# Minor-exclusion in quasi-transitive locally finite graphs

Louis Esperet, Ugo Giocanti, Clément Legrand-Duchesne

Université Grenoble Alpes, Laboratoire G-SCOP, France

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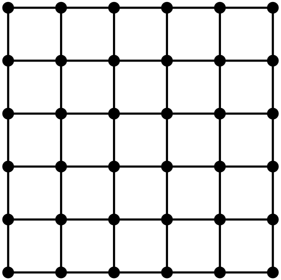

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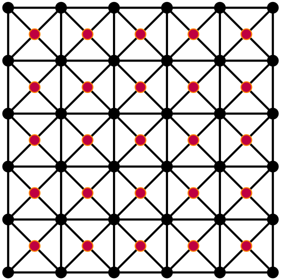

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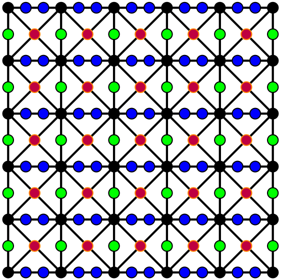

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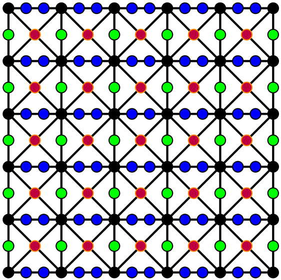

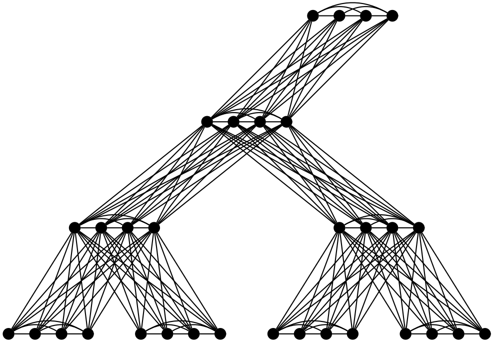

# Canonical tree-decompositions

Tree-decomposition of $G$: $(T, \mathcal{V})$ where $T$: tree, $\mathcal{V} = (V_t)_{t \in V(T)}$ family of subsets $V_t$ of $V(G)$ s.t:

- $V(G) = \bigcup_{t \in V(T)} V_t$;
- for every nodes $t, t', t''$ such that $t'$ is on the unique path of $T$ from $t$ to $t''$, $V_t \cap V_{t''} \subseteq V_{t'}$;
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$(T, \mathcal{V})$ is canonical if $\mathrm{Aut}(G)$ induces an action on $T$ s.t. foreach $t \in V(T), \gamma \in \mathrm{Aut}(G)$, $V_{t \cdot \gamma} = V_t \cdot \gamma$.

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Adhesion sets: the sets $V_t \cap V_{t'}$ for $tt' \in E(T)$.
Torso $G[\![V_t]\!]$: $G[V_t]$ + all edges belonging to the adhesion sets $V_t \cap V_{t'}$ for $t' \in V(T)$.

# Decompositions in components of low connectivity

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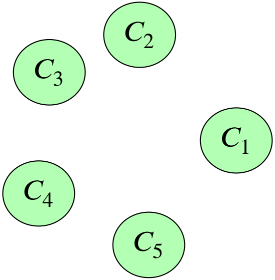

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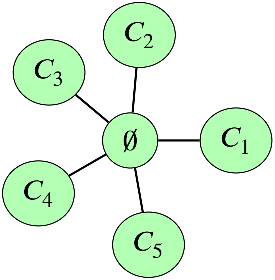

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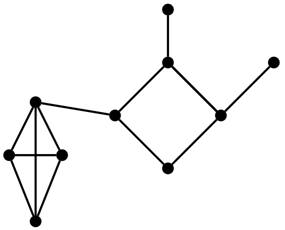

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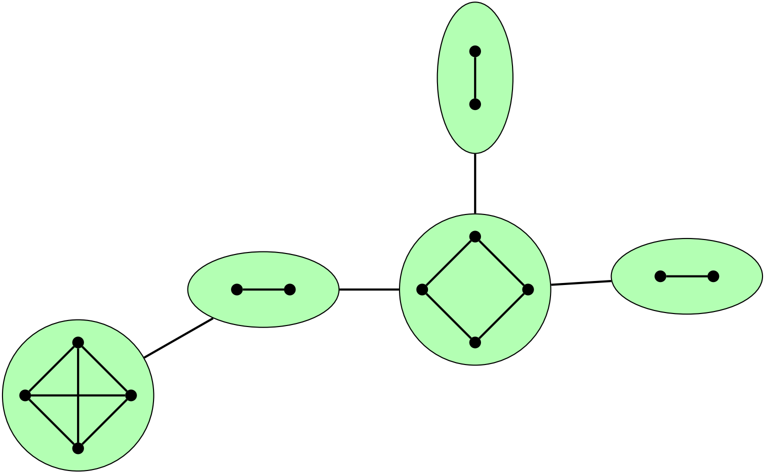
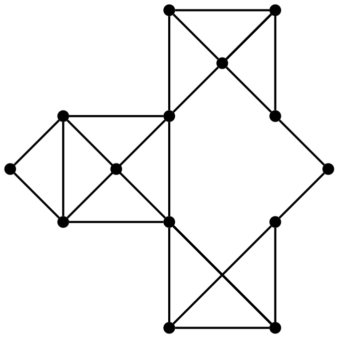


# Decompositions in components of low connectivity

**Theorem (Tutte; DSS)**

*Every $2$-connected locally finite graph $G$ has a canonical tree-decomposition of adhesion at most $2$ whose torsos are either finite cycles, edges, or $3$-connected graphs.*

$G$: 2-connected graph.



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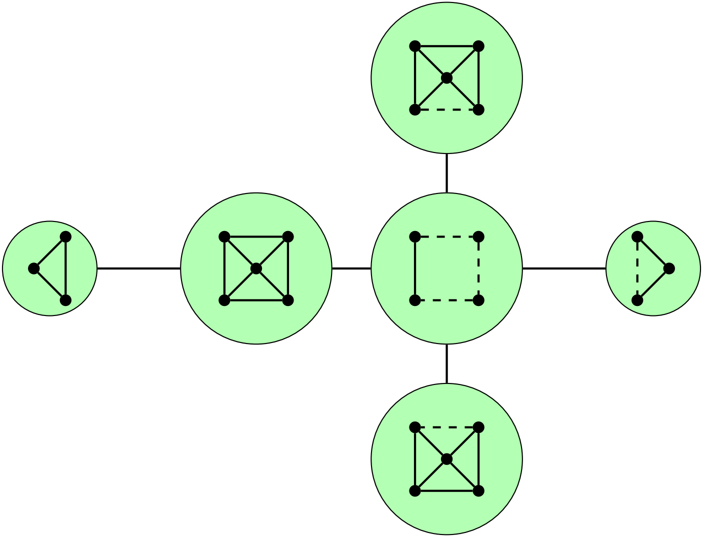

## Decompositions in components of low connectivity

→ What if $G$ is 3-connected?

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The notion of "maximal $4$-connected component" is not the right one to use.

# Robertson-Seymour structure theorem

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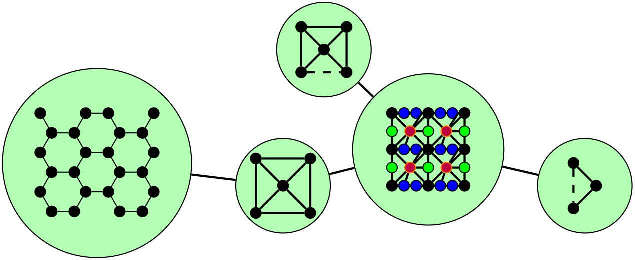
[Diestel-Thomas '99]: "Extends to infinite graphs excluding some finite minor."

# Main result

**Theorem (finite/planar)**

*Let $G$ be a quasi-transitive locally finite graph excluding $K_\infty$ as a minor. Then there is an integer $k$ such that $G$ admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most $k$ whose torsos are either finite or quasi-transitive 3-connected planar minors of $G$.*



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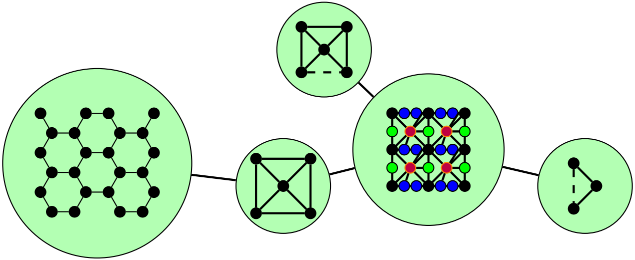

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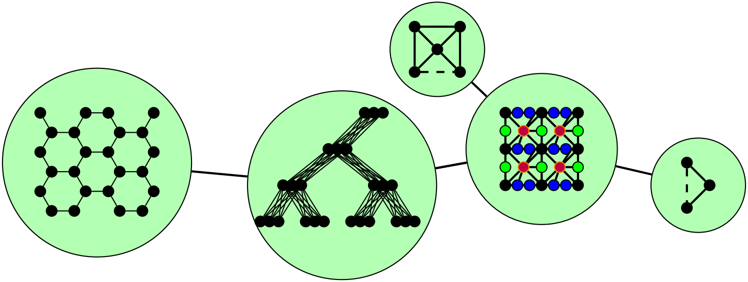

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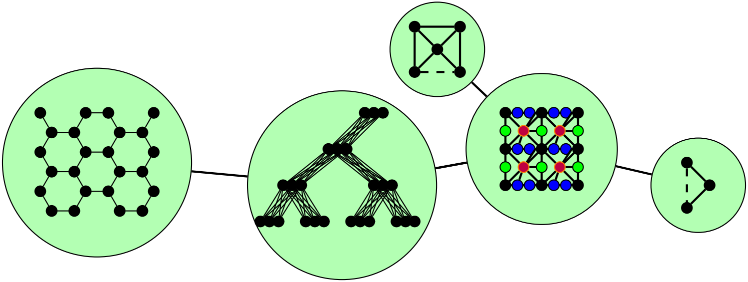

### Corollary

*For every locally finite quasi-transitive graph $G$ avoiding $K_\infty$ as a minor, there is an integer $k$ such that $G$ is $K_k$-minor-free.*

Generalizes [Thomassen '92] dealing with the 4-connected case.

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**Corollary**

*Locally finite quasi-transitive graphs that exclude $K_\infty$ as a minor are accessible.*

## Proof idea

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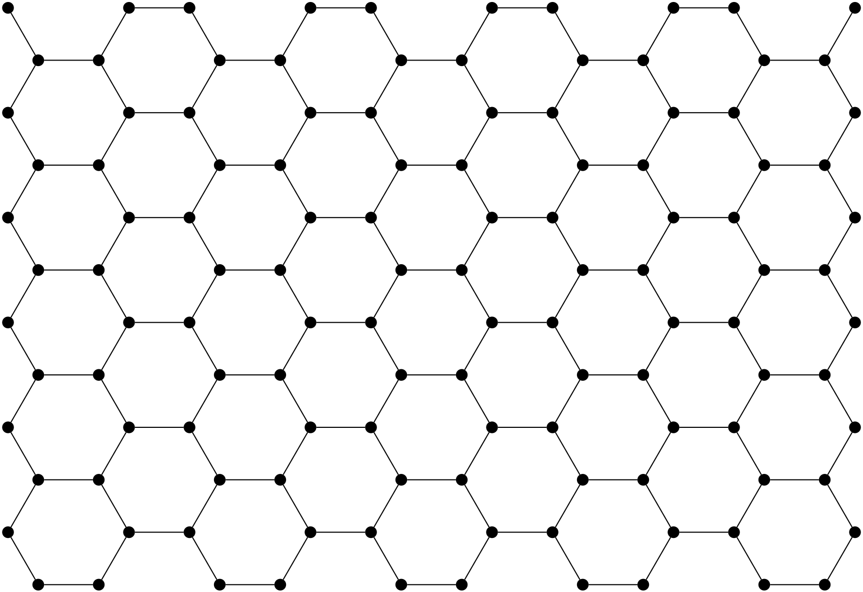
**Theorem (Grohe '16)**

*Every finite graph $G$ has a tree-decomposition of adhesion at most $3$ whose torsos are minor of $G$ and are complete graphs on at most 4 vertices or quasi-4-connected graphs.*

# Grohe's decomposition



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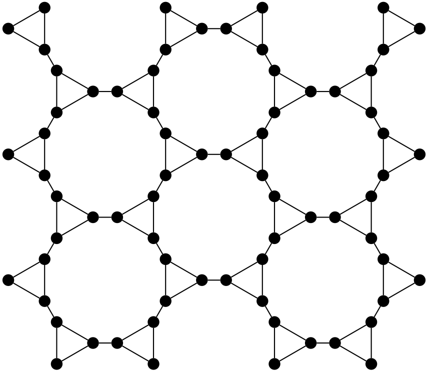

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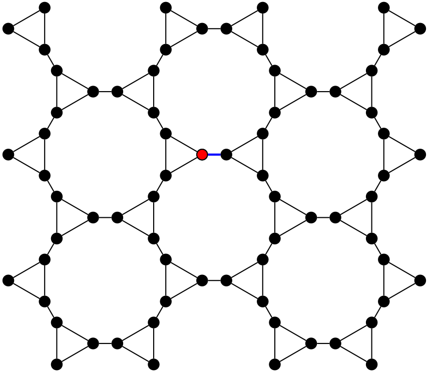

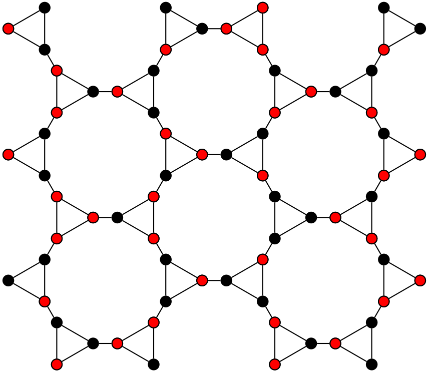

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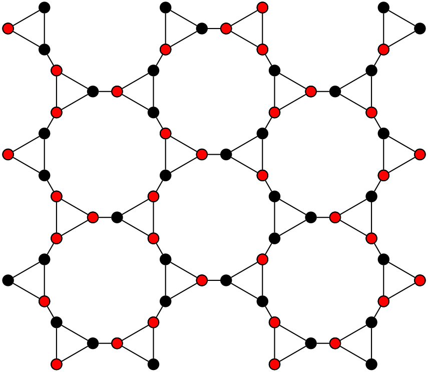

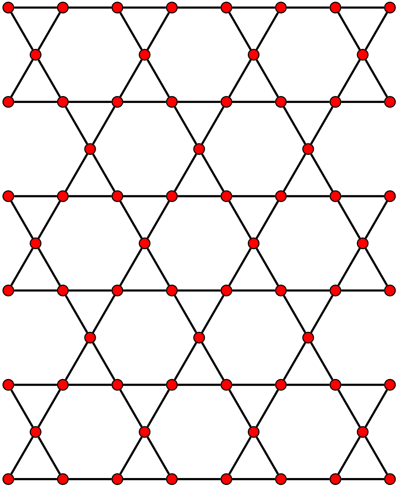

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5. Prove that this torso has either bounded treewidth or is $3$-connected planar.

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$\rightarrow$ $(T, \mathcal{V})$ canonical tree-decomposition of $G$ locally finite of bounded adhesion with tight edge-separations, then $E(T)/\mathrm{Aut}(G)$ finite.

# Some useful tools

**Lemma (HLMR '19)**

*If $G$ quasi-transitive locally finite and $(T, \mathcal{V})$ canonical tree-decomposition of bounded adhesion with $E(T)/\mathrm{Aut}(G)$ finite, then $G[V_t]$ and $G[\![V_t]\!]$ are quasi-transitive locally finite.*

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**Proposition (CHM '22)**

*Let $G$ be quasi-transitive locally finite and $(T, \mathcal{V})$ be a canonical tree-decomposition of $G$ of bounded adhesion with tight separations. If there is a canonical family $(T_t, \mathcal{V}_t)_{t \in V(T)}$ of canonical tree-decompositions of the torsos $G[\![V_t]\!]$ with bounded adhesion and tight separations, then there exists a canonical tree-decomposition $(T', \mathcal{V}')$ of $G$ that refines $(T, \mathcal{V})$ with respect to the family $(T_t, \mathcal{V}_t)_{t \in V(T)}$.*

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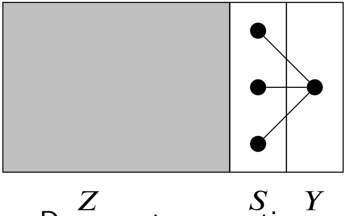


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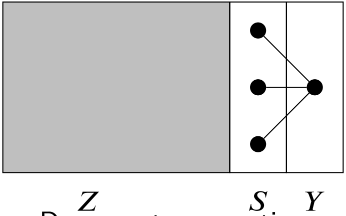


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**Corollary**

*Let $G$ be a 3-connected locally finite graph, and $(T, \mathcal{V})$ be a tree-decomposition of $G$ whose edge-separations have order 3 and are non-degenerate. Then $G[\![V_t]\!]$ is a (faithful) minor of $G$ for each $t \in V(T)$.*

# Plan of the proof (reminder)

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A tangle (of order 4) is a subset $\mathcal{T}$ of $\mathrm{Sep}_{<4}(G)$ such that

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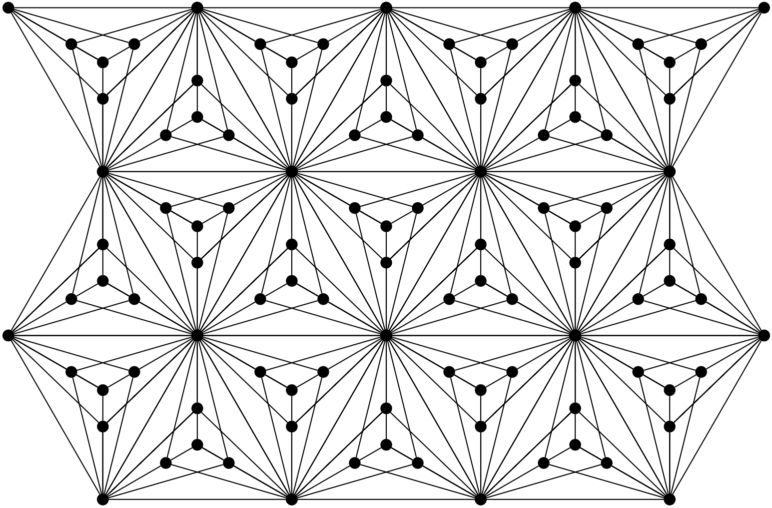
Order $\preceq$ on $\mathrm{Sep}_{<4}(G)$:
$(Y, S, Z) \preceq (Y', S', Z')$ if and only if $S' \cup Y' \subseteq S \cup Y$ and $S \cup Z \subseteq S' \cup Z'$.

# Region/End tangles

A tangle $\mathcal{T}$ of order $4$ is:

- A region tangle if it is w.q.o.
- An end tangle otherwise.



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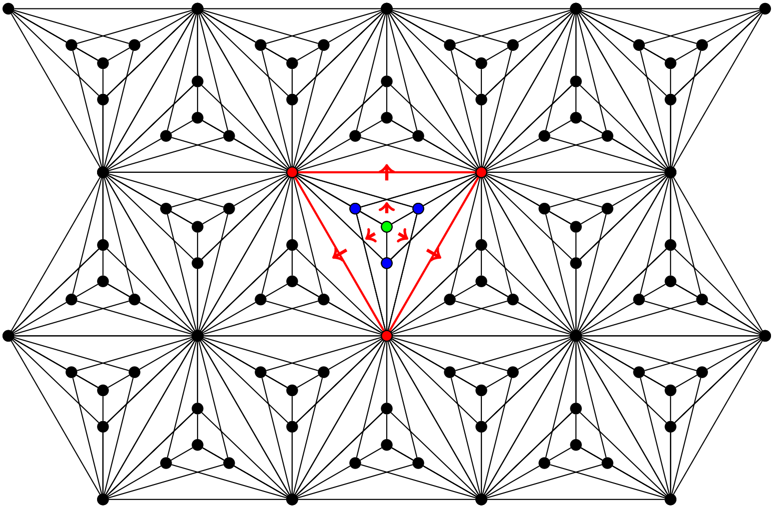

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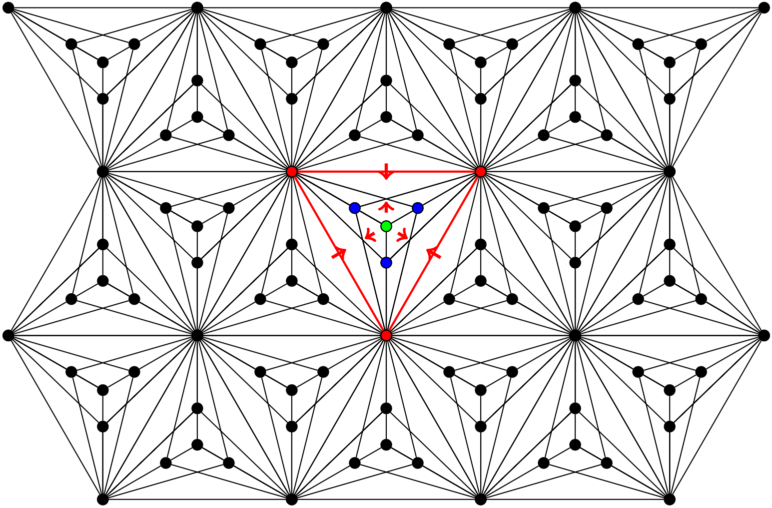

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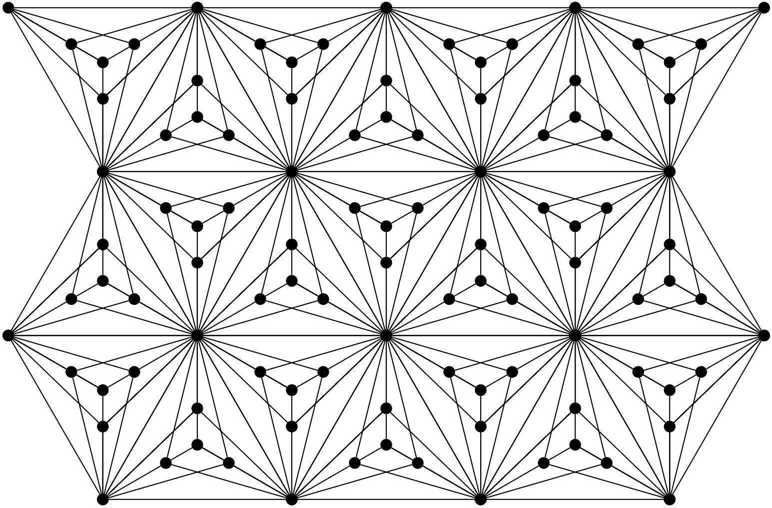

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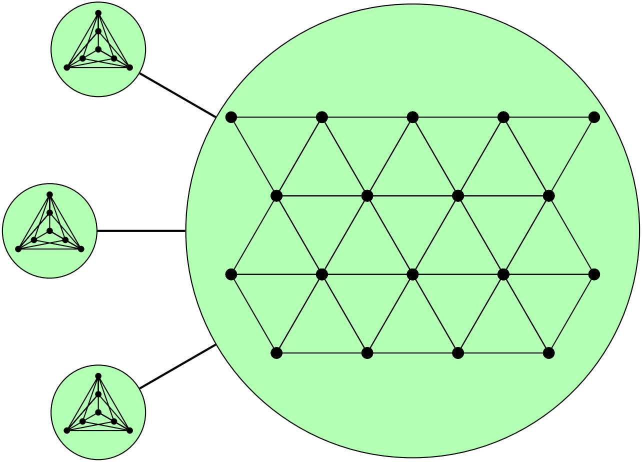

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**Proposition**

*$k \geqslant 1$, $G$ locally finite connected quasi-transitive. Then $G$ cannot have exactly one end of size exactly $k$.*

→ Generalizes [Thomassen '92]: "$G$ has one end ⇒ this end is thick".

# Plan of the proof (reminder)

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**Lemma (Grohe '16)**

*If $(Y_1, S_1, Z_1), (Y_2, S_2, Z_2) \in \mathcal{T}_{nd}$ distinct are crossing, then there are two distinct vertices $s_i \in S_i$, $i \in \{1, 2\}$ such that $s_1 s_2 \in E(G)$ and $S_i \cap Y_{3-i} = \{s_i\}$ and $S_1 \cap S_2 = \emptyset$. Moreover, such crossing edges form a matching in $G$.*

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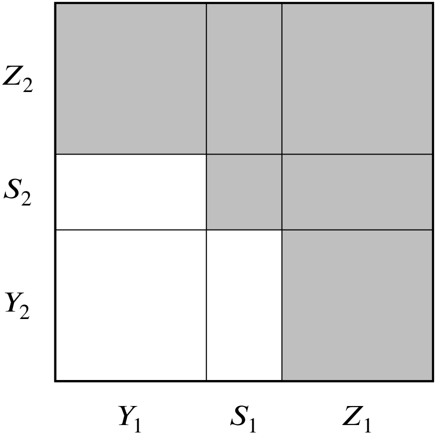
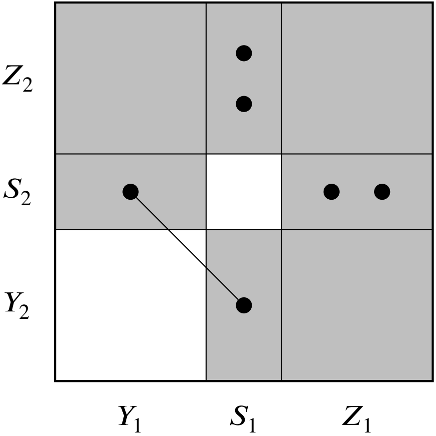



Orthogonal separations




Crossing separations

# Easy case: orthogonal family

If separations of $\mathcal{T}_{nd}$ are pairwise orthogonal:

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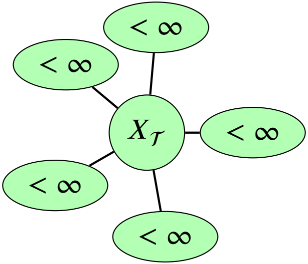

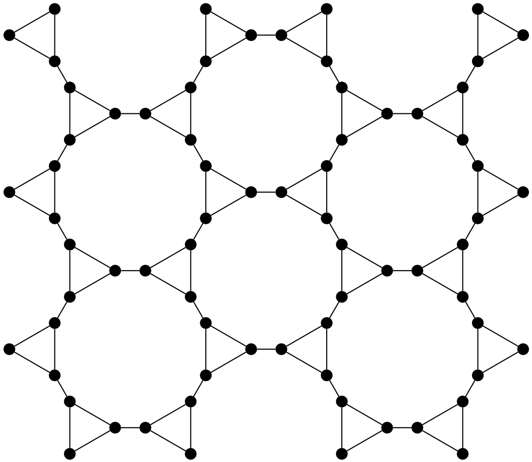

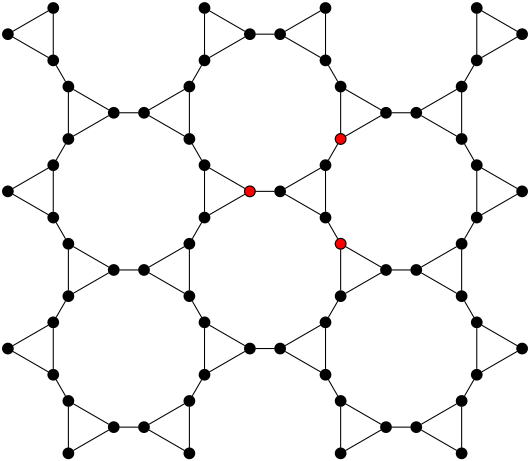

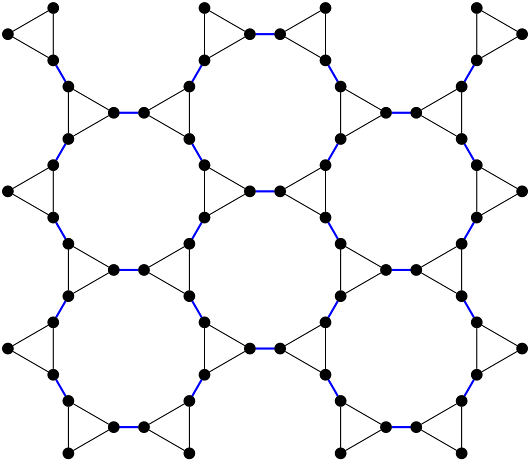

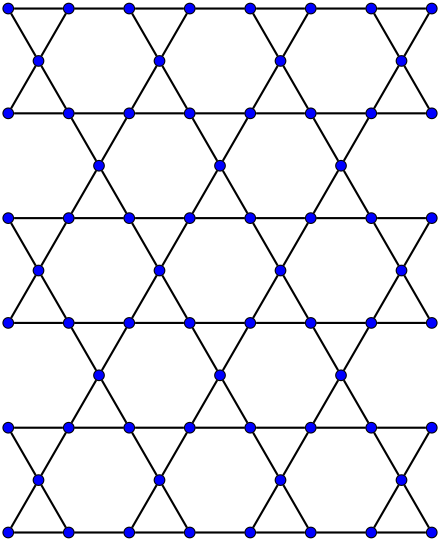

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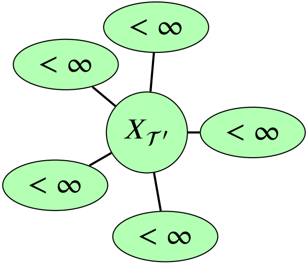
$M \subseteq R_{\mathcal{T}}$ and in $G/M$, there is an induced tangle of order 4 $\mathcal{T}'$ s.t.

$$R_{\mathcal{T}}/M = X_{\mathcal{T}'}.$$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$< \infty$
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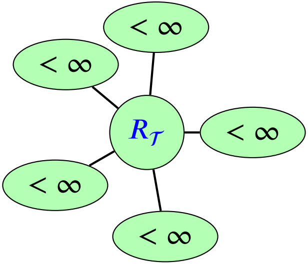

< ∞
< ∞
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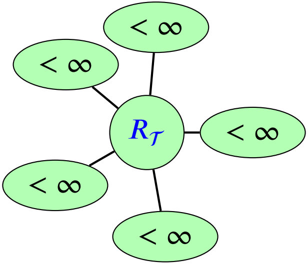


**Lemma**

*If $G/M[\![X_{\mathcal{T}'}]\!]$ is planar or has bounded treewidth, then $G[\![R_{\mathcal{T}}]\!]$ also does.*

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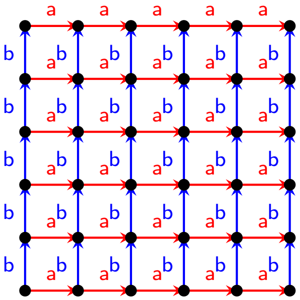
$\mathrm{Cay}(\mathbb{Z}^2, S)$, with $S = \{a, b\}$



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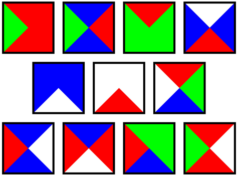
**Corollary**

*Every minor-excluding finitely generated group $\Gamma$ is finitely presented.*

Proof based on the approach of [Hamann '18]

# Domino Problem



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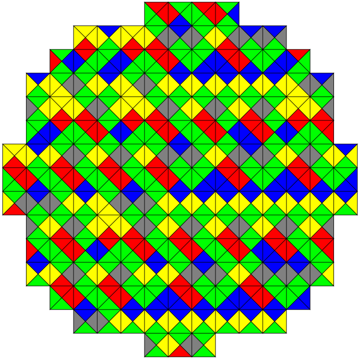

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# Domino Problem

Domino problem on $(\Gamma, S)$:

Input: a finite alphabet $\Sigma$ and a finite set $\mathcal{F} = \{F_1, \dots, F_p\}$ of forbidden *patterns*, where $F_i$ is a $\Sigma$-coloring of the 1-ball around $1_\Gamma$ in $\mathrm{Cay}(G, S)$.
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**Corollary**

*The conjecture is true for planar groups and more generally for minor-excluding groups.*

# Questions

**Question**

*If $G$ is locally finite quasi-transitive, and $M$ is an $\mathrm{Aut}(G)$-invariant matching, is there an orientation of $M$ such that the obtained graph is still quasi-transitive (for the action of some subgroup $\Gamma$ of $\mathrm{Aut}(G)$)?*

Danke