Constraint Programming
Doctoral class

Hadrien Cambazard

(ROSP, G-SCOP)

Constraint Programming

- A framework to model and solve combinatorial optimization problems

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- A toy timetabling problem :
  a. **4** meetings A, B, C, D **of one hour** take place **in the same room** between 8h and 13h
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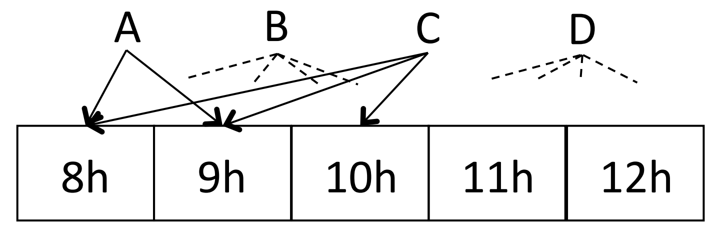

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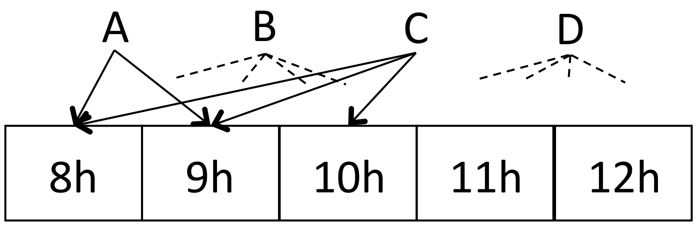

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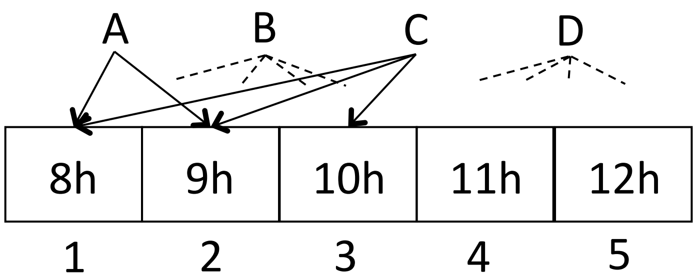

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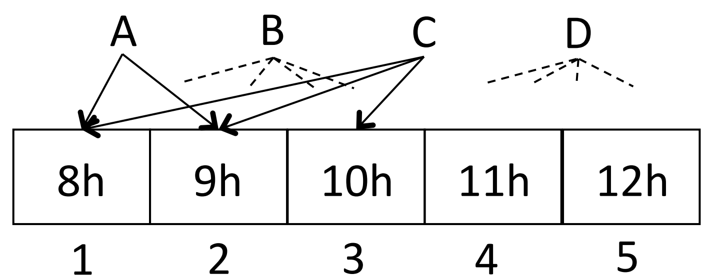

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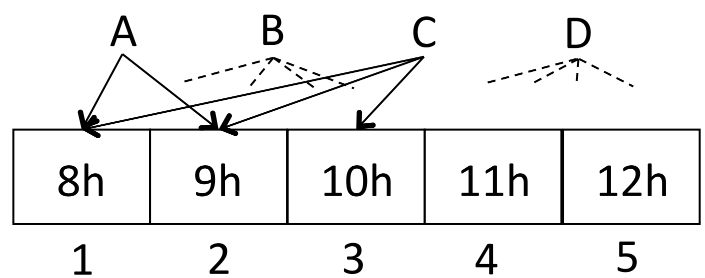

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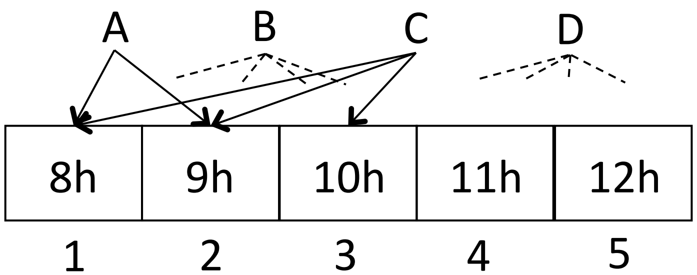

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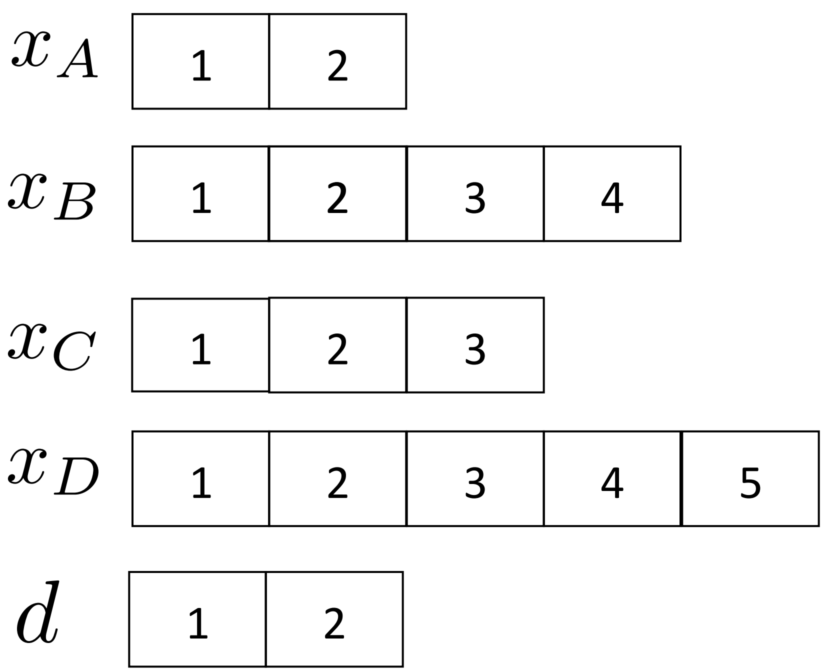

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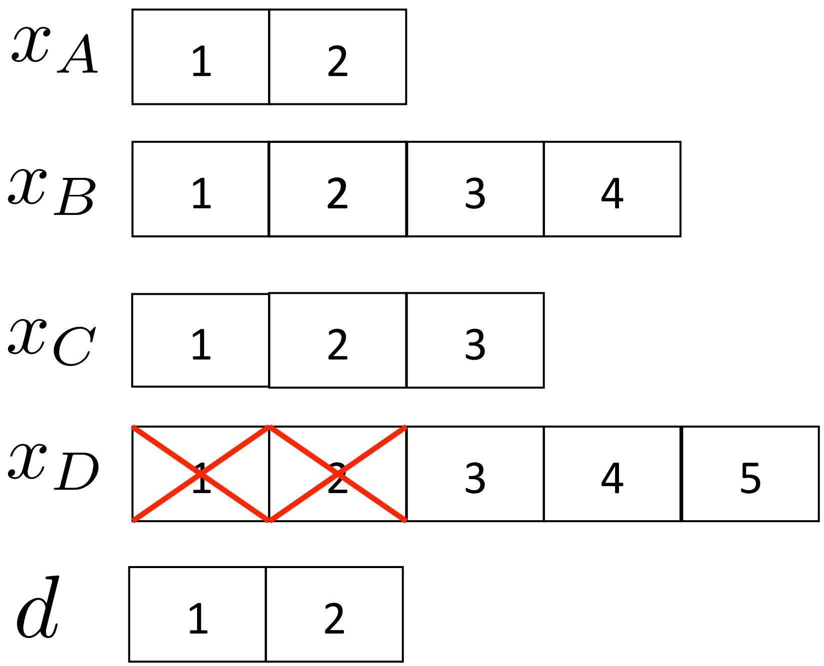

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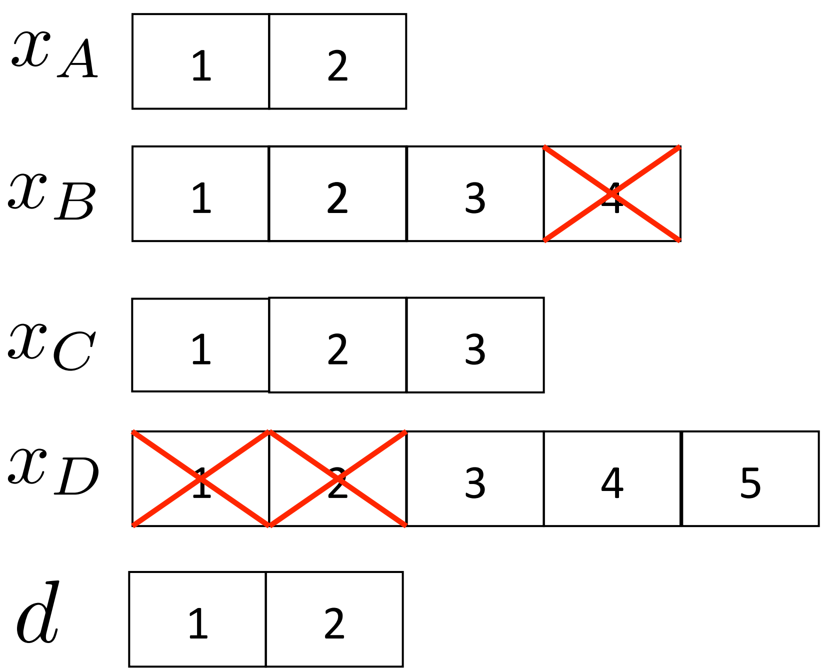

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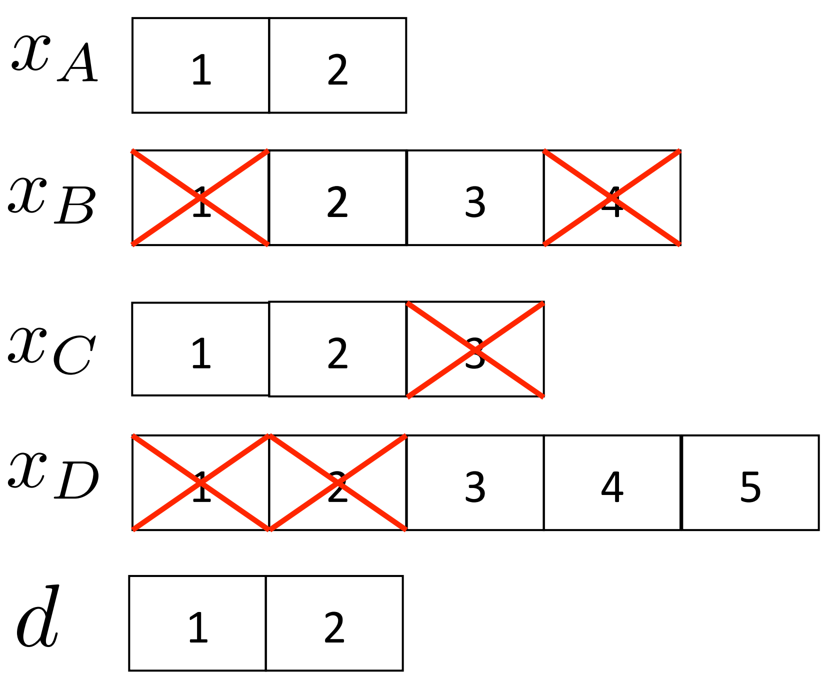

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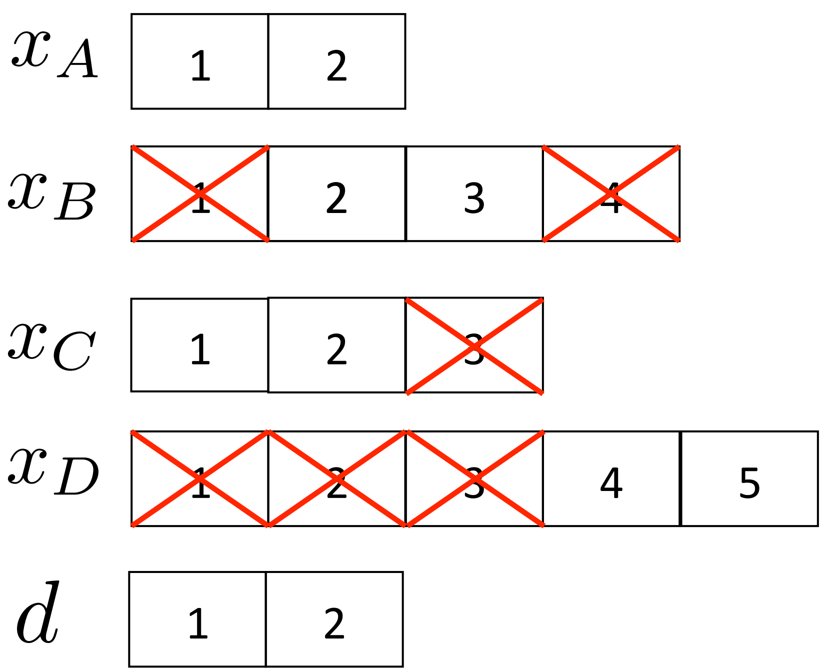

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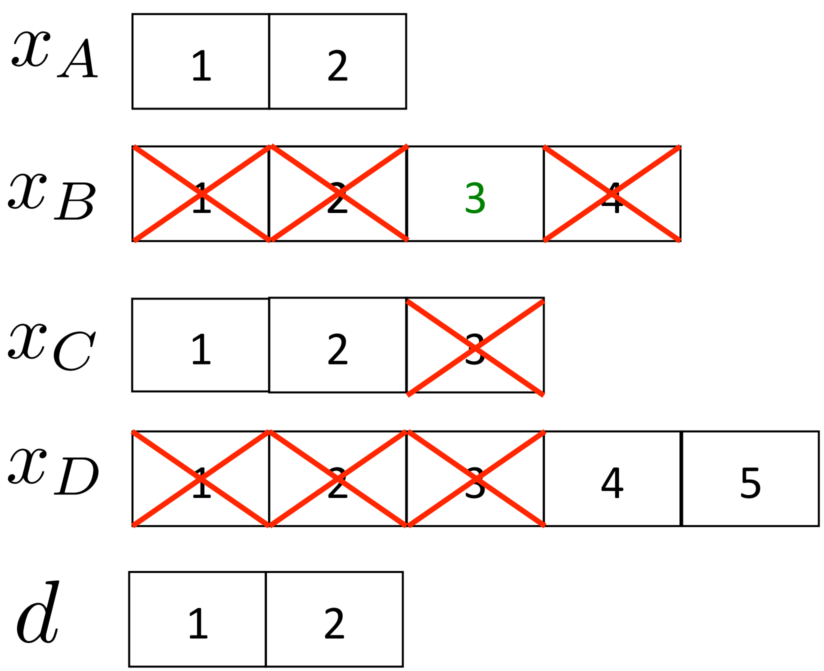

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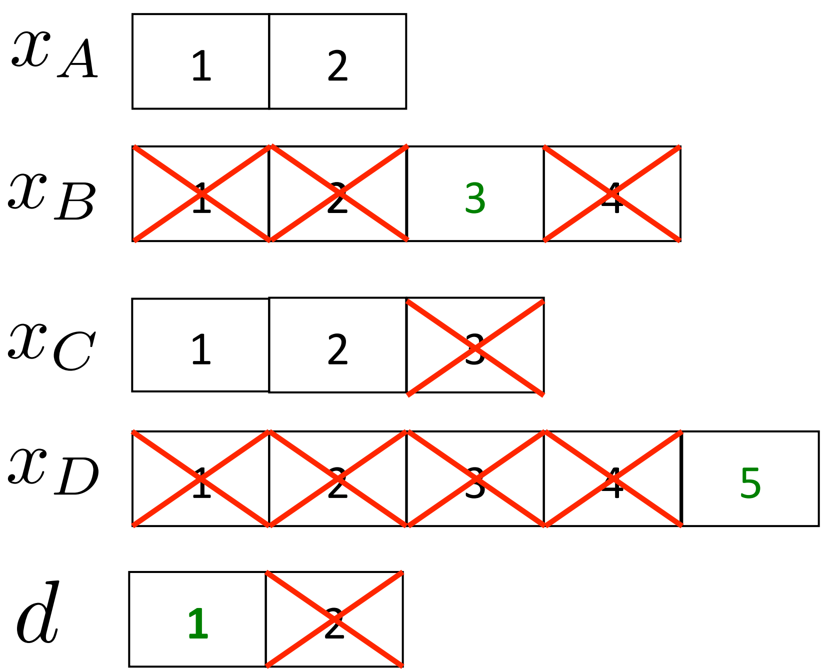

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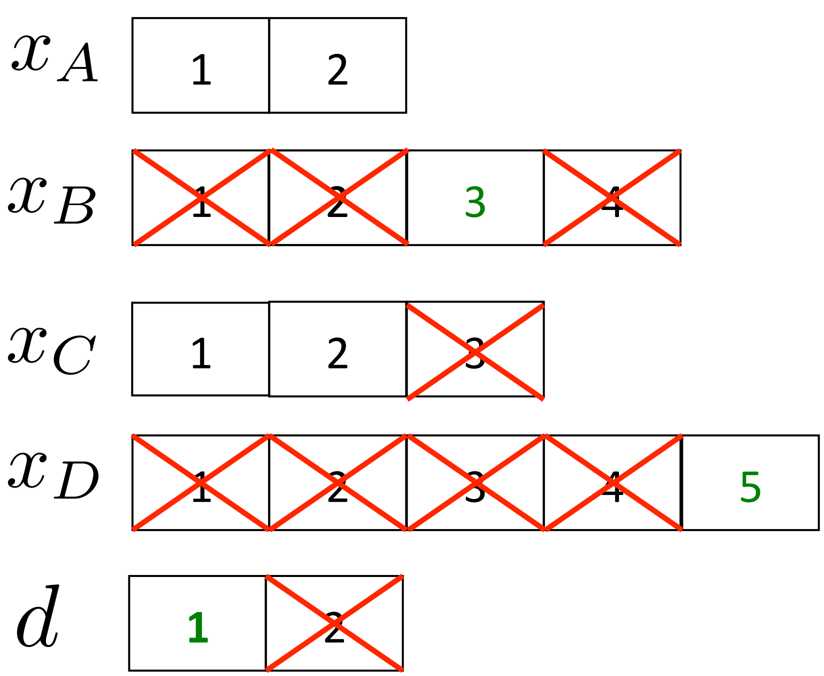
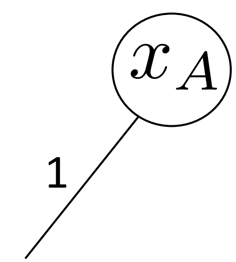


Branching (Tree Search)



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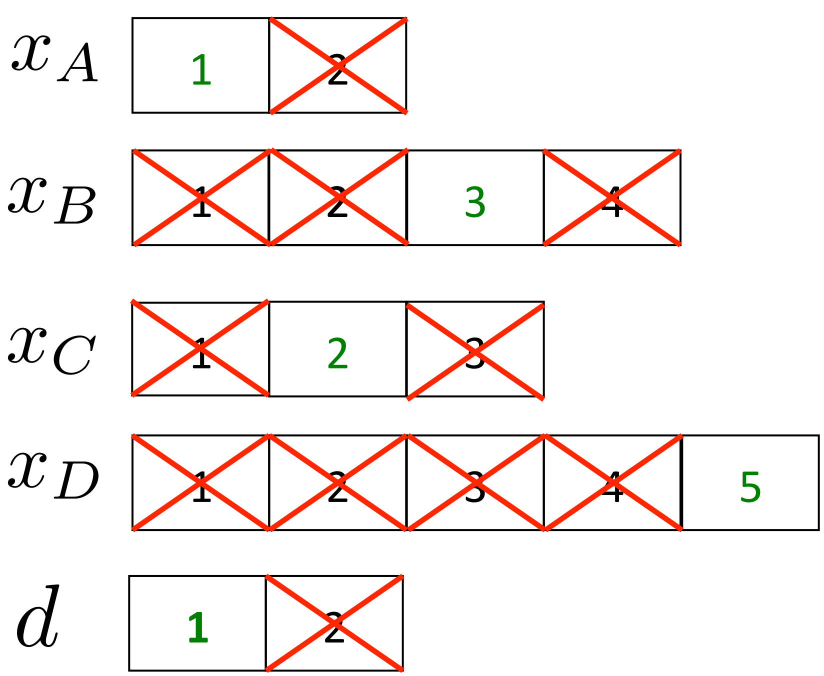

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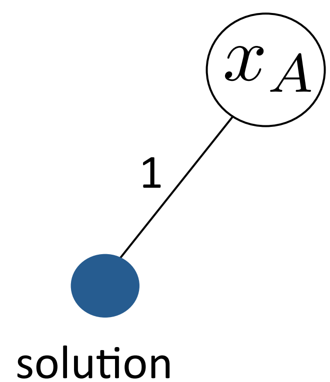
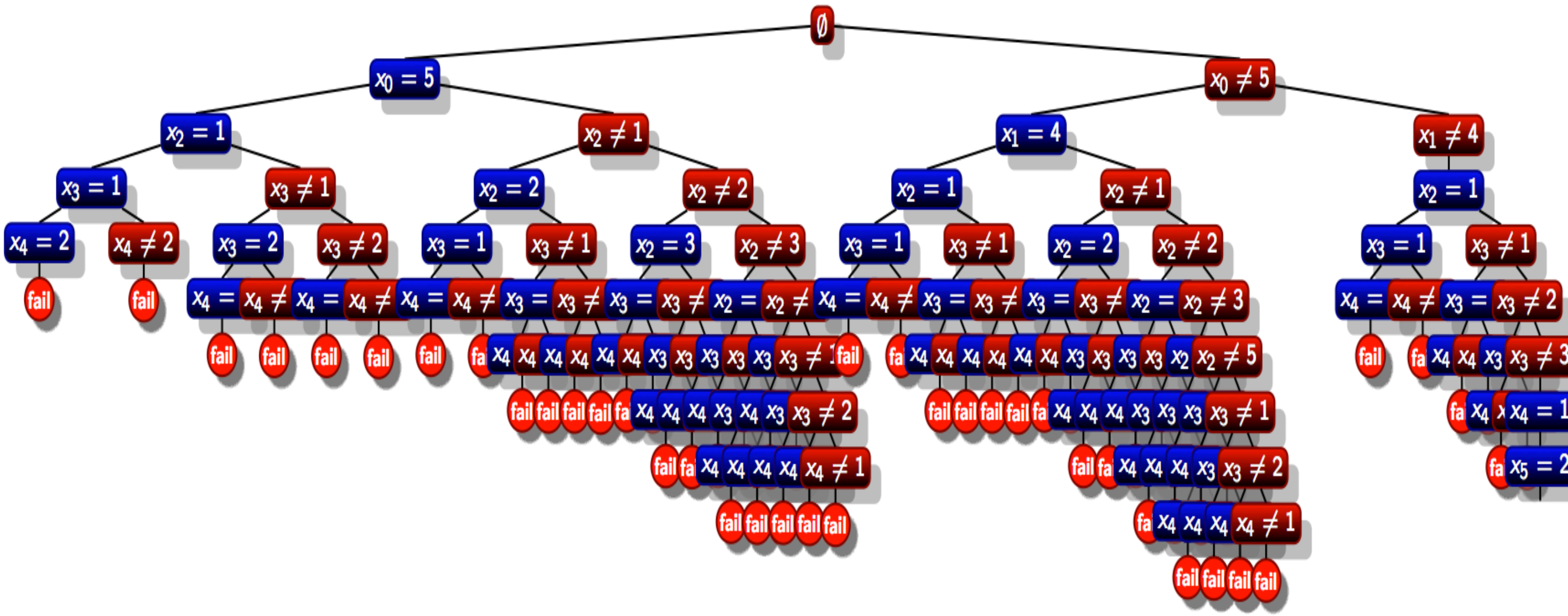


Branching is usually much bigger



Constraint Programming

- *A constraint* :
  1. A **logical relationship** that values of the variables must satisfy in a solution
  2. A **reasoning block** for a **frequent** sub-problem that we can solve efficiently (filtering algorithm)
  3. A reusable **modelling block**
  4. (Almost) no restriction on the constraints :
     - **Checkable in polynomial time**
     - Often a known relation for which an efficient algorithm exists

- *Filtering* : Detect inconsistent variable/value pairs and remove them

- *Constraint Propagation*:
  - **Exchange of information** (trough domains) between constraints (➡ fix point)
  - The inference method to solve the **composite** problem.
  - Used **before each decision** of the branching algorithm

- *Tree search*:
  - Take **decisions** (reductions of the problem) until a solution is found or the proof that none exists is completed (**backtrack** search).

Restrictions of the problem formulation

MILP (Mixed Integer Linear Programming)

- linear inequalities
- Real/integer variables

$$\text{Maximize} \quad P = p_1 x_1 + p_2 x_2 + \cdots + p_k x_k$$

$$\begin{aligned} \text{Subject to:} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k \leq q_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k \leq q_2 \\ & \vdots \\ & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nk}x_k \leq q_n \\ & x_1, x_2, \cdots x_k \geq 0 \end{aligned}$$

SAT (Boolean satisfiability)

- Clauses (conjonctive normal form)
- Boolean variables

$$\Phi = C_1 \wedge \ldots \wedge C_8$$

$$\begin{aligned} C_1 &= (x_1 \vee x_2 \vee \neg x_3) \\ C_2 &= (\neg x_1 \vee \neg x_2 \vee \neg x_4) \\ C_3 &= (x_1 \vee \neg x_2 \vee \neg x_5) \\ C_4 &= (\neg x_1 \vee x_3 \vee \neg x_4) \\ C_5 &= (x_1 \vee \neg x_3 \vee x_5) \\ C_6 &= (x_1 \vee \neg x_4 \vee x_5) \\ C_7 &= (x_2 \vee x_4 \vee x_5) \\ C_8 &= (\neg x_3 \vee x_4 \vee \neg x_5) \end{aligned}$$

CP (Constraint Programming)

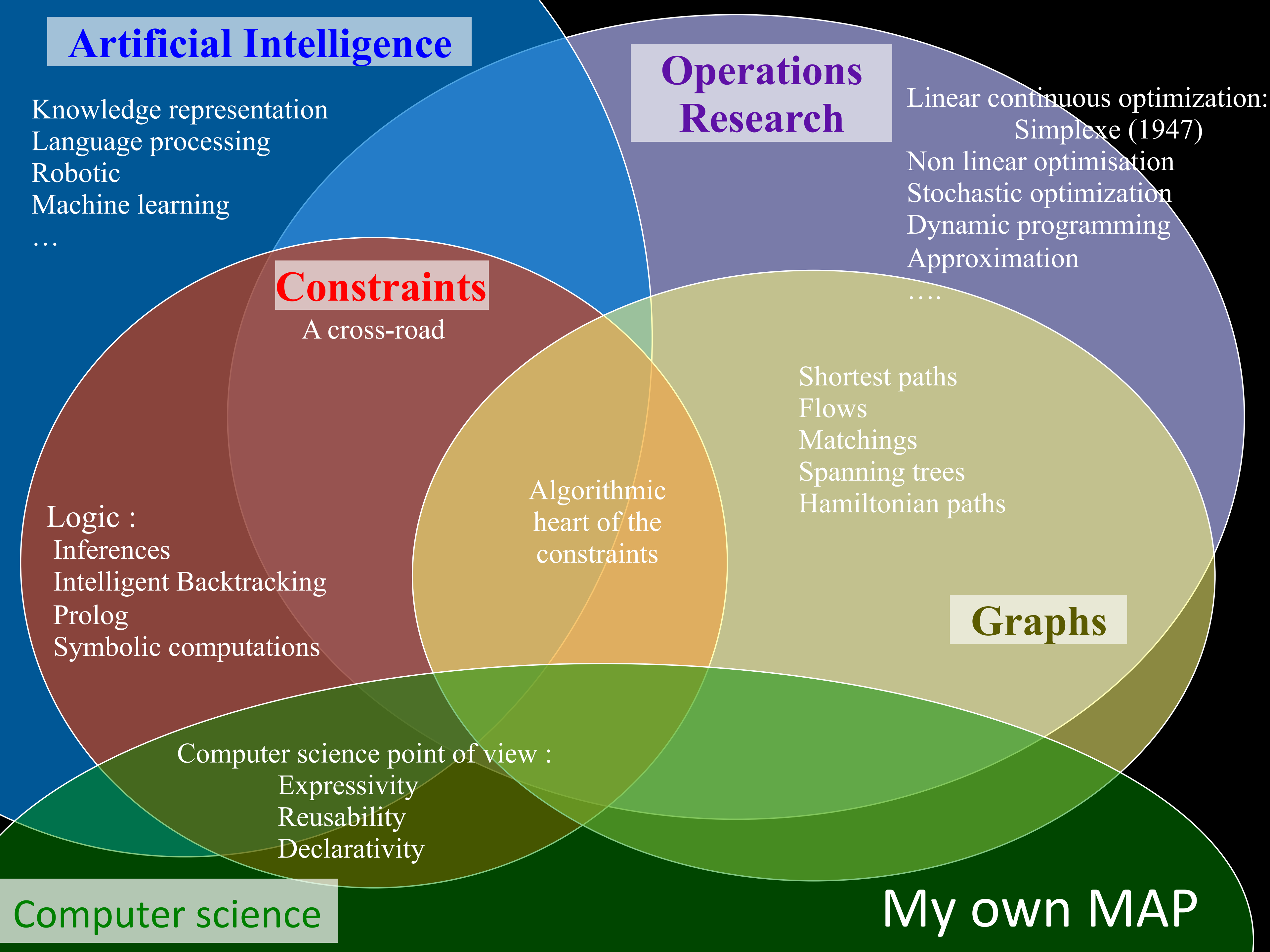
$$\begin{aligned} & \text{AllDifferent}(x_A, x_B, x_C, x_D) \\ & x_B + d + 1 = x_D, \\ & x_C < x_B, \\ & x_A \in \{1,2\}, x_C \in \{1,2,3\}, x_B, x_D \in \{1,2,3,4,5\}, d \in \{1,2\} \end{aligned}$$

Constraint Programming (short history)

- **Language roots**: logic programming (**declarativity**)
  - Planner (Hewitt, 1965) and later Prolog (Colmerauer, 1972)
  - Developed for *language processing* and then used in *Theorem proving*, …, *Planning*.
- **Algorithmic roots:**
  - Machine vision community (“a man-machine graphical communication systems”, 1963)
  - Image processing (“Waltz Filtering”, 1975) (Application: “scene labeling”)

- Constraint Logic Programming: **CLP**
  - Concerns: **Representation** and **Reasoning**
  - Methodology: **Search** (very AI) but also **Inference**
  - **CHIP** (Van Hentenryck, Constraints Handling In Prolog, 1989)

- Constraint Programming: **CP** (strengthened relationship to OR)
  - 1990: birth of a workshop : “*Principles and Practice of Constraint Programming*” (conf in 1995)
  - Global constraints: (Beldiceanu et al, *Extending chip in order to solve complex scheduling and placement problems*, 1992)
  - 1999: birth of a second workshop: “Constraint Programming, Artificial Intelligence and Operations Research” (conf in 2004)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Constraints
A cross-road
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Computer science point of view :
Expressivity
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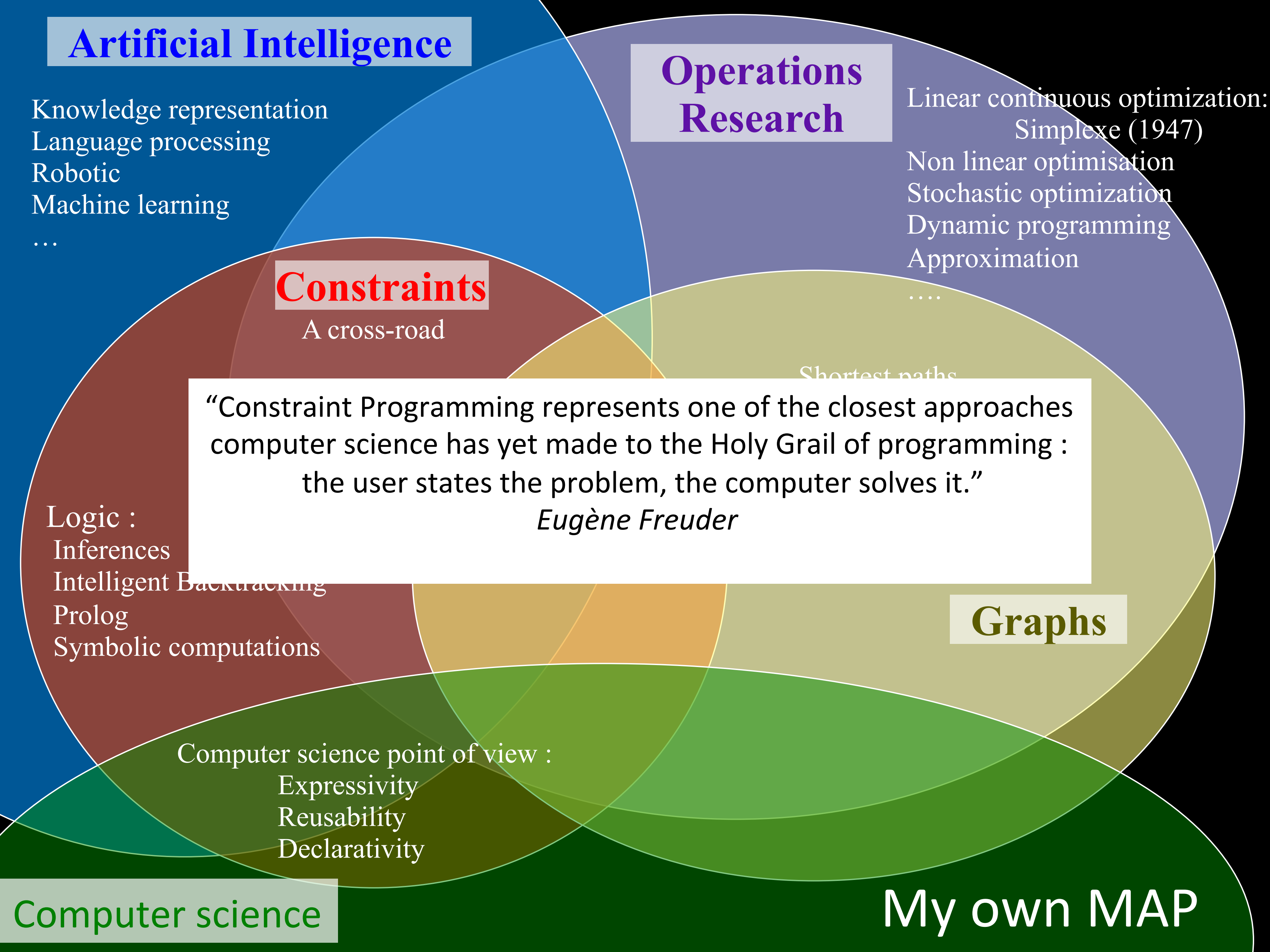
Computer science
My own MAP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Outline