edge or face subdivisions and their inverses. Figure 11 illustrates the process of contracting an edge in this way.

*Exercise* 2.14. The sequence of operations in Figure 11 is still valid when the right endpoint on the figure has degree one, i.e., when the edge is a pendant edge. Propose a simpler sequence of face or edge contractions (and their inverses) equivalent to an edge contraction in that case.

## 2.2 Classification of oriented maps

Compared to topological surfaces, combinatorial maps have the advantage of being of a discrete nature. Combinatorial maps are easier to manipulate, to encode and

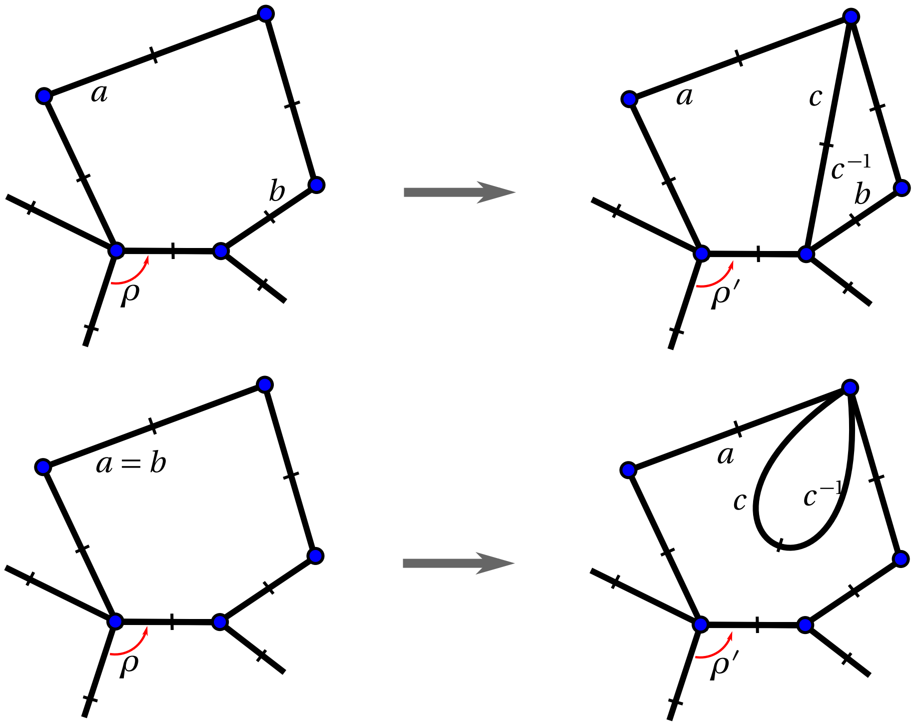


Figure 10: The subdivision of $F(a)$ between the heads of $a$ and $b$. Up, the case $a \neq b$. Down, the case $a = b$.

naturally lead to computations. On the other hand, the topological surface encoded by a map is less apparent and non-isomorphic maps may encode the same topological surface. In other words, the realization functor $\mathscr{M} \to \mathbf{Top}$ from maps to topological surfaces is many to one and is not an equivalence. Map morphisms are too rigid to allow for a full recording of topology. The usual way to circumvent this rigidity is to introduce a combinatorial equivalence.

**Definition 2.15.** **Combinatorial equivalence** of (isomorphisms classes of) maps is the equivalence relation generated by edge and face subdivisions as specified in Section 2.1.

Following Remark 2.13, two maps are combinatorially equivalent if and only if the first map can be obtained from the other one by a finite sequence of the operations described in Section 2: contraction of non-loop edge, regular edge deletion, edge subdivision or face subdivision. By Lemma 2.12, equivalent maps share the same connectedness and Euler characteristic. We shall often say that two maps are equivalent when they are combinatorially equivalent.

The **normal form of a sphere** is the map with a unique loop edge:

$$(\{a, a^{-1}\}, (a, a^{-1}), (a, a^{-1}))$$

The **normal form of a connected sum of $g$ tori**, $g > 0$, is the map with $2g$ edges $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ whose rotation system has a unique cycle

$$\rho_g = (a_1, b_1^{-1}, a_1^{-1}, b_1, \dots, a_g, b_g^{-1}, a_g^{-1}, b_g) \tag{3}$$

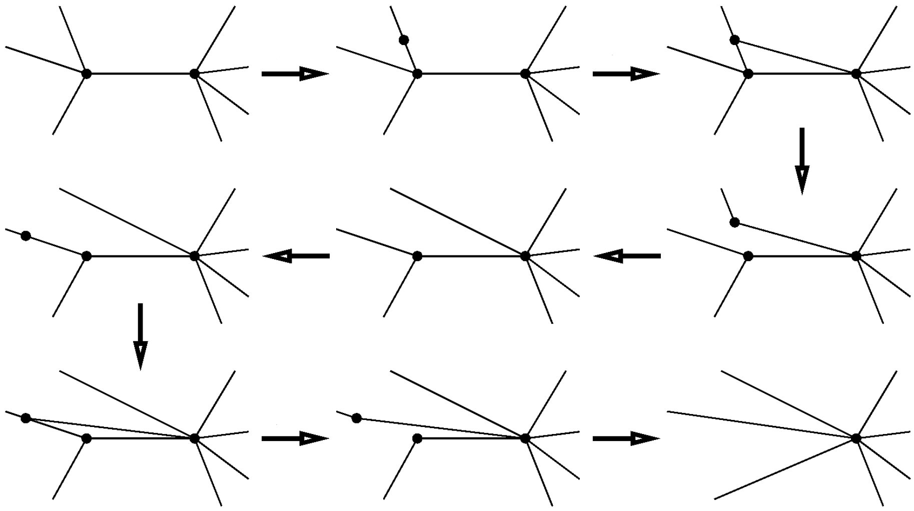

Figure 11: Let us denote by $S_e$ or $S_f$ respectively, a generic edge or face subdivision and by $S_e^{-1}$ and $S_f^{-1}$ the corresponding inverse operations. The contraction of the upper left edge is the result of the sequence of operations: $S_e, S_f, S_f^{-1}, S_e^{-1}, S_e, S_f, S_f^{-1}, S_e^{-1}$.

**Definition 2.16.** A sphere or a connected sum of tori is a map combinatorially equivalent to its corresponding normal form.

We shall prove that every finite map is equivalent to exactly one of the above sphere or tori normal forms.

**Lemma 2.17.** *Every finite connected oriented map is equivalent to a map with a single vertex.*

PROOF. Let $M$ be a finite connected map and let $T$ be a spanning tree of its graph $G(M)$. If $M$ has at least one edge not in $T$, we may contract the edges of $T$ one after the other, in any order. We obtain this way an equivalent map with a single vertex. If $G(M)$ is a tree, we may contract all of its edges but one and note that an oriented map with a single non-loop edge is combinatorially equivalent to a normal sphere. To see this, subdivide the unique face of the map by connecting the endpoints of the non-loop edge and contract that edge. □

A map with a single vertex and either a single face or a single edge is said **reduced**.

**Lemma 2.18.** *Every finite connected oriented map is equivalent to a reduced map.*

PROOF. Let $M$ be a finite connected oriented map. By the previous lemma we may assume that $M$ has a single vertex. Note that every face of $M$ corresponds to a vertex of its dual graph $G(M^*)$ which is connected by Proposition 2.2. As long as $M$ has more than one face, the connectivity of the dual graph implies the existence of an edge $e$ incident to two distinct faces of $M$. If $M$, hence $M^*$, has more than one edge we can delete $e$, thus reducing the number of faces of $M$. By induction on the number of edges, we obtain an equivalent reduced map. □

The classification of maps relies on a repeated use of face subdivisions and edge deletions. We introduce some concise notations to describe these operations. We denote by $\varphi = \rho \circ \iota$ the facial permutation of a map $(A, \rho, \iota)$. We also view a face $F(a) = (a, \varphi(a), \varphi^2(a), \ldots)$ as a circuit in $G(M)$ and write $X\,Y$ for the concatenation of two paths rather than $X \cdot Y$. We shall not distinguish the notation for a circuit or a path; the distinction should be clear from the context. If $F(a) = F(b)$ we thus have $F(a) = X\,a\,Y\,b$ for some subpaths $X, Y$. Following Definition 2.11, the subdivision of $F(a) = X\,a\,Y\,b$ from $a$ to $b$ splits the face into two new faces $X\,a\,c$ and $c^{-1}\,Y\,b$ by the introduction of an edge $\{c, c^{-1}\}$ between the heads of $a$ and $b$. This subdivision is depicted by the following diagram:

$$X\,a\,Y\,b \xrightarrow{c=(a,b)} X\,a\,c + c^{-1}\,Y\,b$$

Conversely, if an edge $e = \{c, c^{-1}\}$ is incident to two distinct faces $X\,c$ and $c^{-1}\,Y$, then the deletion of $e$ is depicted by the diagram

$$X\,c + c^{-1}\,Y \xrightarrow{-e} X\,Y$$

We first state an auxiliary lemma.

**Lemma 2.19.** *Let $M = (A, \rho, \iota)$ be a finite reduced oriented map that is not a sphere. For every arc $a \in A$ there exists an arc $b \notin \{a, a^{-1}\}$ such that the unique face $F$ of $M$ has the form*

$$F = a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$$

*where each ellipsis denotes a possibly empty subpath.*

Proof. Let us write $F = a\,X\,a^{-1}\,Y$, for some possibly empty paths $X, Y$. Observe that $M$ being reduced and not a sphere, $F$ is the unique cycle of $\rho$.

Assume for a contradiction that for every arc $b$ occurring in $X$, its opposite arc $b^{-1}$ also occurs in $X$. Let $U$ be the set of arcs comprising $a^{-1}$ and the arcs in $X$. Note that $a \notin U$. Now, $\rho = \varphi \circ \iota$ leaves $U$ globally invariant, in contradiction with the above observation. □

**Theorem 2.20.** *Every finite connected oriented map is either a sphere or a connected sum of tori.*

Proof. Let $M$ be a finite connected oriented map. We need to prove that $M$ is equivalent to the normal form of a sphere or of a connected sum of tori. By Lemma 2.18, we may assume that $M$ is reduced. If $M$ has a single edge, we trivially check that $M$ is equivalent to the normal form of a sphere. We can thus further assume that $M$ has a single face $F$ and at least two edges. According to Lemma 2.19, we may write

$$F = a\,X\,b\,Y\,a^{-1}\,Z\,b^{-1}\,W\,T_k$$

where each of $X, Y, Z$ and $W$ is a possibly empty path and where

$$T_k = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_k b_k a_k^{-1} b_k^{-1}$$

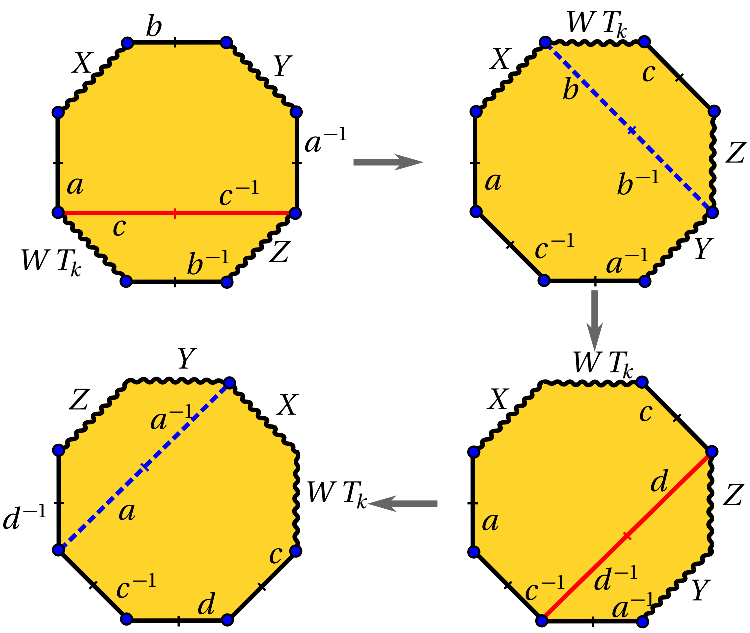


Figure 12: Upper left, A schematic view of the unique face of $M$. The map $M$ is applied a face subdivision by inserting a (red plain) edge. Upper right, An equivalent view of the resulting two faces. The deletion of the dashed (blue) edge merges the two faces in a different way. We further apply a face subdivision (lower right figure) and an edge deletion (lower left figure) to obtain an equivalent reduced map.

for some $k \geq 0$ (by convention $T_0$ is the empty path) and some pairwise distinct edges $\{a_i, a_i^{-1}\}, \{b_i, b_i^{-1}\}$. We apply the following sequence of operations as illustrated on Figure 12.

$$\begin{aligned}
F = Z\,b^{-1}W\,T_k a X\,b\,Y a^{-1} &\xrightarrow{c=(\varphi^{-1}(a),a^{-1})} Z\,b^{-1}W\,T_k c + c^{-1}a X\,b\,Y a^{-1}\\
W\,T_k c Z\,b^{-1} + b\,Y a^{-1}c^{-1}a X &\xrightarrow{-\{b,b^{-1}\}} W\,T_k c Z\,Y a^{-1}c^{-1}a X\\
c Z\,Y a^{-1}c^{-1}a X\,W\,T_k &\xrightarrow{d=(c,a^{-1})} c^{-1}a X\,W\,T_k c\,d + d^{-1}Z\,Y a^{-1}\\
X\,W\,T_k c\,d\,c^{-1}a + a^{-1}d^{-1}Z\,Y &\xrightarrow{-\{a,a^{-1}\}} X\,W\,T_k c\,d\,c^{-1}d^{-1}Z\,Y
\end{aligned}$$

We obtain this way a reduced equivalent map $M'$ whose unique face has facial circuit $Z\,Y\,X\,W\,T_{k+1}$ with $a_{k+1} = c$ and $b_{k+1} = d$. Note that this face has the same length as $F$. Since $|T_{k+1}| > |T_k|$, it follows by induction on $k$ that $M$ must be equivalent to a reduced map $M_g$ whose unique face has the form $T_g$ for $g = |A|/4$. We conclude that up to a renaming of the arcs the rotation system of $M_g$ is $\rho_g$ as in (3). □

# 3 Path homotopy in Oriented Maps

A **path**, **loop** or **circuit** of a map $M$ is a path, loop or circuit of its graph $G(M)$ (See the lecture notes on graphs). The notion of path deformation via elementary homotopies should now take into account that a path can be deformed inside a face since a face should represent a topological disc. This leads to the following definitions.

Let $f$ be a face of a map $M = (A, \rho, \iota)$. Recall from Definition 1.1 that a face is cycle of the the permutation $\rho \circ \iota$. In order to formally differentiate between a face and its boundary circuit, we denote by $\partial f$ the cycle $f$ *viewed* as a circuit in $G(M)$. We call $\partial f$ the **facial circuit** of $f$. If $\partial f = u \cdot v^{-1}$, where $u$ is a possibly constant subpath of $\partial f$, then $u$ and $v$ are said **complementary subpaths**.

**Definition 3.1.** Let $M$ be a map. An **elementary homotopy** in a path $\gamma$ of $M$ consists either in adding or removing a spur in $\gamma$, or in replacing a subpath of $\gamma$ that is also a subpath of a facial circuit by its complementary subpath. In other words, if $\gamma = \lambda \cdot u \cdot \mu$ and $\partial f = u \cdot v^{-1}$ then $\gamma$ is transformed into $\lambda \cdot v \cdot \mu$ by elementary homotopy. A **free elementary homotopy** is an elementary homotopy applied to any of the path representatives of a circuit. The **homotopy** relation is the transitive closure of elementary homotopies. Likewise, **free homotopy** is the transitive closure of free elementary homotopies. We write $\gamma \sim \lambda$ if $\gamma$ and $\lambda$ are homotopic paths and $\gamma \overset{\text{free}}{\sim} \lambda$ when they are freely homotopic circuits. A loop or circuit (freely) homotopic to a constant path is said **contractible**. If the last vertex of a path $\gamma$ coincides with the first vertex of a path $\lambda$, their **concatenation** is the path $\gamma \cdot \lambda$ whose arc sequence is the the arc sequences of $\gamma$ followed by the arc sequence of $\lambda$.

*Remark* 3.2. The homotopy relation is actually generated by the second type of homotopies only, replacing a piece of facial walk by a complementary subpath. Indeed, the addition of a spur $u \cdot v \sim u \cdot a \cdot a^{-1} \cdot v$ can be obtained via elementary homotopies of the above type: if $\partial f = a \cdot w$, then $u \cdot v \sim u \cdot a \cdot w \cdot v \sim u \cdot a \cdot a^{-1} \cdot v$. Here, the first elementary homotopy replaces the constant path $(o(a))$ by $a \cdot w$ and the second elementary homotopy replaces $w$ by the complementary subpath $a^{-1}$.

### 3.0.1 Homotopy versus free homotopy

**Lemma 3.3.** *Two loops $\alpha$ and $\beta$ with a common basepoint on a map $M$ are* freely *homotopic if and only if there exists a loop $\ell$ such that $\alpha$ and $\ell \cdot \beta \cdot \ell^{-1}$ are homotopic.*

PROOF. Let $n$ be the number of free elementary homotopies separating $\alpha$ from $\beta$. We thus have for some loops $\alpha_i$, $i = 1, n-1$, that $\alpha = \alpha_0 \overset{\text{free}}{\to} \alpha_1 \overset{\text{free}}{\to} \ldots \overset{\text{free}}{\to} \alpha_n = \beta$ where each arrow is a free elementary homotopy. We claim that there exist paths $\lambda_1, \ldots, \lambda_n$ such that $\alpha_0 \sim \lambda_1 \alpha_1 \lambda_1^{-1} \sim \ldots \sim \lambda_n \alpha_n \lambda_n^{-1}$. Indeed, setting $\lambda_0$ to the constant path, we can recursively define $\lambda_i$ as follows. Since $\alpha_i \overset{\text{free}}{\to} \alpha_{i+1}$ we have an elementary homotopy $\alpha'_i \to \alpha'_{i+1}$ for some cyclic permutations of $\alpha_i$ and $\alpha_{i+1}$ respectively. If $p$ is the subpath of $\alpha_i$ from its basepoint to the basepoint of $\alpha'_i$, we have $\alpha_i \sim p \cdot \alpha'_i \cdot p^{-1}$. Similarly,

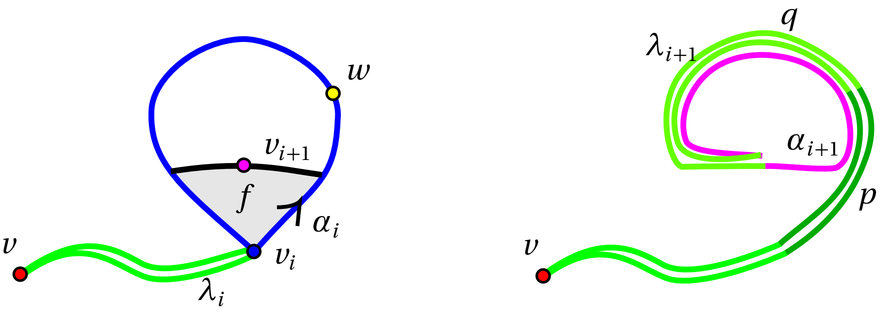


Figure 13: Left, $v$ is the common basepoint of $\alpha$ and $\beta$ while $v_i$ and $v_{i+1}$ are the basepoints of $\alpha_i$ and $\alpha_{i+1}$ respectively, and $w$ is the common basepoint of $\alpha'_i$ and $\alpha'_{i+1}$. The elementary homotopy $\alpha'_i \to \alpha'_{i+1}$ is supposed to use complementary subpaths of $\partial f$. Right, The loop $\lambda_{i+1}\alpha_{i+1}\lambda_{i+1}^{-1}$.

$\alpha_{i+1} \sim q \cdot \alpha'_{i+1} \cdot q^{-1}$ for some subpath $q$ of $\alpha'_{i+1}$. Assuming $\alpha_0 \sim \lambda_i \alpha_i \lambda_i^{-1}$ we can thus write

$$\alpha_0 \sim \lambda_i \cdot p \cdot \alpha'_i \cdot p^{-1} \cdot \lambda_i^{-1} \sim \lambda_i \cdot p \cdot \alpha'_{i+1} \cdot p^{-1} \cdot \lambda_i^{-1} \sim \lambda_{i+1}\alpha_{i+1}\lambda_{i+1}^{-1}$$

with $\lambda_{i+1} = \lambda_i . p \cdot q^{-1}$. See Figure 13. Choosing $\ell = \lambda_n$ we may conclude the lemma. □

**Corollary 3.4.** *Let $\alpha$ and $\beta$ be two circuits and let $v$ be a vertex of a map $M$. The circuits $\alpha$ and $\beta$ are freely homotopic if and only if for any paths $p$ and $q$ from $v$ to the basepoints of $\alpha$ and $\beta$ respectively, there exists a loop $\ell$ such that the loops $p \cdot \alpha \cdot p^{-1}$ and $\ell \cdot q \cdot \beta \cdot q^{-1} \cdot \ell^{-1}$ are homotopic.*

**Corollary 3.5.** *Let $\alpha$ and $\beta$ be two paths on a map $M$. We have the equivalences*

$$\alpha \sim \beta \quad \Leftrightarrow \quad \alpha \cdot \beta^{-1} \sim 1 \quad \Leftrightarrow \quad \alpha \cdot \beta^{-1} \overset{free}{\sim} 1$$

PROOF. We have $\alpha \sim \beta \implies \alpha \cdot \beta^{-1} \sim \beta \cdot \beta^{-1} \sim 1$. Conversely, $\alpha \cdot \beta^{-1} \sim 1 \implies \alpha \cdot \beta^{-1} \cdot \beta \sim \beta$. This takes care of the first equivalence. The second equivalence follows from Lemma 3.3, as $\alpha \cdot \beta^{-1} \overset{\text{free}}{\sim} 1 \Leftrightarrow \alpha \cdot \beta^{-1} \sim \ell \cdot \ell^{-1} \sim 1$ for some (thus any) loop $\ell$. □

## 3.1 The fundamental group of maps

Let $v$ be a vertex of a map $M$. It is easily checked that the path concatenation $\lambda \cdot \mu$ is homotopic to the path concatenation $\lambda' \cdot \mu'$ whenever $\lambda \sim \lambda'$ and $\mu \sim \mu'$. Hence, the set of homotopy classes of loops with basepoint $v$ is a group for the law of path concatenation with (the class of) the constant path $(v)$ as unit. It is called the **fundamental group** of $M$ based at $v$ and denoted by $\pi_1(M, v)$. According to Corollary 3.4, the free homotopy classes correspond to the conjugacy classes in this group.

**Lemma 3.6.** *Let $T$ be a spanning tree (of the graph $G(M)$) of a connected map $M$. Then $\pi_1(M, v)$ is isomorphic to the group with combinatorial presentation*

$$\Pi = \langle E(M) \mid E(T), \{\partial F\}_{F \in F(M)} \rangle$$

*If we denote by $C$ the set of chords of $T$ in $G(M)$ and by $r_F$ the sequence of arcs not in $T$ in the facial circuit of a face $F$, this group is also isomorphic to*

$$\langle C \mid \{r_F\}_{F\in F(M)}\rangle$$

PROOF. Recall from the graph lecture notes the notations $T[v,w]$ for the simple $vw$-path in $T$ and $T[v,a]$ for the loop with basepoint $v$ obtained by joining the endpoints of the arc $a$ to $v$ by simple paths in $T$. With a little abuse of notation we shall use $T[v,a]$ for the loop or for its homotopy class. If $\sigma=(a_1,a_2,\dots,a_k)$ is a sequence of arcs, we write $T[v,\sigma]$ for the concatenation $T[v,a_1]\cdot T[v,a_2]\cdots T[v,a_k]$. When $\sigma$ is the sequence of arcs of a loop with basepoint $w$, we remark that the loop $T[v,\sigma]$ is homotopic in $G(M)$ (by removing spurs) to the loop $T[v,w]\cdot\sigma\cdot T[w,v]$. We assume that each edge has a default orientation so that it can be identified with an arc. We consider the map $\pi: E(M)\to\pi_1(M,v), a\mapsto T[v,a]$. If $a\in E(T)$ we know that $T[v,a]$ is contractible. Let $F$ be a face of $M$. By the preceding remark $T[v,\partial F]\sim T[v,w]\cdot\partial F\cdot T[w,v]$, where $\partial F$ is a path representative with basepoint $w$ of the corresponding circuit. Using an elementary homotopy to replace $\partial F$ by the empty path in this loop, we get $T[v,\partial F]\sim T[v,w]\cdot T[w,v]\sim 1$. The relations $E(T)\cup\{\partial F\}_{F\in F(M)}$ in the above presentation of $\Pi$ are thus satisfied in $\pi_1(M,v)$. It follows that the map $\pi$ extends to a group morphism $\pi:\Pi\to\pi_1(M,v)$. By the above remark any loop $\ell$ with basepoint $v$ is homotopic to $T[v,\ell]$, implying that $\pi$ is onto. It remains to prove that $\pi$ is one-to-one. For two words $\omega,\sigma$ in the free group $\langle E(M)\mid -\rangle$ we write $\omega=_\Pi\sigma$ if $\omega$ and $\sigma$ are equal as elements in $\Pi$. Let $\sigma\in\langle E(M)\mid -\rangle$ such that $\pi(\sigma)$ is contractible. Since $\pi(\sigma)\overset{\text{free}}{\sim}T[v,\sigma]$, it ensues from Corollary 3.5 that $T[v,\sigma]$ can be reduced to the constant path by elementary homotopies. Considering $T[v,\sigma]$ as a word in $\langle E(M)\mid -\rangle$, we thus have $T[v,\sigma]=_\Pi 1$. On the other hand, since the edges of $T$ are relations in $\Pi$, we have $T[v,\sigma]=_\Pi\sigma$, whence $\sigma=_\Pi 1$. The second part of the lemma is immediate by applying Tietze transformations to remove the generators in $E(T)$ from the presentation of $\Pi$.
□

*Example* 3.7. Since the normal form of a connected sum of $g$ tori (see (3)) has a single vertex and a single face we easily deduce that its fundamental group has presentation

$$\langle a_1,b_1,\dots,a_g,b_g \mid [a_1,b_1]\cdots[a_g,b_g]\rangle$$

where $[a,b]=aba^{-1}b^{-1}$.

**The homotopy functor** From Definition 1.6, it is seen that the arc function of a map morphism $f: M\to N$ extends to a map from the loops of $M$ to the loops of $N$. Moreover, the property that $f$ sends faces of $M$ onto faces of $N$ implies that homotopic loops are sent to homotopic loops. Indeed, if two loops $\ell,\ell'$ are related by an elementary homotopy that replaces a subpath of $\ell$ by its complementary subpath in a face of $M$ with ramification index $e$, then $f(\ell)$ is related to $f(\ell')$ by a sequence of $e$ elementary homotopies. It ensues that $f$ induces a group morphism

$$f_*:\pi_1(M,v)\to\pi_1(N,f(v))$$

The following lemma is immediate.

**Lemma 3.8.** *The association of a map morphism to its induced group morphism is functorial. In other words, the induced group morphism of a composition of map morphisms is the composition of the induced group morphisms.*

*Example* 3.9. Since the elementary homotopies in the graph $G(M)$ of a map $M$ are homotopies in the map $M$, we have an epimorphism $\jmath : \pi_1(G(M), v) \twoheadrightarrow \pi_1(M, v)$. More generally, a graph morphism $f : H \to G(M)$ induces a morphism $\jmath \circ f_* : \pi_1(H, w) \to \pi_1(M, f(w))$.

### 3.1.1 The induced morphisms of basic operations

Although the basic operations in Sections 2.1.2, 2.1.3 and 2.1.4 are not morphisms (they could however be interpreted as such in a more relaxed definition of morphisms), we can define an induced morphism for each basic operation.

- Let $e$ be a non-loop edge of a map $M$ with basepoint $v$. The contraction of $e$ yields a map $M/e$ with an evident mapping $C_e$ from the vertices of $M$ to the vertices of $M/e$. This mapping extends to loops as follows. If $\ell$ is a loop of $M$ with basepoint $v$, we define $C_e(\ell)$ as the loop of $M/e$ with basepoint $C_e(v)$ obtained by removing the occurrences of $e$ in $\ell$. Since the edge contraction sends faces to faces, two loops related by an elementary homotopy are sent by $C_e$ to homotopic loops. As $C_e$ trivially commutes with path concatenation, we conclude that $C_e$ induces a morphism $(C_e)_* : \pi_1(M, v) \to \pi_1(M/e, C_e(v))$.

- The subdivision of any edge $e$ of $M$ yields a map $S_e M$. Every loop $\ell$ of $M$ maps to a loop $S_e(\ell)$ of $S_e M$ obtained by replacing each occurrence of $e$ with the sequence of two edges resulting from its subdivision. Similarly to the edge contraction, the mapping $S_e$ induces a morphism $(S_e)_* : \pi_1(M, v) \to \pi_1(S_e M, v)$.

- The subdivision of a face in $M$ from an arc $a$ to an arc $b$ induces an inclusion $S_{(a,b)}$ of the loops of $M$ into the set of loops of $S_{(a,b)}M$. In turn, $S_{(a,b)}$ induces a morphism $(S_{(a,b)})_* : \pi_1(M, v) \to \pi_1(S_{(a,b)}M, v)$.

We note that the morphisms $(C_e)_*, (S_e)_*$ and $(S_{(a,b)})_*$ are obtained from the mappings $C_e, S_e$ and $S_{(a,b)}$ as quotients by the homotopy relation. It follows that the induced morphism of a composition of such mappings is the composition of the induced morphism. For an appropriate notion of morphisms of map, this just means that the association $f \mapsto f_*$ is functorial.

**Lemma 3.10.** *The above group morphisms $(C_e)_*, (S_e)_*$ and $(S_{(a,b)})_*$ are isomorphisms.*

PROOF. The subdivision of an edge $e$ of $M$ replaces this edge by two edges $e_1, e_2$. Using the above notations and identifying $e_1$ with $e$, we get that $S_e$ inserts occurrences of $e_2$ in a loop while $C_{e_2}$ removes such occurrences. It follows that $C_{e_2} \circ S_e$ is the identity on loops, whence $(C_{e_2})_* \circ (S_e)_* = Id$. On the other hand, for each loop $\ell$ of $S_e M$ the loop $S_e \circ C_{e_2}(\ell)$ is obtained from $\ell$ by possible insertions of the spur $(e_2^{-1}, e_2)$. It follows that $(S_e)_* \circ (C_{e_2})_* = Id$, implying that $(S_e)_*$ is an isomorphism.

Let $e$ be the edge added by the face subdivision $S_{(a,b)}$. To every loop $\ell$ of $S_{(a,b)}M$ we associate the loop $D_e(\ell)$ obtained by substituting each occurrence of $e$ with the complementary subpath in one of the two incident faces. In particular the mapping $D_e \circ S_{(a,b)}$ does not modify any loop of $M$ while $S_{(a,b)} \circ D_e(\ell)$ is homotopic to $\ell$. The mapping $D_e$ induces a group morphism $(D_e)_*$ and we conclude that $(S_{(a,b)})_*$ and $(D_e)_*$ are inverse to each other.

The proof that $(C_e)_*$ is an isomorphism can be done analogously by providing an adequate mapping from the set of loops of $C_e M$ to the set of loops of $M$. A less direct proof follows from Remark 2.13. □

As combinatorial equivalence is generated by edge and face subdivisions we conclude that

**Corollary 3.11.** *Combinatorially equivalent maps have isomorphic fundamental groups.*

The fundamental groups of maps are thus given by the presentations in Example 3.7.

# 4 Coverings

**Definition 4.1.** A morphism $p : M \to N$ is a **covering** if its restrictions to stars $\mathrm{Star}(x) \to \mathrm{Star}(p(x))$, for $x$ a vertex or a face of $M$, are bijective.

Most of the properties proved for graph coverings in the previous lecture notes remain valid for map coverings. The definition of a path lift for graphs applies verbatim to maps: a lift of a path $\gamma$ in $N$ is a path $\delta$ in $M$ such that $p(\delta) = \gamma$. The unique lift property also remains true for maps. We also have

**Lemma 4.2.** *Let $M \to N$ be a covering. Let $\tilde\alpha$ and $\tilde\beta$ be respective lifts with a same origin in $M$ of two homotopic paths $\alpha, \beta$ in $N$. Then $\tilde\alpha$ and $\tilde\beta$ are homotopic in $N$.*

Proof. If $\alpha$ and $\beta$ are related by one elementary homotopy, then so are $\tilde\alpha$ and $\tilde\beta$. This is obvious for the insertion or deletion of a spur. Otherwise, thanks to the bijections induced by the covering restrictions to faces (identified with their stars), the replacement in $\alpha$ of a subpath of the facial circuit of a face $F$ by its complementary subpath lifts to the replacement in $\tilde\alpha$ of complementary subpaths in a face above $F$. The lemma now follows by induction on the number of elementary homotopies relating $\alpha$ to $\beta$. □

We thus have a right action of $\pi_1(N, v)$ on the fiber above $v \in V(N)$ given by the final endpoint $w.[\alpha]$ of the lift with origin $w$ of a loop $\alpha$, where $p(w) = v$. Similarly to graph coverings we get

**Corollary 4.3.** *If $p : (M, w) \to (N, v)$ is a covering, then the induced morphism $p_* : \pi_1(M, w) \to \pi_1(N, p(w))$ is one-to-one.*

The fundamental group of $(M, w)$ thus embeds as a subgroup of the fundamental group of $(N, v)$. We also have that every subgroup of $\pi_1(N, v)$ can be realized as the fundamental group of a covering.

**Proposition 4.4.** *Let $v$ be a vertex of the connected map $M$. For every subgroup $U < \pi_1(M, v)$ there exists a connected covering $p_U : (M_U, w) \to (M, v)$ with $p_{U*}\pi_1(M_U, w) = U$.*

PROOF. Fix a spanning tree $T$ of $M = (A, \rho, \iota)$. We write $\gamma_a$ for the loop $T[v, a]$ (see the notations in the proof of Lemma 3.6). Define $M_U = (A_U, \rho_U, \iota_U)$ by

- $A_U = A \times \{Ug\}_{g\in\pi_1(M,v)}$,
- $\rho_U(a, Ug) = (\rho(a), Ug)$, and
- $\iota_U(a, Ug) = (\iota(a), Ug[\gamma_a])$

where $Ug$ denotes the right coset representative in $\pi_1(M, v)$ of $g$ with respect to $U$. $M_U$ is indeed a map: we trivially check that $\iota_U$ is a fixed point free involution and that $\rho_U$ is a permutation of $A_U$. We consider the projection on the first component $p_U : A_U \to A$. The following relations are immediate from the above definitions

$$\iota \circ p_U = p_U \circ \iota_U, \quad \rho \circ p_U = p_U \circ \rho_U$$

Hence, $p_U : M_U \to M$ is a map morphism in the sense of 1.6. We claim that $p_U$ is a covering. Since $\langle \rho_U \rangle (a, Ug) = \langle \rho \rangle a \times \{Ug\}$, the restrictions of $p_U$ to vertex stars are indeed bijective. We also need to prove that the restrictions of $p_U$ to facial circuits are bijective. Let $\varphi = \rho \circ \iota$ and $\varphi_U = \rho_U \circ \iota_U$ be the facial permutations of $M$ and $M_U$ respectively. We have

$$\varphi_U(a, Ug) = \rho_U(\iota(a), Ug[\gamma_a]) = (\rho(\iota(a)), Ug[\gamma_a]) = (\varphi(a), Ug[\gamma_a])$$

Hence, if $(a_1, \dots, a_k)$ is the facial circuit of a face of $M$, then for $0 \le i \le k$ we get $\varphi_U^i(a_1, Ug) = (a_{i+1}, Ug[\gamma_{a_1}] \dots [\gamma_{a_i}])$. In particular, $\varphi_U^k(a_1, Ug) = (a_1, Ug)$ since $\gamma_{a_1} \cdots \gamma_{a_k}$ is freely homotopic to $(a_1, \dots, a_k)$, which is contractible. It follows that the restriction $p_U : F(a_1, Ug) \to F(a_1)$ is indeed bijective.

It remains to prove that $M_U$ is connected and that $p_{U*}\pi_1(M_U, w) = U$ for $w = (v, U)$. The proof is formally identical to the proof of the analogous proposition for graphs. □

*Example* 4.5. When $U = \{1\}$ is the trivial group, $p_U$ is the **universal cover** of $M$. Up to isomorphism, this is the unique simply connected covering of $M$.

*Example* 4.6. When $U = [\pi_1(M, v), \pi_1(M, v)]$ is the derived subgroup of $\pi_1(M, v)$, $p_U$ is the **homology covering**. Choosing for $U$ the subgroup of $\pi_1(M, v)$ generated by the squares, we get the $\mathbb{Z}/2\mathbb{Z}$-homology covering.

The notion of morphism between graph coverings extends to map coverings. Hence, a **morphism** between map coverings $p : M \to N$ and $q : K \to N$ is a map morphism $f : M \to K$ such that $p = q \circ f$. The results in Section 7.1.1 *Covering morphisms* of the lecture notes on graphs apply verbatim to maps. In particular,

**Theorem 4.7.** *The isomorphism classes of connected coverings of a connected map $M$ are in 1-1 correspondence with the conjugacy classes of subgroups of its fundamental group.*

# 5 Quotient Maps

We denote by **Aut**$(M)$ the group of automorphisms of a map $M$.

**Definition 5.1.** Let $\Gamma < \mathrm{Aut}(M)$ acts on a map $M = (A, \rho, \iota)$. We define the **quotient map** $M/\Gamma = (A_\Gamma, \rho_\Gamma, \iota_\Gamma)$ by

- $A_\Gamma = \{\Gamma \cdot a\}_{a \in A}$,
- $\rho_\Gamma(\Gamma \cdot a) = \Gamma \cdot \rho(a)$ and $\iota_\Gamma(\Gamma \cdot a) = (\Gamma \cdot a)^{-1} = \Gamma \cdot a^{-1}$

It readily follows from the definition that the **quotient map** $p_\Gamma : M \to M/\Gamma$ sending an arc to its orbit is a map morphism.

*Remark* 5.2. As for graphs, a subgroup $\Gamma < \mathrm{Aut}(M)$ is said to **act without arc inversion** if for any arc $a$ of $M$, $a^{-1}$ is not in the $\Gamma$-orbit of $a$. If $M$ has no self-opposite arc, then $M/\Gamma$ has no self-opposite arc if and only if $\Gamma$ acts without arc inversion.

*Remark* 5.3. $\mathrm{Aut}(M)$ acts freely on $A$. Indeed if $f \in \mathrm{Aut}(M)$, then $f(a) = a$ for some arc $a \in A$ implies $f(\rho(a)) = \rho(a)$ and $f(\iota(a)) = \iota(a)$, so that $f$ is the identity by the transitive action of the monodromy group of $M$.

**Definition 5.4.** An automorphism $f \in \mathrm{Aut}(M)$, with $M = (A, \rho, \iota)$ is **fixed point free** if $f$ does not fix any vertex or face of $M$. Equivalently,

$$\forall a \in A : f(a) \notin \langle \rho \rangle a \cup \langle \rho \circ \iota \rangle a \tag{4}$$

A subgroup $\Gamma < \mathrm{Aut}(M)$ **acts freely** on $M$ if every $f \in \Gamma \setminus \{Id\}$ is fixed point free.

**Proposition 5.5.** *Let $\Gamma < Aut(M)$. The quotient map $p_\Gamma : M \to M/\Gamma$ is a covering if and only if $\Gamma$ acts freely on $M$.*

Proof. Since $p_\Gamma$ is onto, it is a covering if and only if its restriction to (vertex and face) stars is one-to-one. Using that $\Gamma$ commutes with $\rho$ and $\iota$, this can be rephrased as

$$\forall a \in A, \forall b \in \langle \rho \rangle a \cup \langle \rho \circ \iota \rangle a, \quad \Gamma.b = \Gamma.a \implies a = b$$

By Remark 5.3, this is just saying that (4) holds for every $f \in \Gamma \setminus \{Id\}$. □

All the results, proofs and definitions from Section 7 in the previous lecture on graphs remain valid for maps without self-opposite loops if we add the condition that a group acting on a map should not invert arcs. If we allow maps to have self-opposite loops, we can simply drop this condition everywhere in that section. In particular, the set Aut($p$) of automorphisms of a covering $p: M \to N$ is by definition the set of automorphisms $f$ of $M$ such that $p \circ f = p$. We also have,

**Proposition 5.6.** *Let $p: M \to N$ be a connected covering and let $v$ be a vertex of $M$. Then,*

$$Aut(p) \simeq N\left(p_*\pi_1(M,v)\right)/p_*\pi_1(M,v)$$

*where $N\left(p_*\pi_1(M,v)\right)$ is the normalizer of $p_*\pi_1(M,v)$ in $\pi_1(N,p(v))$.*

## 5.1 Hurwitz's automorphisms theorem

The famous Hurwitz's bound on the number of automorphisms of a Riemann surface, applies to combinatorial maps.

**Theorem 5.7** (Hurwitz, 1893)**.** *Let $M$ be a finite map of genus $g \geq 2$, then $|Aut(M)| \leq 84(g-1)$*

PROOF. Put $\Gamma = \mathrm{Aut}(M)$ and consider the morphism $p_\Gamma : M \to M/\Gamma$. We first observe that, by the very definition of a quotient, $\Gamma$ acts transitively on each fiber of $p_\Gamma$. This is equally true for a fiber above a vertex or a face. It follows that all vertices or faces in a fiber are stars of the same size, hence have the same ramification index. We denote by $e_w$ the ramification index of the fiber above $w$. The Riemann-Hurwitz formula in Theorem 1.12 thus factors as

$$\begin{aligned}
\chi(M) &= \deg(p_\Gamma)\chi(M/\Gamma) - \sum_{w\in V(M/\Gamma)\cup F(M/\Gamma)} |p_\Gamma^{-1}(w)|(e_w - 1)\\
&= \deg(p_\Gamma)\chi(M/\Gamma) - \sum_{w\in V(M/\Gamma)\cup F(M/\Gamma)} |p^{-1}(w)|e_w(1-\frac{1}{e_w})\\
&= |\Gamma|\left(\chi(M/\Gamma) - \sum_{w\in V(M/\Gamma)\cup F(M/\Gamma)} (1-\frac{1}{e_w})\right)
\end{aligned}$$

where, we have used the index formula of Proposition 1.11 for the last equality and the fact that $\Gamma$ acts freely on $A(M)$ (See Remark 5.3). Since $\chi(M) = 2-2g$, we may rewrite this last equation as

$$2(g-1) = |\Gamma|Q$$

where $Q = \sum_{w\in V(M/\Gamma)\cup F(M/\Gamma)}(1-\frac{1}{e_w}) - \chi(M/\Gamma)$. It follows that $Q$ must be positive. We claim that $Q \geq 1/42$, which proves the theorem. Recall that the characteristic of a (closed) oriented map is an even integer.

- If $\chi(M/\Gamma) \leq -2$, then $Q \geq 2$, confirming the claim.

- If $\chi(M/\Gamma) = 0$, then $Q > 0$ implies that one of the term in the sum $\sum_w (1 - \frac{1}{e_w})$ is positive, *i.e.* $e_w \geq 2$ for some $w \in V(M/\Gamma) \cup F(M/\Gamma)$. Is follows that $Q \geq 1/2$, which again confirms the claim.
- It remains the case $\chi(M/\Gamma) = 2$, when $M/\Gamma$ is a sphere. Put $W = \{w \in V(M/\Gamma) \cup F(M/\Gamma) \mid e_w > 1\}$. We must have $\sum_{w \in W}(1 - \frac{1}{e_w}) > 2$. In particular, $|W| \geq 3$. Note that each term in this sum satisfies $1/2 \leq 1 - 1/e_w \leq 1$ and is an increasing function of $e_w$.
  - If $|W| \geq 5$, then $\sum_{w \in W}(1 - \frac{1}{e_w}) \geq 5/2$, whence $Q \geq 1/2$.
  - If $|W| = 4$, then $\sum_{w \in W}(1 - \frac{1}{e_w}) \geq 3(1/2) + 2/3$, whence $Q \geq 1/6$.
  - If $|W| = 3$, then the least possible value of $\sum_{w \in W}(1 - \frac{1}{e_w})$ is $1/2 + 2/3 + 6/7$, whence $Q \geq 1/42$.

This proves the claim in all the possible cases. □

Maps of genus $g$ whose automorphism group size reaches the Hurwitz's bound $84(g-1)$ are called *Hurwitz maps*. From the proof of the above theorem, such maps must ramify over the sphere with three branch values of ramification indices 2,3 and 7, respectively.

# 6 From Maps to Dessins d'Enfants

Here, we try to give an intuition on the concept of *dessins d'enfants* discovered by Grothendieck. The monograph by Girondo and González-Diez [GGD12] is an excellent introduction to the theory of dessins d'enfants.

## 6.1 A note on the monodromy group

### 6.1.1 Branched covering

In surface topology, a **branched covering** is a continuous map $p : S' \to S$ between surfaces $S'$ and $S$ such that the restriction $p : S' \setminus p^{-1}(\Sigma) \to S \setminus \Sigma$ is a (unbranched) covering for some discrete subset $\Sigma \subset S$. The points in the **singular set** $\Sigma$ are called **branch values** and the **degree** of the branched covering is the degree of its restriction, *i.e.* the number of sheets of the corresponding covering. The points in $p^{-1}(\Sigma)$ are the **ramification points** of $p$. In the neighbourhood of each ramification point, $p$ should in addition look like (in some charts around the branching point and its image) the map $z \mapsto z^k$ in the complex plane. The integer $k > 0$ is the **ramification index** of the ramification point.

A morphism of oriented maps $f : M \to N$ can be realized as a branched covering of degree $\deg(f)$, preserving the orientation, between the corresponding topological surfaces $S(M)$ and $S(N)$. To construct this realization, one can for instance triangulate each face of $S(M)$ by coning its boundary from a center point. The resulting triangles inherit the orientation of $S(M)$ and each of them is incident with exactly one arc of $M$ consistently oriented. One can similarly triangulate $S(N)$. A triangle incident with an arc $a$ of $M$ can now be sent homeomorphically to the triangle incident with $f(a)$.

This defines a branch covering whose ramification points are vertices and face centers of $M$ with ramification indices as defined in Lemma 1.8.

When $M$ and $N$ are allowed to have self-opposite arcs, one should refine the above construction in order to define the branched covering properly. First, if an arc of a map is self-opposite it corresponds in the graph to an edge folded back on itself; only one of its endpoints is considered as a vertex. The other endpoint is called a **free end**. The simplest map shown on Figure 14 has a single self-opposite arc. It intuitively corresponds to the Riemann sphere. In order to triangulate a map with self-opposite

Figure 14: Left, the trivial map $(\{a\}, Id, Id)$ has one edge, one face, one (black) vertex and one (white) free end. Right, the canonical triangulation of the trivial map has two triangles.

edges, we first color all the vertices of the map in black and introduce a white vertex in the middle of every edge that is not self-opposite. We also color in white the free ends of the self-opposite edges. As in the above construction we further triangulate each face by coning its boundary from a center point. The number of triangles in a face is doubled compared to the previous construction. Note that the vertices of every triangle include one black vertex, one white vertex (corresponding to an edge) and a face center. Also, the side of a triangle connecting its black and white vertices can be associated with the arc of the map that contains it, oriented from the black to the white vertex. Each arc, self-opposite or not, is now associated with two triangles with opposite orientations with respect to the ordering (black,white,center) of its vertices. The resulting triangulation is the **canonical triangulation** of the map. We finally send homeomorphically a triangle of the canonical triangulation of $M$ associated with an arc $a$ to the triangle of the canonical triangulation of $N$ associated with $f(a)$, and with the same orientation. The resulting branched covering may now ramifies at white vertices, *i.e.* at the middle of edges or at free ends, in addition to black vertices and face centers.

This correspondence between map morphisms and branched coverings can be performed in the realm of Riemann surfaces [GGD12], providing adequate functors.

#### 6.1.2 The canonical morphism of a map and its monodromy group

In order to complete the parallel between combinatorial maps and Riemann surfaces we shall define the combinatorial counterpart of a Belyi function, that is of a holomorphic map to the Riemann sphere with at most three branch values. However, to be defined properly this combinatorial counterpart requires to allow a combinatorial map to have self-opposite arcs. Hence, an oriented map becomes a triple $M = (A, \rho, \iota)$ where, as before, $\rho$ and $\iota$ are permutations of $A$ with $\iota$ an involution, but $\iota$ may now fix some arcs. It now appears that for any map $M = (A, \rho, \iota)$ there is a **canonical morphism** to the **trivial map** $(\{a\}, Id, Id)$ given by the constant function $A \to \{a\}$. It corresponds to a branched covering of the sphere that ramifies above its vertex, the center of its face *and* its free end. Identifying free ends with their arcs, we see that the above correspondence between topological and combinatorial maps must take into account ramifications at edges in addition to vertices and faces [JS78]. The formalisms of constellations and hypermaps permit us to avoid those singular free ends. Those formalisms are sketched in the next section for completeness. However, as far as algorithms on curves on surfaces are concerned the point of view of rotation systems seems more adequate and more intuitive.

Given a topological covering $f : (S, y) \to (B, x)$ there is a right action of $\pi_1(B, x)$ on the fiber $f^{-1}(x)$ obtained by lifting a loop representative of a homotopy class, in the same way as for graph coverings in the previous lecture. The representation of $\pi_1(B, x)$ as a subgroup of permutations of $f^{-1}(x)$ is called the **monodromy group** of the covering. Changing the basepoint produces an isomorphic action, so that the monodromy group is well-defined up to isomorphism. When $f$ is a branched covering, we can still define its monodromy group by considering the restriction $f : S \setminus f^{-1}(C) \to B \setminus C$, where $C$ is the set of branch values (also called critical values) of $f$. This restriction is indeed a (unbranched) covering on which acts the fundamental group of $B \setminus C$ by monodromy.

We can now consider the monodromy group of the branched covering corresponding to the topological realization of the canonical morphism of a combinatorial surface $M$. This branched covering has the form $S(M) \to \mathbb{S}^2$ whose set $C$ of branch values contains the two endpoints of (the embedding of) the unique edge of the trivial map and the center of its unique face. Hence, $\mathbb{S}^2 \setminus C$ is a sphere with three punctures. It is homeomorphic to a **pair of pants** without boundary. Its fundamental group is a free group of rank 2 generated by two loops $\lambda, \mu$, each surrounding one of the edge endpoints (see Figure 15). If we connect the chosen basepoint $x$ to a point $p$ on the

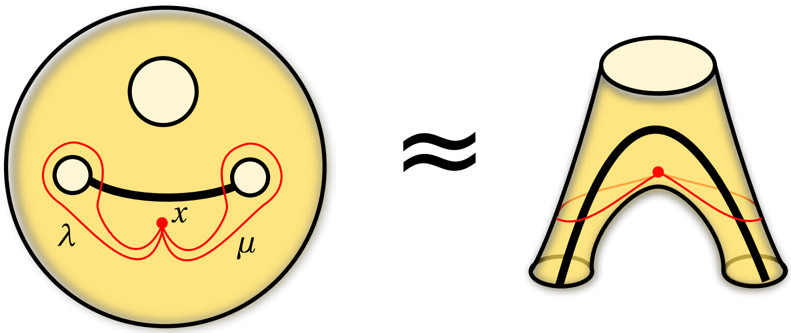


Figure 15: Two views of a pair of pants with two loops generating its fundamental group. The loop $\lambda$ surrounds the vertex of the embedded edge and the loop $\mu$ surrounds its free end.

unique edge, $a$, of the punctured sphere by a path, then the lifts of this path establish a correspondence between the fiber of $x$ and the fiber of $p$. In turn, the points of $p$'s fiber can be identified with the arcs of $M = (A, \rho, \iota)$. Indeed, each lift of $a$ contains a single lift of $p$ and the fiber of $a$ by the canonical morphism is the whole set $A$. Because each lift of $\lambda$ crosses exactly one arc, the action of $\lambda$ on a point in the fiber of $x$, identified with an arc $e$ of $M$, corresponds to a rotation of $e$ about its origin. This action thus corresponds to the rotation system $\rho$. Similarly, the action of $\mu$ corresponds to the involution $\iota$. The monodromy action of $\pi_1(\mathbb{S}^2 \setminus C, x) = \langle \lambda, \mu \rangle$ is thus isomorphic to the monodromy group $\langle \rho, \iota \rangle$, whence the terminology.

## 6.2 Other models of oriented maps

We briefly mention some other combinatorial models of oriented surfaces found in the litterature such as the notion of constellation as presented by Lando and Zvonkin [LZ04].

### 6.2.1 Constellations

**Definition 6.1.** A **constellation** is a finite sequence of permutations $(g_1, \ldots, g_k)$ acting transitively on a finite set $\{1, \ldots, n\}$ and such that the product $g_1 \cdots g_k$ is the identity permutation.

This algebraico-combinatorial object can be interpreted as an $n$-fold branched covering of the Riemann sphere with $k$ branch values indexed by $1, \ldots, k$. The ramification indices of the ramification points above the branch value of index $i$ are given by the length of the cycles of the permutation $g_i$. Given a constellation $(g_1, \ldots, g_k)$ the construction of this branched covering can be performed as follows. We consider a set $C$ of $k$ punctures in the oriented sphere $\mathbb{S}^2$ and a basepoint $x \in \mathbb{S}^2 \setminus C$. We draw a star graph in $\mathbb{S}^2$ connecting $x$ to each point in $C$ (see Figure 16). We obtain a generating set

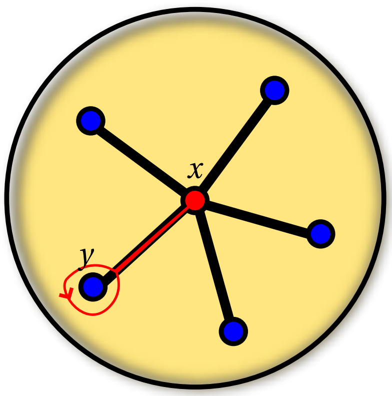


Figure 16: A sphere with five (blue) punctures. The loop $\gamma_y$ with basepoint $x$ surrounds the puncture $y$.

for $\pi_1(\mathbb{S}^2 \setminus C, x)$ by forming a loop $\gamma_y$ for each $y \in C$; this loop follows the edge $xy$ in the star graph and stops just before reaching $y$, goes around $y$ in the counterclockwise direction and travels back to $x$. The product of the $\gamma_y$ in the counterclockwise order of the star edges is clearly contractible. Denoting by $1, \ldots, k$ the points of $C$ in *clockwise* order, we thus have a presentation $\langle [\gamma_1],\dots,[\gamma_k]; [\gamma_1]^{-1}\cdots[\gamma_k]^{-1}=1\rangle$ for $\pi_1(\mathbb{S}^2\setminus C, x)$. Since the unique relation of the $[\gamma_i]$ is satisfied by the $g_i^{-1}$, the map $[\gamma_i]\mapsto g_i^{-1}$ induces a group morphism $\phi:\pi_1(\mathbb{S}^2\setminus C,x)\to G$ where $G=\langle g_1,g_2,\dots,g_k\rangle$ is the **monodromy group** of the constellation. Let $U=\{g\in G; g(1)=1\}$ be the stabilizer of 1 (recall that $G$ acts on $\{1,\dots,n\}$). Similarly to the case of graphs in the previous lecture notes, the preimage $U_\phi := \phi^{-1}(U) < \pi_1(\mathbb{S}^2\setminus C,x)$ determines a covering $p_U: S_U\to\mathbb{S}^2\setminus C$ whose fiber elements are indexed by the right cosets of $U_\phi$ in $\pi_1(\mathbb{S}^2\setminus C,x)$. These right cosets are put in 1-1 correspondence with $\{1,\dots,n\}$ thanks to the map

$$F:\pi_1(\mathbb{S}^2\setminus C,x)\to\{1,\dots,n\},\quad \alpha\mapsto\phi(\alpha^{-1})(1)$$

Indeed, $F(\alpha)=F(\beta)$ is equivalent to $\beta\in U_\phi\alpha$ and $F$ is onto because $\phi$ is onto and $G$ acts transitively on $\{1,\dots,n\}$. In turn, $F$ "quotients" to a bijection $F_x$ between the fiber of $p_U$ above $x$ and $\{1,\dots,n\}$ given by $F_x(x,U_\phi\alpha)=F(\alpha)$.

We now check that $F_x$ transforms the action of the monodromy group of $p_U$ into the action of the monodromy group of the constellation. As seen in the previous lecture notes on graphs, the action of $\pi_1(\mathbb{S}^2\setminus C,x)$ on the fiber above $x$ is given by $(x,U_\phi\alpha).[\gamma_i]=(x,U_\phi\alpha[\gamma_i])$. Using the correspondence $F_x$, we compute

$$\begin{aligned}F_x\left((x,U_\phi\alpha).[\gamma_i]\right)&=F_x(x,U_\phi\alpha.[\gamma_i])=F(U_\phi\alpha.[\gamma_i])=\phi\left((\alpha[\gamma_i])^{-1}\right)(1)\\&=\phi[\gamma_i]^{-1}\phi(\alpha^{-1})(1)=g_i(F_x(x,U_\phi\alpha)),\end{aligned}$$

which shows the correspondence between the monodromy group actions.

We finally compactify $S_U$ and $\mathbb{S}^2\setminus C$ to extend $p_u$ to a branched covering $\overline{S}_U\to\mathbb{S}^2$. To this end, we consider small punctured discs $D_y^*$ centered at each puncture $y\in C$ and note that the restriction $p_U: p_U^{-1}(D_y^*)\to D_y^*$ being a covering of finite degree, $p_U^{-1}(D_y^*)$ must be a disjoint union of punctured discs. We formally add a center to those punctured discs and extend $p_U$ trivially by sending the added centers to $y$. We obtain this way a compact branched covering of the sphere $\mathbb{S}^2$. We claim that its ramification indices are the cycle lengths of the $g_i$. To see this, we may further assume that in each $\gamma_y$, the small loop around $y$, call it $\lambda_y$, is contained in $D_y^*$. Let $y'$ be the basepoint of $\lambda_y$ (on the edge $xy$). Consider a compactified component $D$ of $p_U^{-1}(D_y^*)$. By definition, the monodromy action of $[\lambda_y]$ on the fiber $p_U^{-1}(y')$ restricts to a cyclic permutation of $p_U^{-1}(y')\cap D$ whose order is the ramification index of the center of $D$. Now, it is a simple exercise to check that the action of $[\lambda_y]$ on $p_U^{-1}(y')$ is isomorphic to the action of $[\gamma_y]$ on $p_U^{-1}(x)$. In turn, this action is isomorphic to the action of $g_i$ on $\{1,\dots,n\}$ by the preceding discussion.

We can plug these ramification indices into the Riemann-Hurwitz formula of theorem 1.12 to obtain the Euler characteristic of $\overline{S}_U$. This branched covering comes with a cellular embedding of a graph obtained by lifting the star graph on $\mathbb{S}^2$. This graph is bipartite, the partition being given by the fiber of the basepoint on the one side and the union of the fibers of the branch values on the other side. It can be seen as a union of stars of degree $k$ centered at the vertices in the basepoint fiber, whence the name of constellation.

### 6.2.2 Hypermaps

Other cellular embeddings can be obtained starting with a different graph on the sphere. One possibility is to start with a chain graph going trough the branch values $y_1, y_2, \ldots, y_{k-1}$, leaving the point $y_k$ aside. In this case, the covering has branch points at vertices and at the center of faces (the points above $y_k$). When $k = 3$, that is when we start with a 3-constellation $(g_1, g_2, g_3)$, we obtain a single edge on the sphere as for the trivial map on Figure 14. The partition corresponding to the two fibers of $y_1$ and $y_2$ again make the lifted graph bipartite. If we further impose that $g_2$ is a fixed point free involution, then $y_2$ lifts to degree two vertices. Viewing the edges as arcs oriented from (lifts of) $y_1$ to $y_2$ we get exactly the same picture as for the canonical morphism of a map, where the lifts of $y_2$ become the middle-point of the edges. Formally we have an identification of the combinatorial maps as 3-constellations given by the correspondence $(A, \rho, \iota) \mapsto (\rho, \iota, (\rho \circ \iota)^{-1})$, where the set $A$ of arcs should be identified with $\{1, \ldots, n\}$. We can thus identify the fibers of $y_1$, $y_2$ and $y_3$ as vertices, edges and faces respectively. If we still keep this terminology for general 3-constellations, where $g_2$ may be any permutation, then each "edge" in the fiber of $y_2$ becomes incident to possibly more than two vertices. Traditionally, the vertices above $y_1$ are colored in black and the vertices above $y_2$ are colored in white. The lifted graph can thus be interpreted as a hypergraph[1] whose ground set is the set of black vertices and whose hyperedges are the sets of neighbours of the white vertices. This is why 3-constellations are usually called **hypermaps**. The cellular embedding of this hypergraph has a **canonical triangulation** defined in the same way as in Section 6.1.1 for maps with self-opposite edges: the vertices of each triangle are respectively black, white, and at the center of a face.

As noted above, any map can be considered as a hypermap. The converse is also true, so that there is no real loss of generality by considering maps instead of hypermaps: If one subdivide every black-white edge of a hypermap by introducing a grey vertex in the middle, we obtain a map where the arcs corresponds to the edges of the subdivide hypermap and the opposite of a black-grey edge is the incident grey-white edge.

### 6.2.3 Intrinsic algebraic formalism

Given a connected combinatorial map $(A, \rho, \iota)$, its monodromy group $\langle \rho, \iota \rangle$ acts transitively on the set $A$ of arcs that can thus be identified with the left cosets of the stabilizer $S_a = \{\tau \in \langle \rho, \iota \rangle \mid \tau(a) = a\}$ of some fixed arc $a \in A$. Indeed, it is easily seen that the correspondence $\langle \rho, \iota \rangle / S_a \to A$ given by $\tau S_a \mapsto \tau(a)$ is well defined and one-to-one. To obtain an isomorphic action of the monodromy group, one should consider its left action on the left cosets $\langle \rho, \iota \rangle / S_a$. A map can thus be represented by a (monodromy) group $\Gamma$, a (stabilizer) subgroup $S$, and two generators $\rho, \iota$ of $\Gamma$ such that $\iota^2 = 1$. However, to represent a map, one should make sure that $\Gamma$ acts faithfully on $\Gamma/S$. Indeed, since the monodromy group was originally defined as a subgroup of permutations of $A$ it acts faithfully, meaning that each element of the monodromy group is uniquely determined by its action. In general, if we are given $(\Gamma, S, \rho, \iota)$ as above there is no reason why $\Gamma$ should act faithfully on $\Gamma/S$. Note that for $h, g \in \Gamma$, having $h(gS) = gS$

[1] Note that a hypergraph, or set system, is just another name for a bipartite graph where one part is chosen as the ground set.

is equivalent to $h \in gSg^{-1}$. So, any $h$ in the intersection of the conjugate subgroups of $S$ acts as the identity on $\Gamma/S$. This intersection is the largest normal subgroup of $\Gamma$ contained in $S$ and is usually denoted by $\text{core}_\Gamma(S)$. Hence, $\text{core}_\Gamma(S)$ should be trivial if we want $\Gamma$ to act faithfully. When this is not the case this can be enforced by considering the action of $\Gamma/\text{core}_\Gamma(S)$ on $\Gamma/S$ with the condition $\iota^2 \in \text{core}_\Gamma(S)$. See [BS85] and [BW09, Ch. 10] for further details.

We close this section by noting that constellations should not be compared with combinatorial maps but rather with map morphisms to maps of genus zero endowed with fixed embedded graphs (like star graphs). The two formalisms are equivalent but thanks to its symmetry the formalism of constellations is more powerful when dealing with algebraic properties. However, the graph embedding we can associate to a constellation is not really encoded in the constellation as it depends on the graph drawn on the base sphere. When dealing with combinatorial curves on surfaces, the embedded graph itself becomes the main object of study and combinatorial maps provide this graph more directly.

## 6.3 Dessins d'enfants

The theory of Dessins d'enfants, as named by A. Grothendieck, is the result of beautiful connections between combinatorial maps, Riemann surfaces and the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on combinatorial (hyper)maps. The correspondence between combinatorial maps and Riemann surfaces was developed at the end of the 1970's by G. Jones and D. Sigerman [JS78], but the full connection with the the absolute Galois group was recognized by Grothendieck in his famous Esquisse d'un programme. We end these notes on the topology of combinatorial surfaces by a brief and sketchy introduction to these correspondences.

### 6.3.1 From maps to Belyi functions

As observed in Section 6.1, a map comes with a canonical morphism onto the trivial map $(\{a\}, Id, Id)$. Moreover, this morphism can be realized as a topological branched covering over the sphere, with three branch values respectively at the vertex, free end and face center of the trivial map. The ramification index above the free end can be 2, or 1 when the given map has self-opposite arcs. To avoid this limitation for the ramification index, one can resort to hypermaps as defined in Section 6.2.2. An analogous construction indeed holds for hypermaps. To see this, first replace the trivial map by the **trivial hypermap** composed of a single black-white edge corresponding to the 3-constellation $(Id, Id, Id)$ acting on the singleton $\{1\}$. Let $\bullet, \circ$ and $\star$ denote respectively the black vertex, the white vertex and the face center of the corresponding (cellular) embedding of the black-white edge into the sphere. The obvious projection of the canonical triangulation, $T$, of a hypermap $(g_1, g_2, g_3)$ (see Section 6.2.2) to the canonical triangulation, $T_0$, of the trivial hypermap provides a topological ramified covering $f : T \to T_0$ with $\bullet, \circ, \star$ as its three branch values and whose ramification indices are the lengths of the cycles of the $g_i$'s. We can actually realise this ramified covering in the realm of Riemann surfaces. For this, consider the Riemann sphere

$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Send the two triangles of $T_0$ to $\widehat{\mathbb{C}}$ so that their common boundary is sent to the circle $\mathbb{R} \cup \{\infty\}$ with $\bullet, \circ, \star$ mapped respectively to 0, 1 and $\infty$ and choose which triangle goes to which component of $\mathbb{C} \setminus \mathbb{R}$ in a way that the mapping preserves orientations. This mapping, call it $h$, endows the trivial hypermap with a holomorphic structure: An atlas is provided by the two charts $(T_0 \setminus \{\bullet\}, 1/h_{|T_0 \setminus \{\bullet\}})$ and $(T_0 \setminus \{\star\}, h_{|T_0 \setminus \{\star\}})$. On the other hand, given the canonical triangulation $T$ of a hypermap, the topological ramified covering $f : T \to T_0$ restricts to an unramified covering when removing the ramification points above $\bullet, \circ, \star$. We can lift the above holomorphic structure on $T_0 \setminus \{\bullet, \circ, \star\}$ to a holomorphic structure on $T \setminus f^{-1}(\bullet, \circ, \star)$. As in Section 6.2.1, this structure can be extended to the whole $T$ so that $h \circ f : T \to \widehat{\mathbb{C}}$ now appears as a morphism of Riemann surfaces. Such a morphism, from a compact Riemann surface to the Riemann sphere with three branch values at 0, 1 and $\infty$, is called a **Belyi function**. The above construction leads to a well defined Belyi function up to equivalence, where two morphisms are considered equivalent if they are equal up to a composition with a (holomorphic) automorphism of the source surface ($T$). See [GGD12, Sec 4.2]. Note that by construction the reciprocal image of $[0, 1] \subset \widehat{\mathbb{C}}$ by the Belyi function $h \circ f$ is the graph of the hypermap as discussed in Section 6.2.2.

### 6.3.2 From Belyi functions to algebraic curves defined over $\overline{\mathbb{Q}}$

In order to define an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, one first needs to establish a correspondence between Riemann surfaces and complex algebraic curves.

**From complex algebraic curves to Riemann surfaces.** Given an irreducible polynomial $F \in \mathbb{C}[X, Y]$, we may look at the set $\{F = 0\}$. This complex curve can be given a holomorphic structure thanks to the following theorem [GGD12, th. 1.86].

**Theorem 6.2.** *Let $F \in \mathbb{C}[X, Y]$ be irreducible with $F(X, Y) = q_0(Y)X^m + \ldots + q_m(Y) = p_0(X)Y^n + \ldots p_n(X)$, assuming $n, m > 0$. Let*

$$\begin{aligned} S_X &= \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0,\ \frac{\partial F}{\partial Y}(x, y) \neq 0,\ p_0(x) \neq 0\} \\ S_Y &= \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0,\ \frac{\partial F}{\partial X}(x, y) \neq 0,\ q_0(y) \neq 0\} \end{aligned}$$

1. *$S_X$ and $S_Y$ are connected Riemann surfaces on which the coordinates* $\mathbf{x}, \mathbf{y}$ *are holomorphic functions.*
2. *There exists a unique compact Riemann surface $S_F$ containing $S_X \cup S_Y$ and* $\mathbf{x}, \mathbf{y}$ *extend to meromorphic functions.*
3. *The ramification points of* $\mathbf{x}$ *and* $\mathbf{y}$ *are in $S_F \setminus S_X$ and $S_F \setminus S_Y$, respectively.*

**From Riemann surfaces to complex algebraic curves.** Given a compact Riemann surface $S$, we consider the associated **field of meromorphic functions** $\mathscr{M}(S)$, which is the field of (holomorphic) morphisms $S \to \widehat{\mathbb{C}}$ for the law of function multiplication. Choose a non-constant function $f \in \mathscr{M}(S)$. Note that $f$ taking infinitely many values, for every polynomial $p \in \mathbb{C}[X]$ the function $p \circ f$ can not be identically zero. In other words, the field extension $\mathbb{C} \subset \mathbb{C}(f)$ is transcendental (here, $\mathbb{C}$ is identified with the field of constant functions).

**Proposition 6.3.** *Let $f \in \mathcal{M}(S)$ have degree $n$. Then, the field extension $\mathbb{C}(f) \subset \mathcal{M}(S)$ has degree $n$.*

PROOF SKETCH. The idea of the proof is to show that every $h \in \mathcal{M}(S)$ is the solution of a polynomial of degree at most $n$ with coefficients in $\mathbb{C}(f)$. For this, let $x \in \widehat{\mathbb{C}}$ and consider the points $y_i(x)$ above $x$ with respect to the ramified covering $f : S \to \widehat{\mathbb{C}}$, *i.e.* $f(y_i(x)) = x$. The polynomial $P_x$ with roots $h(y_i(x))$, counted with multiplicities, has degree $n$ and its coefficients are symmetric functions in the $h(y_i(x))$. It can be shown that those functions are meromorphic on $\mathbb{C}$. Since the meromorphic functions on $\mathbb{C}$ are the rational functions, so are the coefficients of $P_x$. Moreover, choosing $x = f(y)$ we get that $y$ is one of the $y_i(x)$, so that $P_{f(y)}(h(y)) = 0$. This shows that $h$ is algebraic of degree at most $n$ in $\mathbb{C}(f)$. □

By the theorem of the primitive element the algebraic extension $\mathbb{C}(f) \subset \mathcal{M}(S)$ has the form $\mathcal{M}(S) = \mathbb{C}(f)[h] = \mathbb{C}(f, h)$ for some $h \in \mathcal{M}(S)$. Now, let $F \in \mathbb{C}[X, Y]$ be an irreducible polynomial such that $F(f, h) = 0$. Then,

$$\begin{array}{rcl} S & \xrightarrow{\Phi} & S_F \\ p & \mapsto & (f(p), h(p)) \end{array}$$

is a well defined isomorphism. The proof goes by showing that the restriction of $\Phi$ to $(S_F)_X$ is a degree one covering and then applying the extension of morphisms.

**Corollary 6.4.** *With the above notations and those of Theorem 6.2, $f \mapsto \mathbf{x}$ and $h \mapsto \mathbf{y}$ defines a $\mathbb{C}$-algebra isomorphism from $\mathcal{M}(S)$ to $\mathcal{M}(S_F) = \mathbb{C}(\mathbf{x}, \mathbf{y})$.*

**Belyi's theorem.** A Riemann surface $S$ is **defined over a field** $K$ if $S$ is isomorphic to $S_F$ for an irreducible polynomial $F(X, Y) = \sum a_{ij} X^i Y^j$ with coefficients $a_{ij}$ in $K$. The last step to make $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ act on hypermaps is to prove that the source surface of a belyi function is defined over $\overline{\mathbb{Q}}$.

**Theorem 6.5** (Belyi, 1979)**.** *A compact Riemann surface $S$ is defined over $\bar{\mathbb{Q}}$ if and only if $\mathcal{M}(S)$ contains a Belyi function, i.e. a morphism $S \to \widehat{\mathbb{C}}$ with at most 3 branch values at $0, 1$ and $\infty$.*

Note that the existence of a meromorphic function with less than three branch values implies that $S$ is isomorphic to $\widehat{\mathbb{C}}$ (study the different induced coverings of the sphere minus 0, 1 or 2 points).

PROOF SKETCH OF THE DIRECT IMPLICATION OF THE THEOREM. By assumption, $S$ is isomorphic to $S_F$ with $F \in \overline{\mathbb{Q}}[X, Y]$ irreducible. Consider the coordinate function $\mathbf{x}$ on $S_F$. We have $\mathbf{x} \in \mathcal{M}(S_F)$ by point (2) of Theorem 6.2. Denote by $BV(\mathbf{x})$ the set of branch values of $\mathbf{x}$. Writing $F(X, Y) = p_0(X)Y^n + p_1(X)Y^{n-1} + \cdots + p_n(X)$, we have from point (3) in Theorem 6.2 that the elements of $BV(\mathbf{x})$ are either roots of $p_0$, or the first coordinate of a solution to the system $\{F = 0 \text{ and } \frac{\partial F}{\partial Y} = 0\}$, or the point $\infty \in \widehat{\mathbb{C}}$. Since the coefficients of $F$ are in the algebraically closed field $\overline{\mathbb{Q}}$, these branch values must be in $\overline{\mathbb{Q}} \cup \{\infty\}$.

First suppose that $BV(\mathbf{x}) \subset \mathbb{Q} \cup \{\infty\}$. Since PSL(2, $\mathbb{C}$) acts (as Möbius transformations) transitively on triplets of distinct points in $\widehat{\mathbb{C}}$ we may assume, composing with a Möbius transformation with rational coefficients, that $BV(\mathbf{x})$ contains $\{0,1,\infty\}$. If $BV(\mathbf{x})$ does not contain any other branch value, then we are done. Otherwise, let $\lambda$ be such a value. Composing if necessary with the Möbius transformations $z \mapsto 1-z$ or $z \mapsto 1/z$ we may further assume that $\lambda = m/(m+n)$ for some positive integers $m, n$. Consider the Belyi polynomial ($\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$)

$$P_{m,n}(X) = \frac{(m+n)^{m+n}}{m^m n^n} X^m (1-X)^n$$

From Theorem 6.2, $P_{m,n}$ ramifies at $0, 1, \infty$ and $\lambda$ while its branch values are $P_{m,n}(0) = 0, P_{m,n}(1) = 0, P_{m,n}(\infty) = \infty$ and $P_{m,n}(\lambda) = 1$. It is not hard to see that

$$BV(P_{m,n} \circ \mathbf{x}) = BV(P_{m,n}) \cup P_{m,n}(BV(\mathbf{x})) \tag{5}$$

It follows that $P_{m,n} \circ \mathbf{x} : S_F \to \widehat{\mathbb{C}}$ ramifies over one value less than $\mathbf{x}$. By induction on the number of branch values, we deduce that $\mathscr{M}(S_F)$ contains a Belyi function.

It remains to deal with the case where $BV(\mathbf{x})$ contains values in $\overline{\mathbb{Q}} \setminus \mathbb{Q}$. We consider the non-rational values in $BV(\mathbf{x})$ and their minimal polynomial $m_1(X)$: This is the product of the minimal polynomials of the non-rational values without repetition of factors. Using the same argument as for (5), we have

$$BV(m_1 \circ \mathbf{x}) = m_1(\{m_1' = 0\}) \cup \{0, \infty\}$$

Let $m_2$ be the minimal polynomial of $m_1(\{m_1' = 0\})$. We can show that $m_2$ has a degree strictly less that $m_1$. We then consider $m_2 \circ m_1 \circ \mathbf{x}$ and continue until all the branch values are rational. This must happen since $\deg m_i$ is decreasing. We are thus brought back to the case $BV(\mathbf{x}) \subset \mathbb{Q} \cup \{\infty\}$. □

### 6.3.3 The action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on hypermaps

We are now ready to describe how the absolute Galois group acts on hypermaps. Given a hypermap $M$, we saw in Section 6.3.1 how to construct a canonical Belyi function $f_M : S(M) \to \mathbb{C}$ with $S(M)$ a Riemann surface. By Belyi's theorem 6.5, $S(M)$ is isomorphic to some $S_F$ for an irreducible polynomial $F(X,Y) = \sum a_{ij} X^i Y^j$ with coefficients $a_{ij}$ in $\overline{\mathbb{Q}}$. This isomorphism transforms $f_M$ to a Belyi function ($S_F \to \widehat{\mathbb{C}}$) $\in \mathscr{M}(S_F)$ that we still denote by $f_M$. By Corollary 6.4, $f_M$ is a rational function in $\mathbf{x}, \mathbf{y}$. Applying an isomorphism if necessary, it can be proved that the coefficients of this rational function are also in $\overline{\mathbb{Q}}$.

Let $\sigma \in$ Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) be an automorphism of $\overline{\mathbb{Q}}$. Applying $\sigma$ to the coefficients of $F$ we get a new polynomial $\sigma(F)$ defined over $\overline{\mathbb{Q}}$. Similarly, applying $\sigma$ to the coefficients of the rational function $f_M$ gives a new Belyi function $\sigma(f_M) : S_{\sigma(F)} \to \widehat{\mathbb{C}}$. The reciprocal image of segment $[0,1]$ by $\sigma(f_M)$ viewed as graph cellularly embedded in $S_{\sigma(F)}$ gives a hypermap $\sigma(M)$ whose set of black (resp. white) vertices is the fiber above 0 (resp. 1). This is how Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) acts on hypermaps. It can be proved that this action preserves

- the number and degrees of the white vertices, black vertices and faces,
- the number of edges,
- the genus,
- the monodromy group,
- the automorphism group.

This action is moreover faithfull, meaning that the corresponding morphism from $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to the group of permutations of hypermaps is a monomorphism. This obviously gives more ground to the idea that studying this action would allow to understand the structure of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in more details. In fact, since the genus is preserved, this action leaves globally invariant the hypermaps of fixed genus. It can be proved that the restriction of this action to such hypermaps is also faithfull. Also, since the automorphism group is preserved, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ leaves invariant the regular hypermaps, those for which the automorphism group acts transitively on the edges. It was shown recently [GDJZ15] that the restriction to regular hypermaps also leads to a faithfull action.

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