Graphs and Coverings

Francis Lazarus

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Groups. . .

Definition

Definition A group is a set with a binary operation such that

- the order of successive operations does not matter (in time, *not* in space),
- there is a *unit*,
- every element has an inverse.

Example

The permutation groups

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Groups and morphisms constitute a category

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Subgroups

- The operation of a group G induces a group structure on the left cosets $\{gH\}_{g\in G}$ of $H < G$ iff H is a normal subgroup. Then, $p : G \twoheadrightarrow G/H$, $g \mapsto gH$ and $\ker p = H$.
- Conversally, the kernel of a morphism $f : G \to J$ is normal and $G/\ker f \simeq Imf$.

Example

The (derived) subgroup $[G, G]$ of commutators of G. It is the smallest subgroup D such that G/D is commutative.
$\forall f : G \to H$ with H commutative, $\exists! \bar{f}$:

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr] & & H \\ & G/[G,G] \arrow[ur, "\bar{f}"] & \end{tikzcd}$$

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Categories

Eilenberg - Mac Lane, 1945

Definition

A category consists of

- a class of **objects**,
- for any two objects a, b, a set $\mathbf{Hom}(\mathbf{a}, \mathbf{b})$ of morphisms with an obvious associative law of composition, such that $\mathbf{Hom}(\mathbf{a}, \mathbf{a})$ contains an identity element.

Example

- **Grp**,
- any group G with a single object a and $Hom(a, a) = G$,
- any preordered set with $|Hom(a, b)| = 1 \Leftrightarrow a \leq b$,
- any oriented graph with $Hom(a, b) = \{$ oriented $a \to b$ paths $\}$.

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Functors

Definition

A functor F between two categories C and D consists of

- A map $Objects(C) \to Objects(D)$,
- maps $Hom(a, b) \to Hom(F(a), F(b))$ that preserve identities and the composition laws.

Example

- by forgetting the group structure, we get: **Grp**$\to$**Set**,
- a group morphism $f : G \to H$ induces a functor between the corresponding categories,
- Algebraic topology is mainly concerned with **Top**$\to$**Grp** and **Top**$\to$**Ab**.

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Free Groups

Definition

The free group $F(S)$ on a set S is defined by the *universal property* : $\forall f : S \to G, \exists!\, \varphi$:

$$\begin{tikzcd} S \arrow[rr, "\iota"] \arrow[dr, "f"'] & & F(S) \arrow[dl, "\varphi"] \\ & G & \end{tikzcd}$$

$F(S)$ can be realized as the set of **freely reduced words** in S :
$F(S) = \{s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \mid s_i \in S,\ s_{i+1}^{\varepsilon_{i+1}} \neq s_i^{-\varepsilon_i}\}$.

Example

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Group Presentations

Definition

For a set S and a set $R \subset F(S)$ of relators, the groups with presentation $\langle S\,;\, R\rangle$ is the quotient $F(S)/N$ where N is the *normal closure* or R in $F(S)$.

$\langle S\,;\, R\rangle = (S \cup S^{-1})^*/\sim$ with
$uv \sim uss^{-1}v \sim urv, \forall s \in S \cup S^{-1}, \forall r \in R$.

Example

- $F(S) = \langle S\,;\, -\rangle$,
- For any group, $G = \langle G\,;\, xyz^{-1}, z = xy\rangle$
- $\langle \{s\}\,;\, s^n\rangle \simeq \mathbb{Z}/n\mathbb{Z}$,
- $\langle S\,;\, \{[s,t]\}_{s,t \in S}\rangle \simeq \mathbb{Z}^n$.

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What is a Graph?

Basic definitions and notation

Formally, a *general graph* Γ consists of three things: a set $V\Gamma$, a set $E\Gamma$, and an incidence relation, that is, a subset of $V\Gamma \times E\Gamma$. An element of $V\Gamma$ is called a *vertex*, an element of $E\Gamma$ is called an *edge*, and the incidence relation is required to be such that an edge is incident with either one vertex (in which case it is a *loop*) or two vertices. If every

Algebraic Graph Theory, Biggs, 1974/1993

1.1 Graphs

A *graph* X consists of a *vertex* set $V(X)$ and an *edge* set $E(X)$, where an edge is an unordered pair of distinct vertices of X. We will usually use xy rather than $\{x, y\}$ to denote an edge. If xy is an edge, then we say that

Algebraic Graph Theory, Godsil and Royle, 2001

What *Really* is a Graph

Definition

A graph is a quadruple $(V, A, o, ^{-1})$, with $o : A \to V$ and $^{-1}$ is a fixed-point free involution of A.

Trees, Serre, 1977 (translated by Stillwell)

Graph Morphisms

1.4 Homomorphisms

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is a *homomorphism* if $f(x)$ and $f(y)$ are adjacent in Y whenever x and y are adjacent in X. (When X and Y have no loops, which is our usual case, this definition implies that if $x \sim y$, then $f(x) \neq f(y)$.)

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"There is a evident notion of morphisms for graphs",
Trees, Serre.

Definition I

A morphism $(V, A, o, ^{-1}) \to (W, B, o, ^{-1})$ is given by $f : V \to W$, $g : A \to B$ with $o \circ g = f \circ o$ and $g \circ ^{-1} = ^{-1} \circ g$.

A non-loop edge contraction is not a morphism for Definition I.

Definition II

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Homotopy

A loop with basepoint v in $G = (V, A, o, ^{-1})$ is a sequence of arcs $(a_1, \ldots, a_n)$ with
$o(a_1) = o(a_n^{-1}) = v$ and $o(a_i) = o(a_{i+1}^{-1})$.

Definition

We say that $(a_1, \ldots, a, a^{-1}, \ldots, a_n)$ and $(a_1, \ldots, a_n)$ are elementarily homotopic.
Homotopy is the transitive closure of elementary homotopies.

Lemma

The set of homotopy classes with basepoint v is a group for the concatenation of paths. It is denoted $\pi_1(G, v)$.

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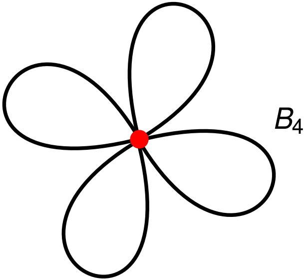
Homotopy

Lemma

For a bouquet B_n of n cycles: $\pi_1(B_n) \simeq F(n)$

PROOF. Paths in B_n are words in $A(B_n)$. Two paths are homotopic iff they freely reduce to the same word. So, $\pi_1(B_n) \simeq F\left(A_+\left(B_n\right)\right)$. $\square$



The Homotopy Functor

A morphism $f : (G, v) \to (H, w)$ induces a group morphism $f_* : \pi_1(G, v) \to \pi_1(H, w)$ using

$$(a_1, \dots, a_n) \mapsto (f(a_1), \dots, f(a_n))$$

Example

A non-loop edge contraction induces a group isomorphism: $\beta \sim \beta'$ with $\beta = f(\alpha)$ and$\beta' = f(\alpha') \implies \alpha \sim \alpha'$.

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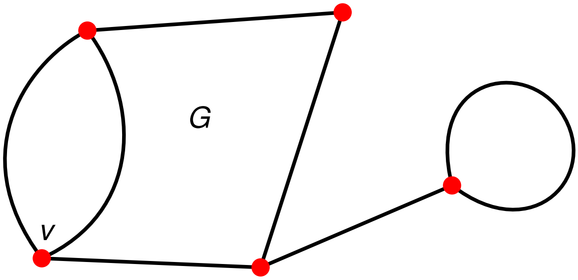
π_1 of Graphs

Let T be a spanning tree of a graph G with basepoint v.



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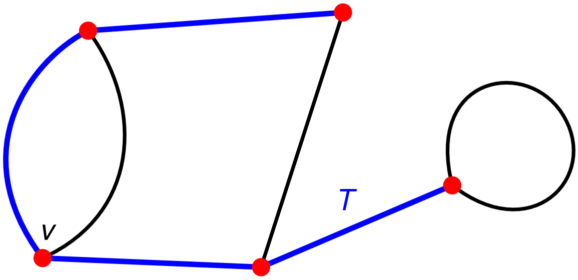

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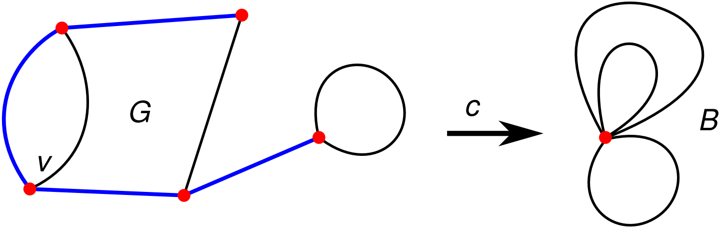
The contraction $c : G \to B$ induces an isomorphism.



$\implies \pi_1(G, v) \simeq F(A_+(B))$

The edges of B are the chords of T in G.

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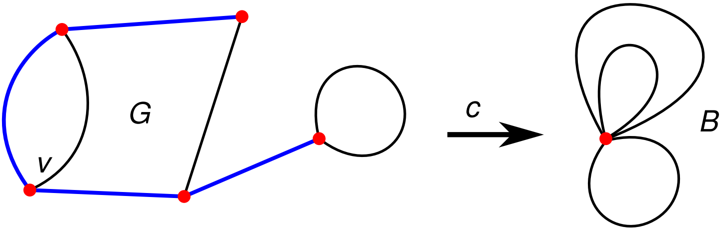

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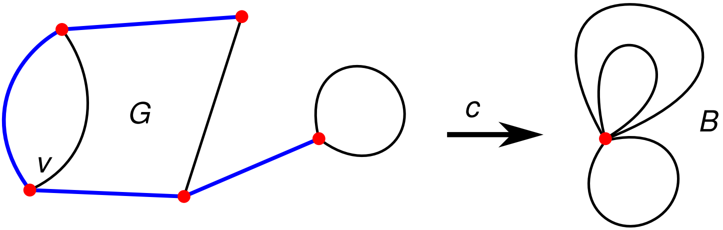

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π_1 of Graphs

Theorem

If G is connected,

$$\pi_1(G, v) \simeq \langle A_+(G \setminus T)\,;\, -\rangle$$

$$\text{rank } \pi_1(G, v) = |A_+(G \setminus T)| = \frac{|A|}{2} - |V| + 1$$

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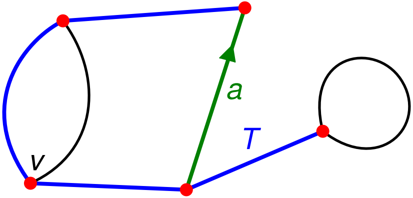

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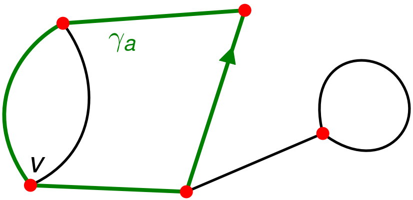
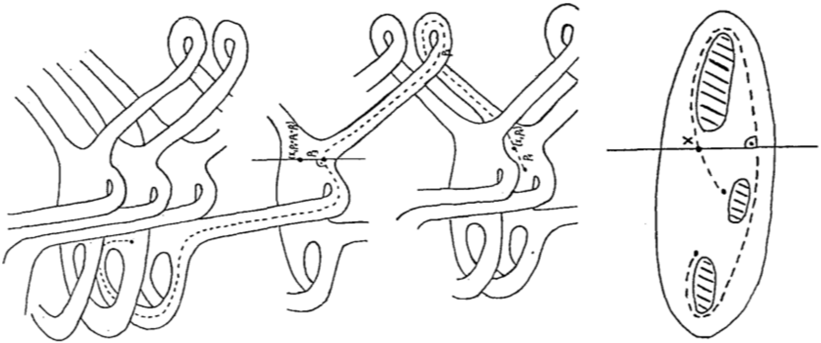


$$\pi(\gamma_a) = a \quad (in\, B) \implies \pi_1(G, v) = \langle \{\gamma_a\}_{a \in A_+(G\setminus T)} \,;\, - \rangle$$

Graph Coverings



Gao et al., 1998

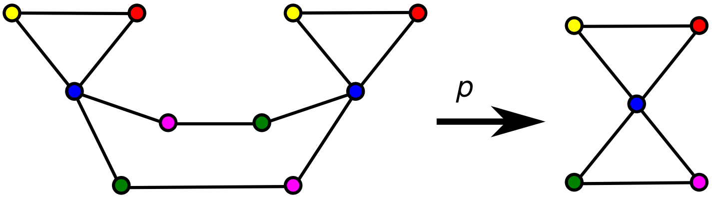
Covering

Let G be a graph. For $v \in V(G)$, let

$$Star(v) := \{a \in A(G) \mid o(a) = v\}.$$

Definition

- A graph (epi)morphism $p : H \to G$ is a covering if the restriction $p : Star(w) \to Star(p(w))$ is bijective for all $w \in V(H)$.
- G is the base of p. For $v \in V(G)$, the set $p^{-1}(v)$ is the fiber above v.



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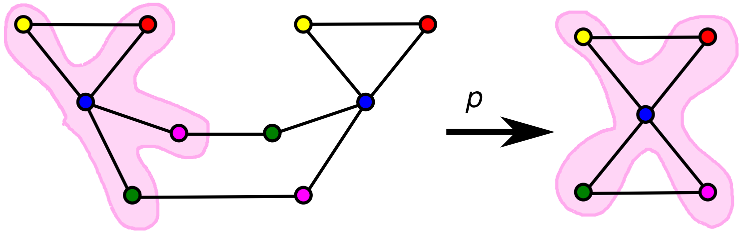


Unique Lift Property

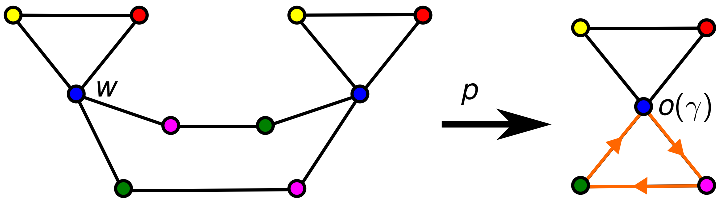
Let $p : H \to G$ be a covering and let γ be a path in G.

Definition

A path δ in H with $p(\delta) = \gamma$ is called a lift of γ.

Lemma

Let $w \in V(H)$ with $p(w) = o(\gamma)$. There exists a *unique* lift of γ with origin w.



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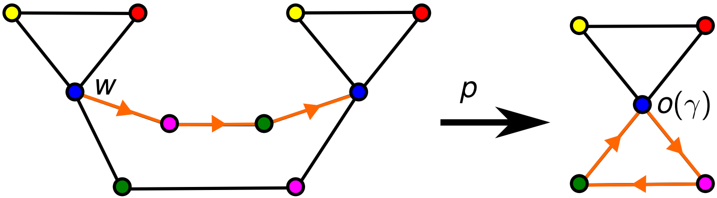

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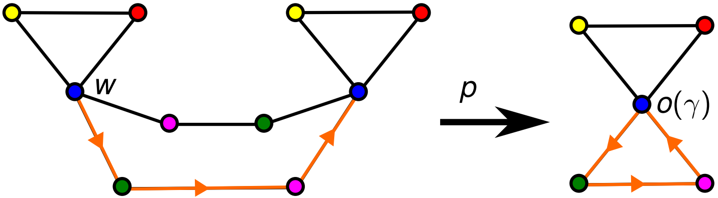

Lift of Homotopies

Let $p : H \to G$ be a covering.

Lemma

Let $\alpha \sim \beta$ be two homotopic paths in G. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be respective lifts with the same origin. Then $\tilde{\alpha} \sim \tilde{\beta}$.

PROOF. By induction on the number of elementary homotopies separating α and β. $\square$

Corollary

p_* is injective.

PROOF.

$$p_*[\alpha] = p_*[\beta] \Leftrightarrow p(\alpha) \sim p(\beta) \Longrightarrow \alpha \sim \beta \Longrightarrow [\alpha] = [\beta]. \quad \square$$

Lift of Homotopies: Application

Lemma

$F(\aleph_0) < F(n) < F(2)$

Remark: $\gamma \in p_*\pi_1(Flower_n) \Leftrightarrow |\gamma|_a \equiv 0 \mod n \implies p_*\pi_1(Flower_n) \lhd \pi_1(Flower_2)$, i.e., $F(n) \lhd F(2)$. What is $F(2)/F(n)$?

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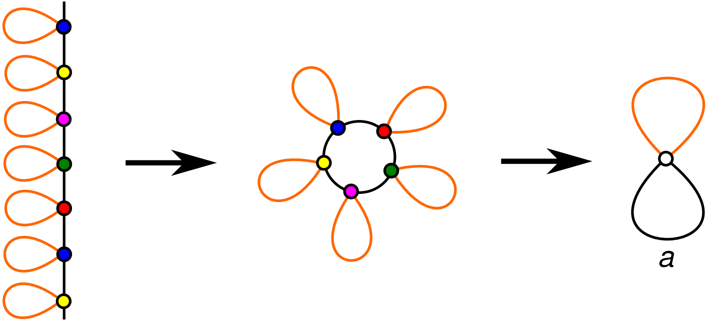

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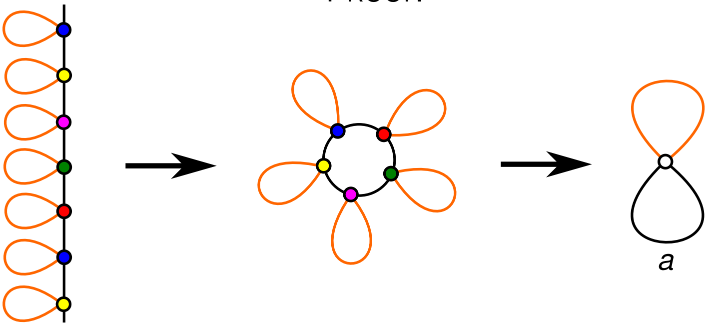

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Subgroups and Coverings

Proposition

Let G be a connected graph. For every subgroup $U < \pi_1(G, v)$, there exists a connected covering $p_U : (G_U, w) \to (G, v)$ with $p_{U*}\pi_1(G_U, w) = U$.

Let T be spanning tree of G and $\gamma_a = T[v, o(a)].a.T(o(a^{-1}), v]$. Define G_U, p_U by

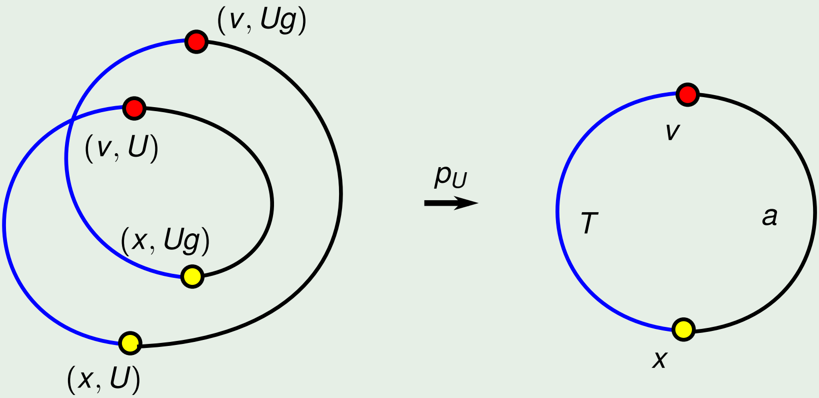
- $V(G_U) = V(G) \times \{Ug\}_{g \in \pi_1(G,x)}$, $A(G_U) = A(G) \times \{Ug\}_{g \in \pi_1(G,x)}$
- $o(a, Ug) = o(a)$ and $(a, Ug)^{-1} = (a^{-1}, Ug[\gamma_a])$,
- p_U is the proj. on first component.

$$(o(a), Ug)\bullet \underset{(a^{-1}, Ug[\gamma_a])}{\overset{(a, Ug)}{\rightleftarrows}} \bullet(o(a^{-1}), Ug[\gamma_a])$$

Subgroups and Coverings

Example

Put $g := [\gamma_a]$, so $\pi_1(G, v) =< g >$. Let $U =< g^2 >$.

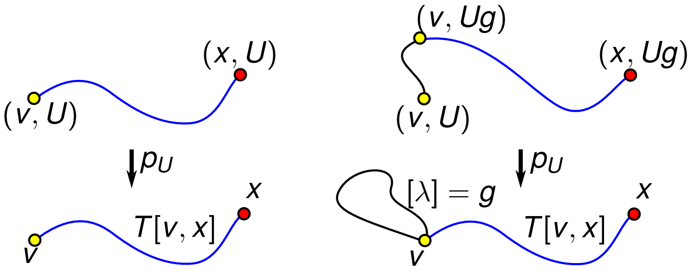


Subgroups and Coverings

- **p_U is a covering:** $Star(x, Ug) = Star(x) \times \{Ug\}$
- **G_U is connected:** For a path $\alpha = (a_1, \ldots, a_n)$, put $\gamma_\alpha = \gamma_{a_1} \cdots \gamma_{a_n}$. Observe that $(v, U).[\alpha] = (t(\alpha), U[\gamma_\alpha])$.
- **$p_{U*}\pi_1(G_U, (v, U)) = U$:**
  $[\lambda] \in \mathtt{Im} p_{U*} \Leftrightarrow (v, U).[\lambda] = (v, U) \Leftrightarrow U[\lambda] = U \Leftrightarrow [\lambda] \in U$

Subgroups and Coverings

- p_U **is a covering:** $Star(x, Ug) = Star(x) \times \{Ug\}$
- G_U **is connected:** For a path $\alpha = (a_1, \ldots, a_n)$, put $\gamma_\alpha = \gamma_{a_1} \cdots \gamma_{a_n}$. Observe that $(v, U).[\alpha] = (t(\alpha), U[\gamma_\alpha])$.



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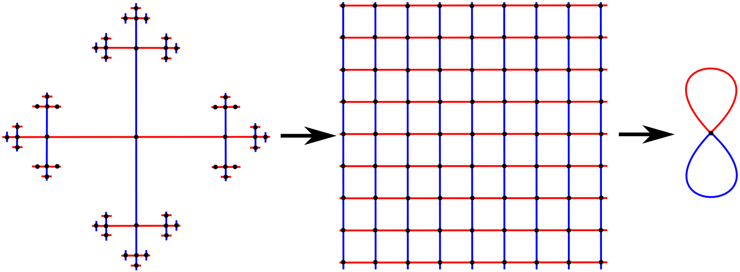
Subgroups and Coverings: Examples

Definition

When $U = \{1\}$, G_U is the universal cover.



we have
$\alpha \in p_*\pi_1(\mathbb{Z}^2 Grid) \Leftrightarrow |\alpha|_a = |\alpha|_b = 0 \Leftrightarrow \alpha \in [F(2), F(2)]$. So, for $G = B_2$, $G_{[G,G]} = \mathbb{Z}^2 Grid$.

Subgroups and Coverings: Application

Nielsen-Schreier theorem, mid 1920's

Every subgroup of a free group is free.

PROOF. Realize $F(S)$ as the π_1 of a bouquet of $|S|$ circles. A subgroup of $F(S)$ is the π_1 of a covering graph, which we know to be free. $\square$

Note: Another proof uses the fact that a group is free iff it acts freely on a tree (Bass-Serre). Any subgroup acts obviously freely on the same tree.

Covering Morphisms

Definition

A morphism f between coverings $p : H \to G$ and $q : K \to G$ sends fibers to fibers. It satisfies:

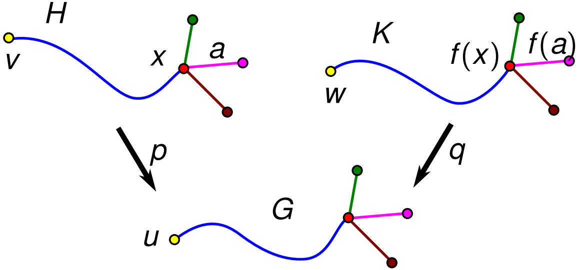
$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "p"'] & & K \arrow[dl, "q"] \\ & G & \end{tikzcd}$$

Covering Morphisms

Lemma

There is a morphism f between coverings $p : (H, v) \to (G, u)$ and $q : (K, w) \to (G, u)$ iff $p_*\pi_1(H, v) < q_*\pi_1(K, w)$ in $\pi_1(G, u)$.

PROOF. For $x \in V(H)$, $\gamma : v \rightsquigarrow x$, set $f(x) = w.p_*[\gamma]$.
If $\lambda : v \rightsquigarrow x$ then $w.p_*[\lambda] = w.p_*[\lambda.\gamma^{-1}].p_*[\gamma] = w.p_*[\gamma]$



- We have $q \circ f = p$: $q(f(x)) = q(w.p_*[\gamma]) = t(p(\gamma)) = p(x)$.
- We also check that f commutes with o and $^{-1}$.

Covering Morphisms

Lemma

There is a morphism f between coverings $p : (H, v) \to (G, u)$ and $q : (K, w) \to (G, u)$ iff $p_*\pi_1(H, v) < q_*\pi_1(K, w)$ in $\pi_1(G, u)$.

Corollary

There is an isomorphism f between coverings $p : H \to G$ and $q : K \to G$ iff $p_*\pi_1(H, v)$ and $q_*\pi_1(K, w)$ are in the same conjugacy class in $\pi_1(G, u)$ for $p(v) = q(w) = u$.

PROOF. $\Longrightarrow$: By the lemma we must have $p_*\pi_1(H, v) = q_*\pi_1(K, f(v))$ in $\pi_1(G, u)$.
$\Longleftarrow$: Suppose $p_*\pi_1(H, v) = [\gamma]^{-1}.q_*\pi_1(K, w).[\gamma]$. But $[\gamma]^{-1}.q_*\pi_1(K, w).[\gamma] = q_*\pi_1(K, w.[\gamma])$ and we can apply the lemma with $f(v) = w.[\gamma]$ $\square$

Lemma

Covering Morphisms

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Lemma

A morphism between coverings is a covering.

PROOF. Since restrictions of p and q to stars are one-to-one and since $q \circ f = p$ it must be the case for f. $\square$

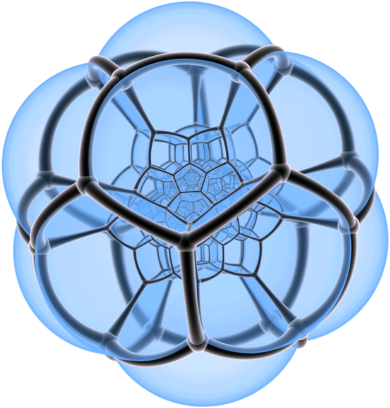
The Set of Coverings

Theorem

The set of coverings of a graph G, *up to isomorphism*, corresponds to the set of conjugacy classes of subgroups of $\pi_1(G)$ with the preorder relation $H \geq K$ if $\exists g \in \pi_1(G)$ with $g^{-1}Hg \subset K$.

The universal covering is the maximal element.

Actions and Quotient Graphs



Jenn3D, F. Obermeyer

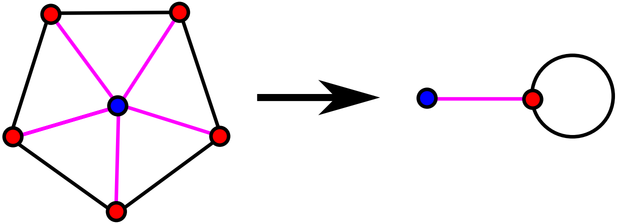
Quotient Graphs

Definition

Let $\Gamma < Aut(G)$ acts without (arc) inversion. The quotient graph G/Γ is given by

- $V(G/\Gamma) = \{\Gamma \cdot v\}_{v \in V(G)}$,
- $A(G/\Gamma) = \{\Gamma \cdot a\}_{a \in A(G)}$,
- $o(\Gamma \cdot a) = \Gamma \cdot o(a)$ and $(\Gamma \cdot a)^{-1} = \Gamma \cdot a^{-1}$

Note: Γ acts without inversion $\Leftrightarrow (\Gamma \cdot a)^{-1} \neq \Gamma \cdot a$



Free Actions

Definition

$\Gamma < Aut(G)$ acts freely if it acts without inversion and $g \in \Gamma \setminus \{1\}$ does not fix any vertex.

Proposition

If Γ acts without inversion then $p_\Gamma : G \to G/\Gamma$ is an epimorphism. It is a covering iff Γ acts freely on G.

PROOF. p_Γ restricted to Stars must be injective. Let $g \neq Id$ fix a vertex. Then $\exists a \in A(G) : a \neq g(a)$ and $o(a) = o(g(a))$.

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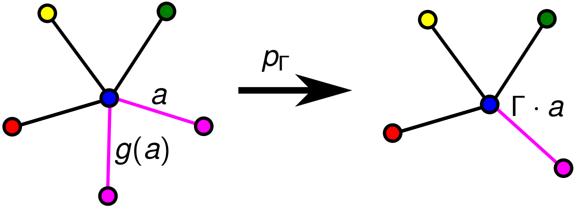
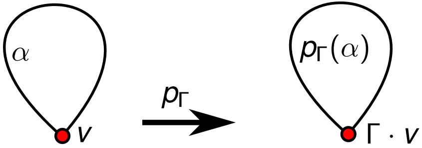


Free Actions

Lemma I

If Γ acts freely on G then $(p_\Gamma)_*\pi_1(G, v) \lhd \pi_1(G/\Gamma, \Gamma \cdot v)$

PROOF. $p_\Gamma(v.\beta) = p_\Gamma(v) \implies \exists g \in \Gamma : g(v) = v.\beta$. So, $v.(\beta p_\Gamma(\alpha)\beta^{-1}) = g(v).(p_\Gamma(\alpha)\beta^{-1}) = (g(v).p_\Gamma(\alpha))\beta^{-1} = g(v).\beta^{-1} = v$. $\square$



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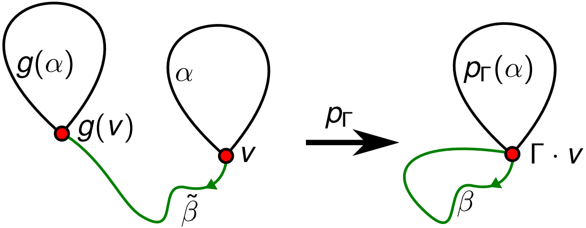

Action of the Covering Automorphisms

Definition

A p-(auto)morphism of a covering $p : H \to G$ satisfies:

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "p"'] & & H \arrow[dl, "p"] \\ & G & \end{tikzcd}$$

$Aut(p) :=$ set of p-automorphisms.

Lemma

$Aut(p)$ acts freely on H.

Let $f \in Aut(p)$. $f(a) = a^{-1} \implies p(a) = p(a^{-1}) = p(a)^{-1}$, contradiction.

$f(v) = v \implies \forall \alpha : v \leadsto x, f(x) = f(v).p(\alpha) = v.p(\alpha) = x$.

Quotients and Coverings

Lemma II

If Γ acts freely on G then $Aut(p_\Gamma) = \Gamma$.

PROOF. Obviously, $\Gamma \subset Aut(p_\Gamma)$ and Γ acts transitively on the fibers of p_Γ. Since $Aut(p_\Gamma)$ acts freely, $Aut(p_\Gamma) \subset \Gamma$. $\square$

Lemma III

If $p : (H, v) \to (G, u)$ is a covering with $p_*\pi_1(H, v) \lhd \pi_1(G, u)$ then $Aut(p)$ acts transitively on fibers.

PROOF. $p(w) = p(v) \implies p_*\pi_1(H, w) = p_*\pi_1(H, v)$.
We construct $f \in Aut(p)$ such that $f(v) = w$: If $\alpha : v \rightsquigarrow x$ set $f(x) = w.[p(\alpha)]$. If $\beta : v \rightsquigarrow x$ then
$w.[p(\beta)] = w.[p(\beta\alpha^{-1})][p(\alpha)] = w.[p(\alpha)]$. $\square$

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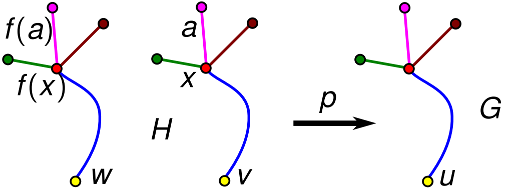

Quotients and Coverings

Proposition

Let $p : H \to G$. If $\Gamma < Aut(H)$ then
$$\begin{tikzcd} & H \arrow[dl, "p_\Gamma"'] \arrow[dr, "p"] & \\ H/\Gamma \arrow[rr, "\simeq"] & & G \end{tikzcd}$$
iff

1. $\Gamma = Aut(p)$
2. $p_*\pi_1(H, v) \lhd \pi_1(G, p(v))$

PROOF. $\Longrightarrow$:

1. p_Γ covering $\Longrightarrow$ Γ acts freely $\Longrightarrow$ $\Gamma = Aut(p_\Gamma) = Aut(p)$ by lemma II.
2. By lemma I, we also have ${p_\Gamma}_*\pi_1(H, v) \lhd \pi_1(H/\Gamma, p_\Gamma(v))$ whence $p_*\pi_1(H, v) \lhd \pi_1(G, p(v))$.

$\Longleftarrow$: In that case $Aut(p)$ acts transitively by Lemma III, so $H/Aut(p) \simeq G$. $\square$

Galois Coverings

Definition

A covering as above is said Galois or regular or normal.

Theorem

If $p : H \to G$ is a covering then
$Aut(p) \simeq N\left(p_*\pi_1(H, v)\right)/p_*\pi_1(H, v)$.
If p is Galois $Aut(p) \simeq \pi_1(G, p(v))/p_*\pi_1(H, v)$.

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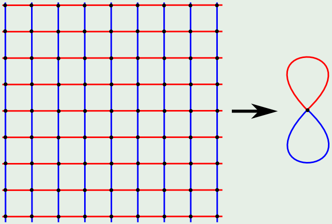
Example



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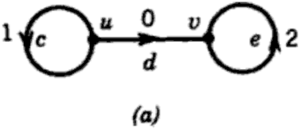
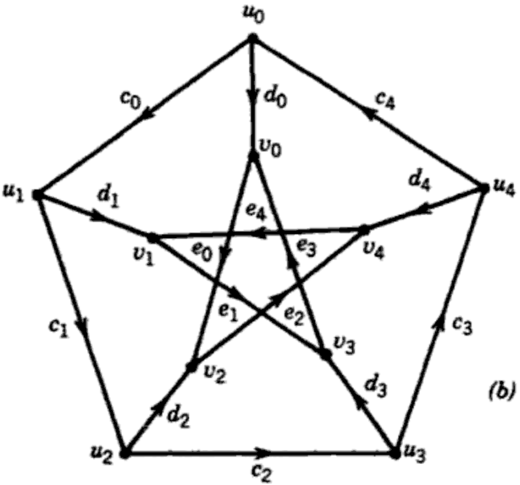
PROOF. $Aut(p) \stackrel{F}{\to} p^{-1}(v), f \mapsto f(v)$. Put $f_w := F^{-1}(w)$, i.e. $f_w(v) = w$. Let $\pi_1(G,p(v)) \stackrel{M}{\to} Aut(p), \alpha \mapsto f_{v.\alpha}$.
M is a morphism: $M(\alpha\beta)(v) = f_{v.\alpha\beta}(v) = v.\alpha\beta = (v.\alpha).\beta = f_{v.\alpha}(v).\beta = f_{v.\alpha}(v.\beta) = f_{v.\alpha} \circ f_{v.\beta}(v)$. So,
$M(\alpha\beta) = f_{v.\alpha} \circ f_{v.\beta} = M(\alpha) \circ M(\beta)$.
$\ker M = \{\alpha \mid v.\alpha = v\} = p_*\pi_1(H,v)$. □

Voltage Graphs





Gross and Tucker, 1987

Voltage Graphs, Gross 1974

Definition

A voltage on a graph G with values in group B is a map $\kappa : A(G) \to B$ with

$$\kappa(a^{-1}) = \kappa(a)^{-1}, \quad \forall a \in A(G)$$

If B acts on the right on the set S, the voltage induces a covering $p_\kappa : G_\kappa \to G$ where

- $V(G_\kappa) = V(G) \times S$ and $A(G_\kappa) = A(G) \times S$, and
- $o(a, s) := (o(a), s)$ and $(a, s)^{-1} := (a^{-1}, s.\kappa(a))$

$$(o(a), s)\bullet \underset{(a^{-1}, s.\kappa(a))}{\overset{(a,s)}{\rightleftarrows}} \bullet(o(a^{-1}), s.\kappa(a))$$

Voltage Graphs and Coverings

Lemma

Every covering $p : H \to G$ is ($\simeq$ to) the covering induced by a voltage on G.

PROOF. Let T be a spanning tree of (G, v). Define $\kappa : A(G) \to \pi_1(G, v)$ by $\kappa(a) = \gamma_a = T[v, o(a)].a.T[o(a^{-1}), v]$ with $\pi_1(G, v)$ acting on the fiber $p^{-1}(v)$.

Check that
$$\begin{tikzcd} G_\kappa \arrow[rr, "\simeq"] \arrow[dr, "p_\kappa"'] & & H \arrow[dl, "p"] \\ & G & \end{tikzcd}$$
$\square$

Voltage Graphs and Coverings

Proposition

A voltage $\kappa : A(G) \to B$ with B acting on itself and $B = < \mathrm{Im}\kappa >$ induces a Galois covering.

PROOF. Note that for $\alpha \in \pi_1(G, v)$, $(v, 1_B).\alpha = (v, \kappa(\alpha))$, so that for the induced morphism $\kappa : \pi_1(G, v) \to B$ we have $\ker \kappa = p_{\kappa *}\pi_1(G_\kappa, (v, 1_B))$. $\square$

Proposition

Conversely, a Galois covering $p : H \to G$ is induced by a voltage $\kappa : A(G) \to B$ with B acting on itself.

PROOF. Let T be a spanning tree of G. For every $a \in A(G)$ there is a unique $f_a \in Aut(p)$ with $f_a(v) = v.\gamma_a$. We let $B = Aut(p)$ and $\kappa(a) = f_a$.

Voltage Graphs and Coverings

Proposition

A voltage $\kappa : A(G) \to B$ with B acting on itself and $B =< \mathtt{Im}\kappa >$ induces a Galois covering.

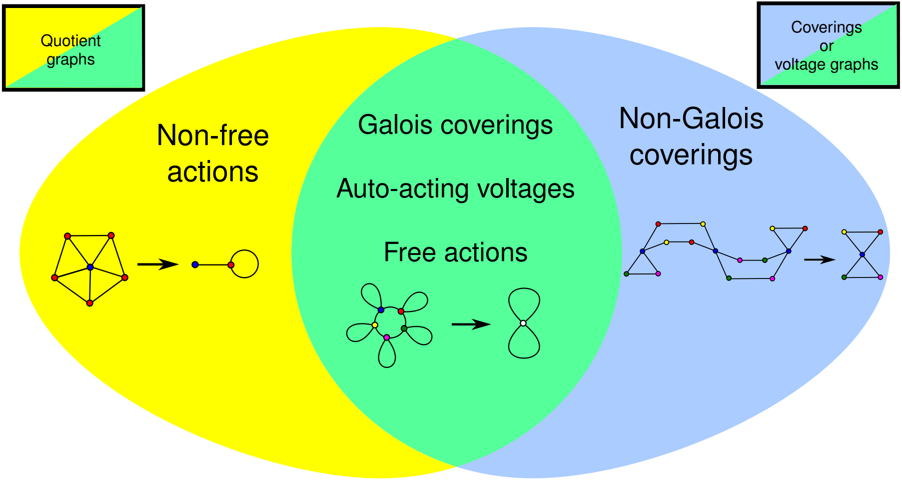
Proposition

Conversely, a Galois covering $p : H \to G$ is induced by a voltage $\kappa : A(G) \to B$ with B acting on itself.

PROOF. Let T be a spanning tree of G. For every $a \in A(G)$ there is a unique $f_a \in Aut(p)$ with $f_a(v) = v.\gamma_a$. We let $B = Aut(p)$ and $\kappa(a) = f_a$.

Check that
$$\begin{tikzcd} G_\kappa \arrow[rr, "\simeq"] \arrow[dr, "p_\kappa"'] & & H \arrow[dl, "p"] \\ & G & \end{tikzcd}$$
$\square$

Summary



Quotient graphs
Coverings or voltage graphs
Non-free actions
Galois coverings
Auto-acting voltages
Free actions
Non-Galois coverings

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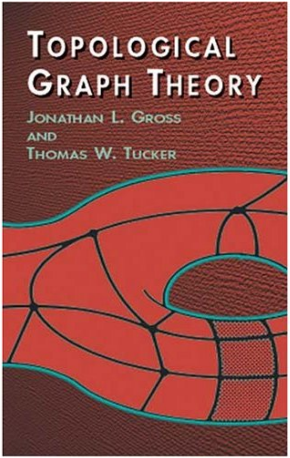

Gross and Tucker, 1987

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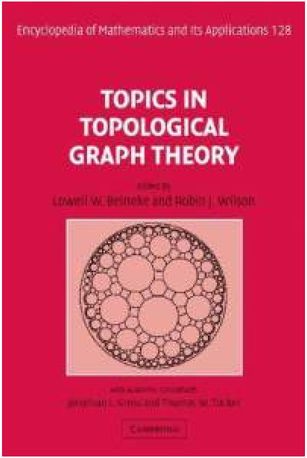

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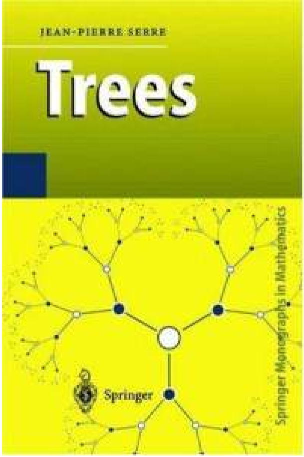

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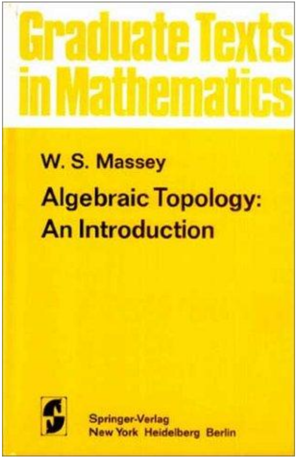

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THANK YOU!