The structure of quasi-transitive graphs avoiding a minor with applications to the Domino Conjecture.

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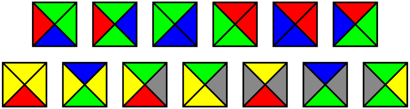
Wang tiling problem



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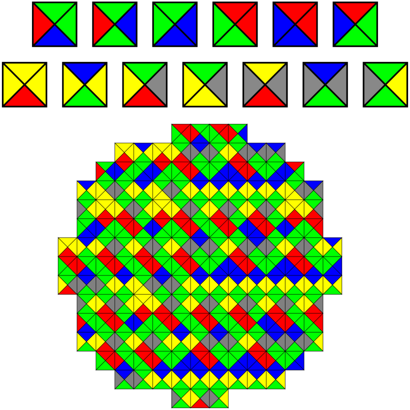

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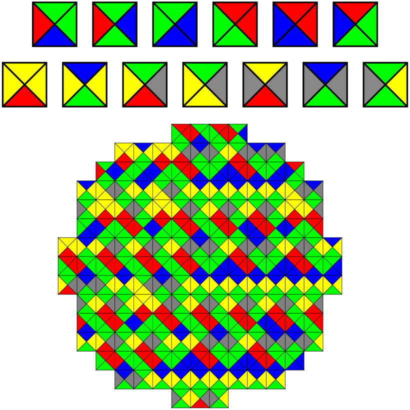

Theorem (Berger, '66)

The Wang tiling problem is undecidable.

Cayley graphs

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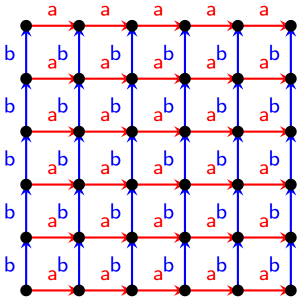
Cay$(\mathbb{Z}^2, S)$,
with $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$



Domino Problem on groups

Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.

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Domino problem on (Γ, S):

Input: a finite alphabet Σ and a finite set $\mathcal{F} = \{p_1, \dots, p_t\}$ of forbidden patterns.

Question: Is there a coloring $c : V(G) \to \Sigma$ avoiding $\mathcal{F}$?

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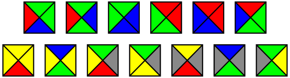

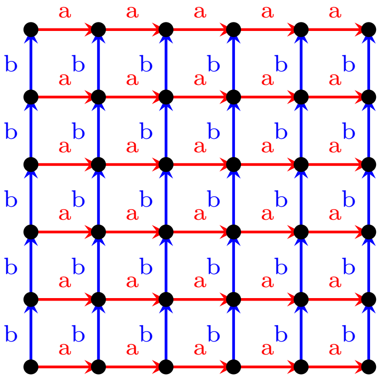

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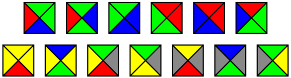

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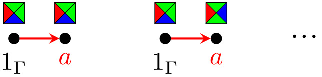
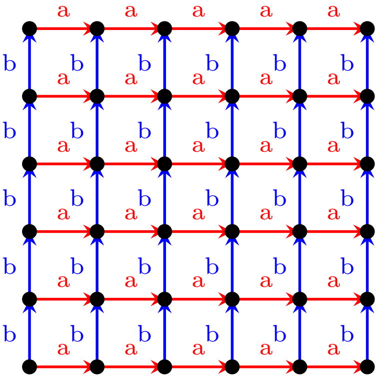
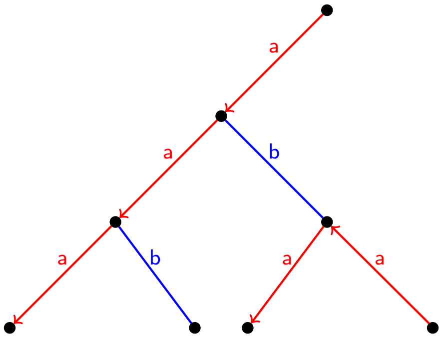




Virtually-free groups

Free-groups $\simeq$ groups Γ that admit a tree as a Cayley graph.



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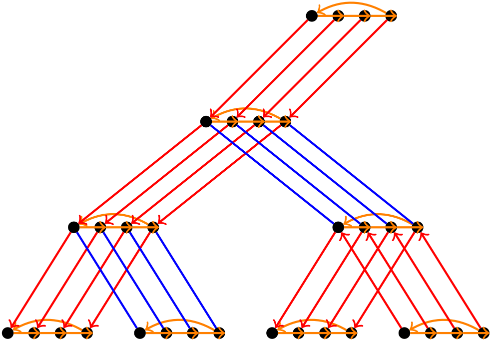

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Theorem (Karass, Pietrowski, Solitar '73)

Γ *is virtually-free if and only if one/all its Cayley graphs have bounded treewidth.*

Claim: If G has bounded degree, then G has bounded treewidth if and only if G is a subgraph of a k-blow up of a tree for some $k \geqslant 0$.

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Conjecture (Ballier-Stein 2018)

The domino problem on Γ is decidable if and only if Γ is virtually-free.

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Theorem

The conjecture is true for planar groups.

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Ray: 1-ended infinite path $r = (x_1, x_2, x_3, \ldots)$ in a graph G.

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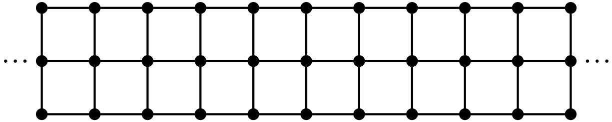

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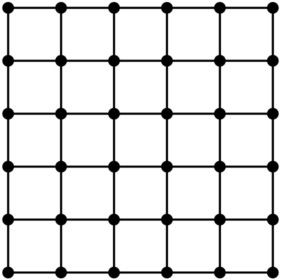

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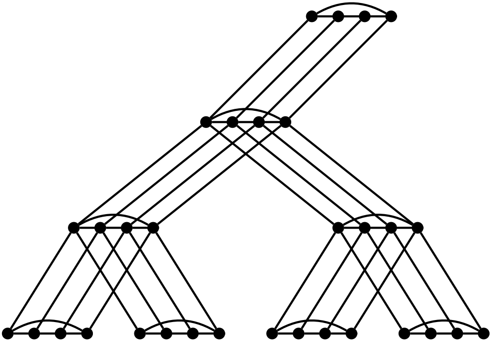

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Theorem (Freudenthal, '44)

A Cayley graph has either 0, 1, 2 *or infinitely many ends.*

One-ended planar groups

Theorem (Bundgaard-Nielsen '42, Fox '52)

If a group Γ is planar with one end, then it contains the fundamental group of a surface as a subgroup of finite index.

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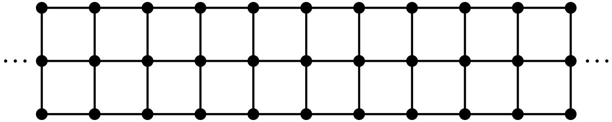

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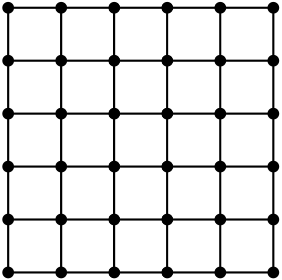

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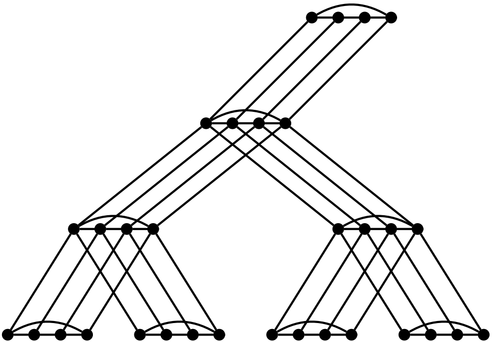

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Theorem (Bass-Serre theory)

If Γ is accessible, then

- *either Γ is virtually free*
- *or Γ contains a finitely generated subgroup with one end.*

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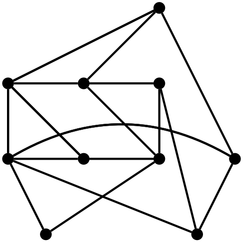

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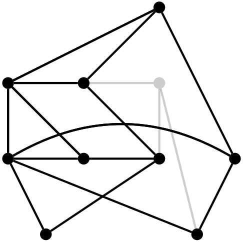

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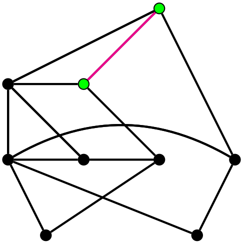

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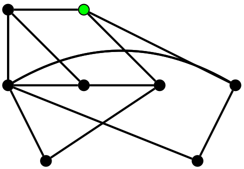

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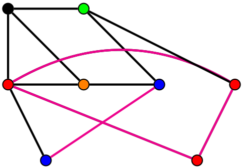

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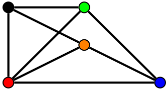

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Theorem

The Domino conjecture is true for planar groups and more generally for minor-excluding groups.

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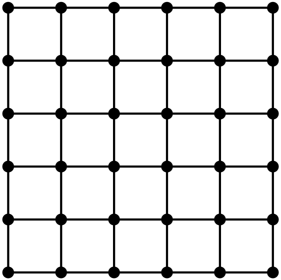

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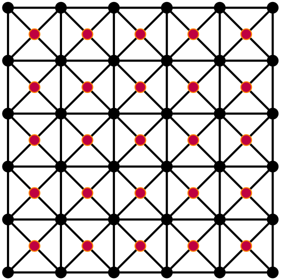

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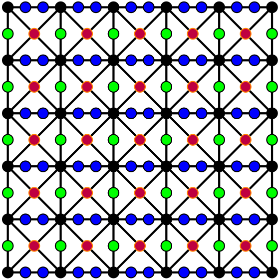
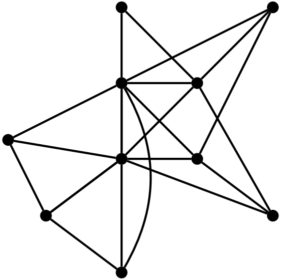


Canonical tree-decompositions



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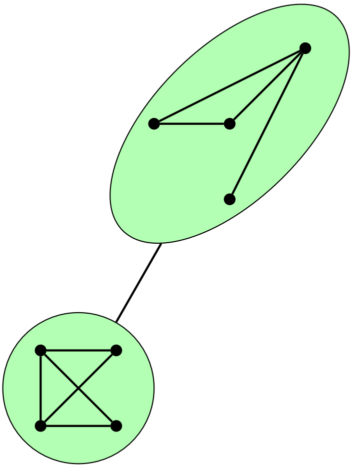

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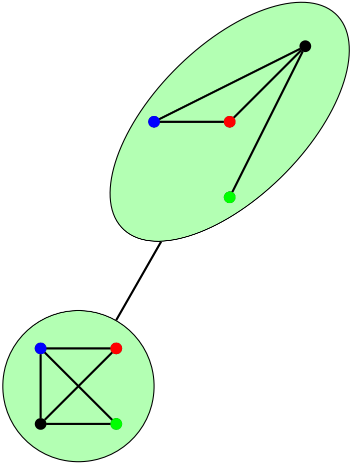

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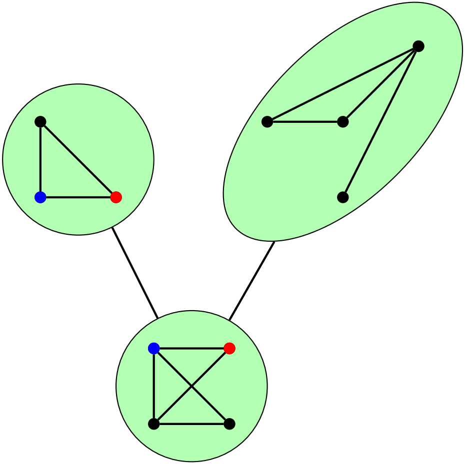

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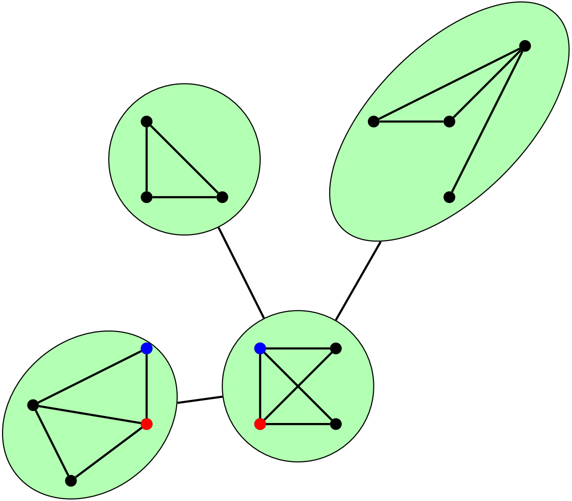

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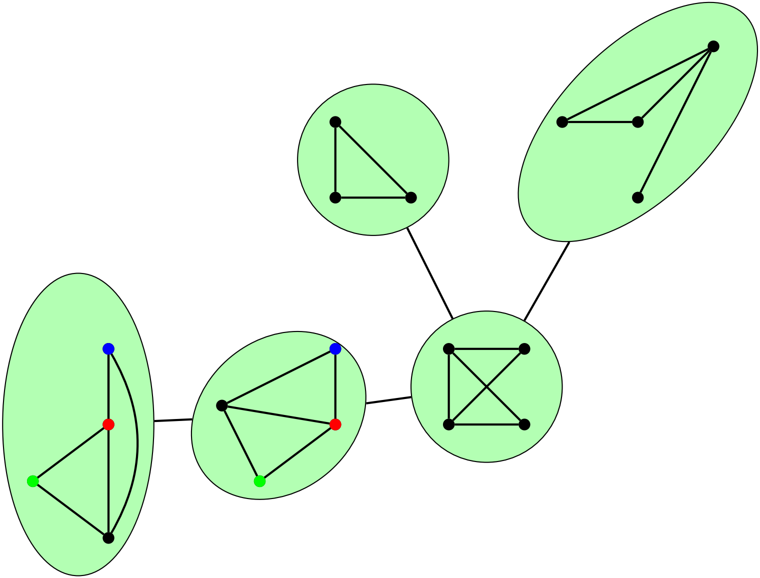

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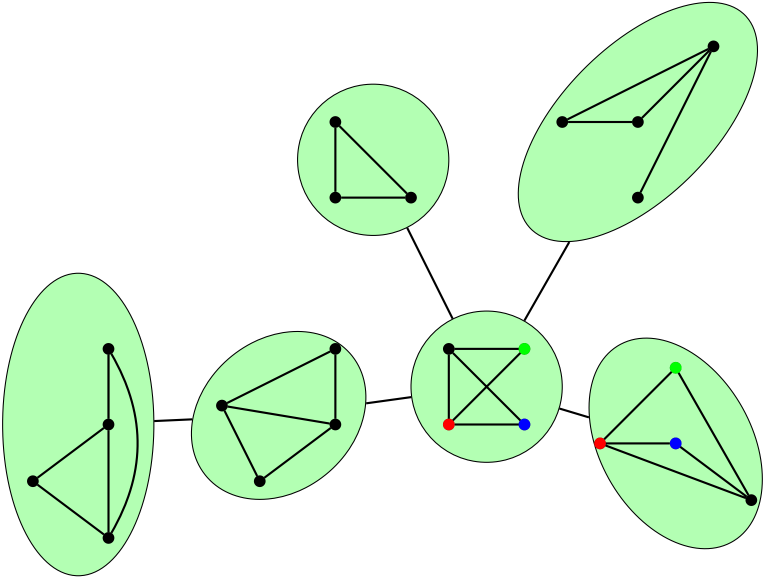

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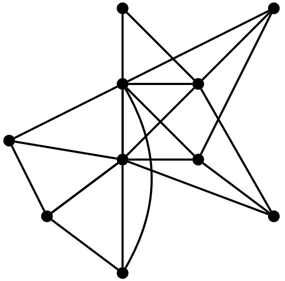
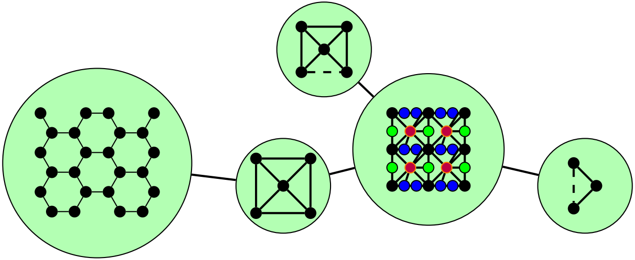


Main result

Theorem (finite/planar)

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G.



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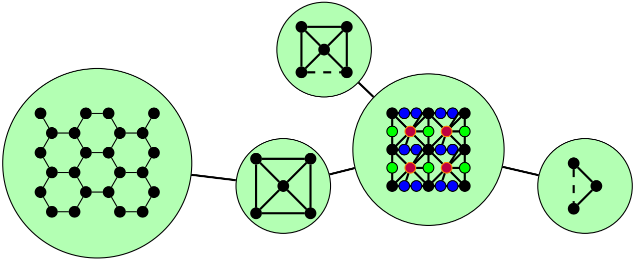

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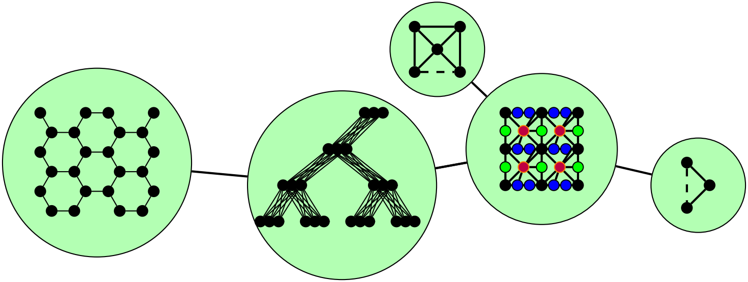

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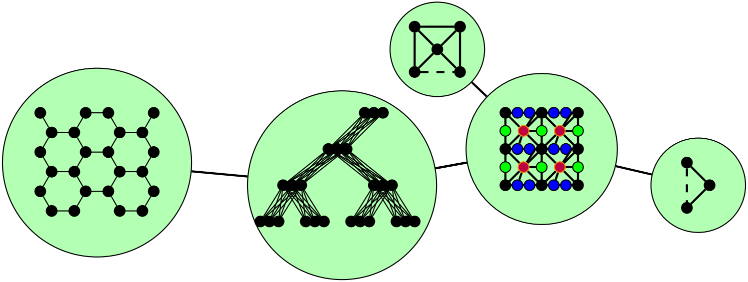

A graph application

Corollary

For every locally finite quasi-transitive graph G avoiding K_∞ as a minor, there is an integer k such that G is K_k-minor-free.

Generalizes [Thomassen '92] dealing with the 4-connected case.

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- Prove results on groups by working in the more general world of quasi-transitive graphs.
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Questions:

- A quasi-transitive graphical reformulation of Domino's conjecture?
- If G is quasi-transitive, is there a proper colouring of G with a finite number of colours such that the colored graph G is quasi-transitive?

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- Key tool: canonicity (allows to do induction in the context of tree-decompositions).

Questions:

- A quasi-transitive graphical reformulation of Domino's conjecture?
- If G is quasi-transitive, is there a proper colouring of G with a finite number of colours such that the colored graph G is quasi-transitive?

Thanks

Proof idea

G is $k+1$-connected if $|V| \geqslant k+1$ and for every set X of at most k vertices, $G \setminus X$ is connected.
G is quasi-4-connected if it is 3-connected and the only vertex-cuts of order 3 separate exactly 2 components, and one of them have size 1.

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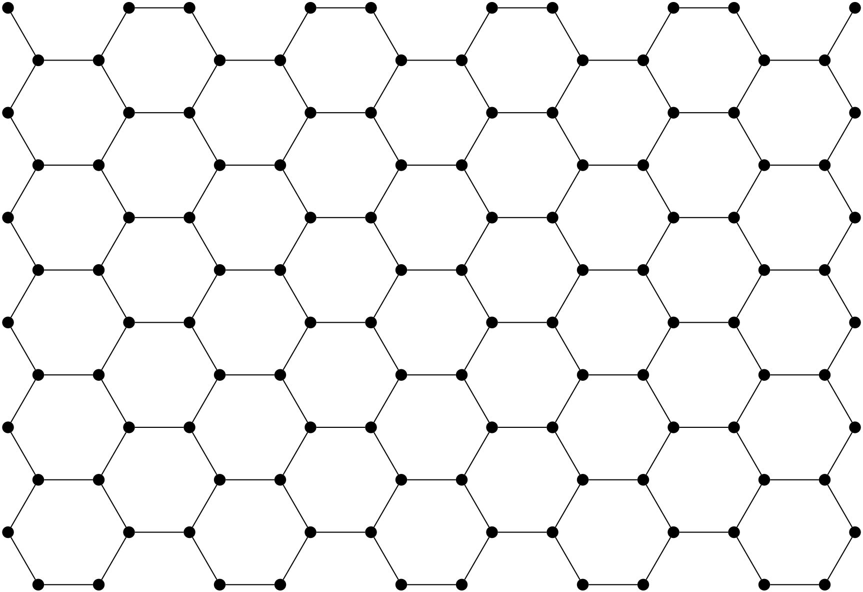
4-connected $\Rightarrow$ quasi-4-connected $\Rightarrow$ 3-connected $\Rightarrow$ 2-connected

Proof idea



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Try to combine the following two results:

Theorem (Thomassen '92)

Let G be a quasi-transitive, quasi-4-connected, locally finite graph which excludes K_∞ as a minor. Then G is planar or has finite treewidth.

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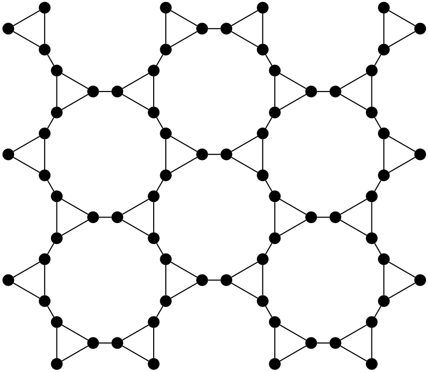
In this case there is nothing to decompose!

Grohe's decomposition

Theorem (Grohe '16)

Every finite 3-connected graph G has a tree-decomposition of adhesion at most 3 whose torsos are minor of G and are complete graphs on at most 4 vertices or quasi-4-connected graphs.

Grohe's decomposition



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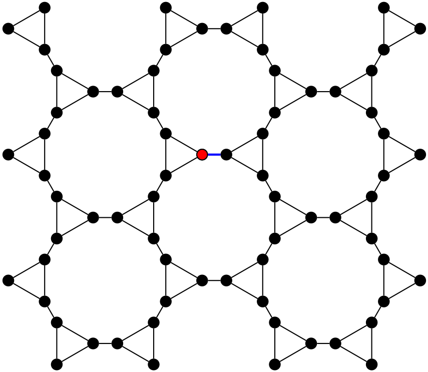

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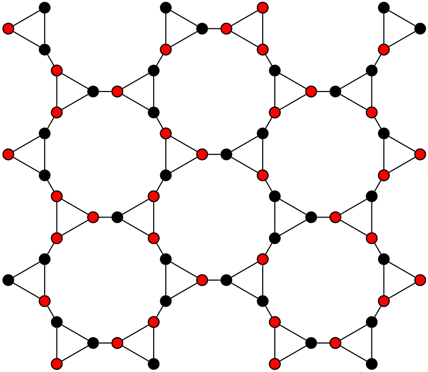

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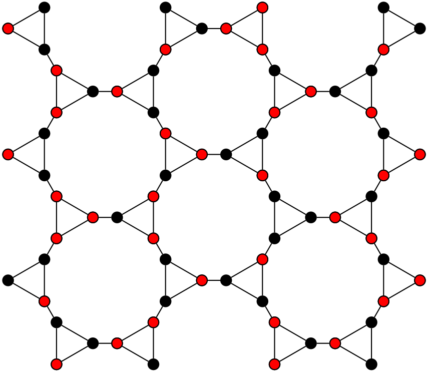

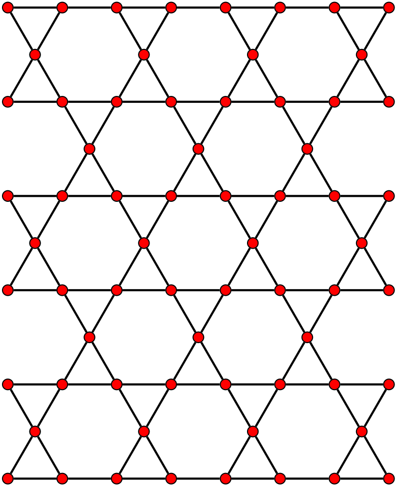

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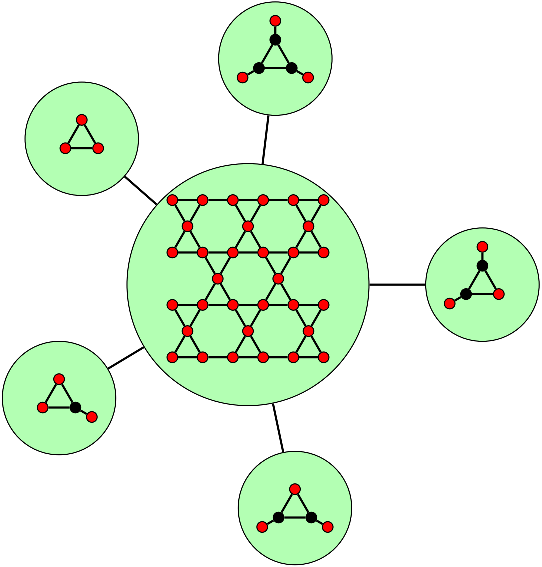

Grohe's decomposition

Theorem (Grohe '16)

Every finite 3-connected graph G has a tree-decomposition of adhesion at most 3 whose torsos are minor of G and are complete graphs on at most 4 vertices or quasi-4-connected graphs.

Bad news: only applies to finite graphs and no canonicity.

Application: Finite presentability.

Theorem (Droms '06)

Planar groups are finitely presented.

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Theorem (Droms '06)

Planar groups are finitely presented.

Corollary

Every minor-excluding finitely generated group Γ *is finitely presented.*

Proof based on the approach of [Hamann '18]

Application: accessibility

Accessibility: first defined in the context of groups.
A ray in a graph G is an infinite path $r = (x_1, x_2, x_3, \dots)$.
$r \simeq r'$ iff for every finite $S \subseteq V(G)$, there is an inifinite component of G containing an infinite subpath of both r and r'.
An end ω is a class of equivalence of rays in a graph.
ω and ω' are k-distinguishable if there exist $S \subseteq V(G)$ of size at most k separating all their rays.

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G is accessible if there exists $k \in \mathbb{N}$ such that every two distinct ends are k-distinguishable.

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Corollary

Locally finite quasi-transitive graphs that exclude K_∞ as a minor are accessible.