The Erdős-Hajnal Conjecture for Long Holes and Anti-holes

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Abstract

Erdős and Hajnal conjectured that, for every graph H, there exists a constant c_H such that every graph G on n vertices which does not contain any induced copy of H has a clique or a stable set of size n^{c_H}. We prove that for every k, there exists $c_k > 0$ such that every graph G on n vertices not inducing a cycle of length at least k nor its complement contains a clique or a stable set of size n^{c_k}.

1 Introduction

Let $G = (V, E)$ be a graph. In the following n will denote the size of $V(G)$. A class $\mathcal{C}$ of graphs (in this paper, a graph class is closed under induced subgraphs) is said to satisfy the *(weak) Erdős-Hajnal property* if there exists some constant $c > 0$ such that every graph in $\mathcal{C}$ on n vertices contains a clique or a stable set of size n^c. The Erdős-Hajnal conjecture [9] asserts that every strict class of graphs satisfies the Erdős-Hajnal property. Alon, Pach and Solymosi proved in [2] that the Erdős-Hajnal conjecture is preserved by modules (a *module* is a subset V_1 of vertices such that for every $x, y \in V_1$, we have $N(x) \setminus V_1 = N(y) \setminus V_1$): in other words, it suffices to prove that the class of graphs which do not contain any copy of H satisfy the Erdős-Hajnal property, for every prime graph H (a *prime graph* is a graph with only trivial modules). The conjecture is satisfied for every prime graph of size at most 4. For $k = 5$, the conjecture is satisfied for bulls [4] but remains open for two prime graphs: the path and the cycle on 5 vertices. Recently, a new approach for tackling this conjecture has been introduced: forbidding both a graph and its complement. This approach provides a large amount of results for paths (see [3, 8, 6, 7] for instance). In particular Bousquet, Lagoutte and Thomassé proved that, for every k, the class of graphs with no P_k nor its complement satisfies the Erdős-Hajnal property. A survey of Chudnovsky [5] details all the known results about this conjecture.

In this paper, we explore the case where long holes and their complements are forbidden (a *hole* is an induced cycle of length at least 4). A long outstanding open problem due to Gyarfás [12] asks if, for every integer k, the class of graphs with no hole of length at least k is χ-bounded.

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Equivalently, can the chromatic number of a graph with no hole of length at least k be bounded by a function of its maximal clique and k? This question is widely open, since it is even open to determine if a triangle-free graph with no long hole contains a stable set of linear size. Several links exist between Erdős-Hajnal property and χ-boundedness. In particular, if the chromatic number of any graph of a class $\mathcal{C}$ is bounded by a polynomial of the maximum clique, the Erdős-Hajnal property holds. Here, we prove that graphs which contain neither a hole of length at least k nor its complement have the Erdős-Hajnal property.

Theorem 1. *For every integer k, the class of graphs with no holes or anti-holes of length at least k has the Erdős-Hajnal property.*

The remaining of this paper is devoted to a proof of Theorem 1.

2 Dominating tree

Let $G = (V, E)$ be a connected graph. The *neighborhood* of a set of vertices X, denoted by $N(X)$, is the set of vertices at distance one from X. The *closed neighborhood* of X is $\overline{N}(X) = N(X) \cup X$. By abuse of notation we drop the braces when X contains a single element. We select a root r in V. Let X be a set of vertices in G. A vertex y in $N(X)$ is *active* for X if it has a neighbor which is not in $\overline{N}(X)$. The following algorithm returns a subtree T of G rooted at r which dominates G, i.e. such that every vertex of G is at distance at most 1 of T.

Algorithm 1 Find a dominating tree.

Require: A graph G, a root r.
Ensure: A dominating tree T rooted at r.

STACK:= $\{r\}$; $T := \{r\}$
While STACK is non empty do
  x :=top(STACK)
  If there exists y active for T such that the only neighbor of y in STACK is x
    Add y to the top of STACK
    Set x as the father of y in the tree T
  Else remove x from STACK and keep track of the deletion order
Return T and the deletion order

First note that there are two orders on T inherited from Algorithm 1: the descendance order (well defined since T is rooted) and the deletion order. Observe that STACK is at every step of the algorithm an induced path originated from the root. Indeed, when a vertex is added to STACK its only neighbor in STACK is the top vertex. Conversely, every path in T containing the root r corresponds to the set of vertices in STACK at a given step of the algorithm. Note also that T dominates G at the end of the algorithm (since G is connected).

For every set N of nodes of T, we denote by $m(N)$ the minimal nodes of N with respect to the descendance order. In other words, $m(N)$ is a minimum subset of N such that every node of N is a descendant in T of some (unique) node of $m(N)$. The *root* $R(N)$ of N is the minimum element of $m(N)$ with respect to the deletion order. Equivalently, if we picture the tree T as being built from top to bottom and left to right (when a new vertex is added in STACK, it is drawn just beneath

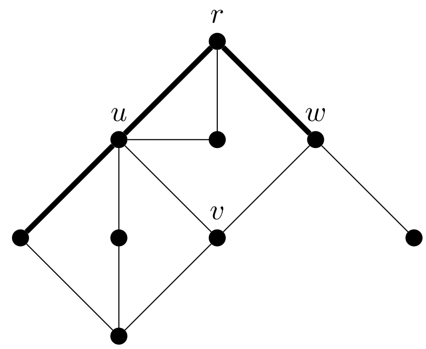

Figure 1: An illustration of Algorithm 1. The tree T is represented with thick edges. The vertices u and w are unrelated, and we have $r(v) = u$.

its only neighbor x in T, and to the right of any other neighbor of x), then $R(N)$ is the leftmost top element of N in T.

To any vertex v in G, we associate a node $r(v)$ in T by letting $r(v) := R(\overline{N}(v) \cap T)$ (see Figure 1 for an illustration). Note that when v is a node of $T \setminus r$, the vertex $r(v)$ is the father of v. Observe also that $r(r) = r$, and that $r(x)$ is well defined since T is a dominating set. Two nodes of the tree are *related* if one is a descendant of the other, otherwise they are *unrelated*. By extension, two sets A and B of nodes are *unrelated* if every $a \in A$ and $b \in B$ are unrelated.

Lemma 2. *If xy is an edge of G, then $r(x)$ and $r(y)$ are related.*

Proof. Without loss of generality, we can assume that $r(x)$ was first added in STACK. If $r(x) = x$ then $r(x)$ is r and $r(x)$ and $r(y)$ are related, so we can assume that $r(x) \neq x$. If $r(y)$ is added in STACK before $r(x)$ is deleted, then they are related (all the nodes added in STACK between the addition and the deletion of x are descendants of x since x is on paths from r to these nodes). If $r(y)$ has not been added in STACK when $r(x)$ is deleted, then x is an active neighbor of $r(x)$, as y is a neighbor of x and has no neighbor in the current T (otherwise $r(y)$ would already have appeared in T). Consequently, x is added to STACK, a contradiction to the fact that y has no neighbor in T until $r(y)$ is added to STACK. □

3 Dominating path

We think that the following result holds for $\frac{1}{3}$, but in our case this easy proof for $\frac{1}{4}$ suffices.

Lemma 3. *Let T be a tree and w be a nonnegative weight function defined on the nodes of T. We assume that $w(T) = 1$. Then there is a path P from the root with weight at least $\frac{1}{4}$ (i.e. $w(V(P)) \geq \frac{1}{4}$) or two unrelated sets A and B both with weight at least $\frac{1}{4}$.*

Proof. We grow a path P from the root r by inductively adding to the endvertex x of P the root of the heaviest subtree among the sons of x. If the path P has weight at least $\frac{1}{4}$, we are done.

Otherwise if there exists a connected component A (i.e. a subtree) of $T \setminus V(P)$ which has weight at least $\frac{1}{4}$, we conclude as follow: the father x of the root z of A belongs to P. In particular, we did not choose z to extend P from x. Thus x has another son which is the root of a subtree B of weight at least $\frac{1}{4}$. We now have our A and B (they are unrelated since the two subtrees do not intersect).

In the last case, every component of $T \setminus V(P)$ has weight less that $\frac{1}{4}$, and the total weight of $T \setminus V(P)$ is at least $\frac{3}{4}$. We can then group the components of $T \setminus V(P)$ into two unrelated sets of weight at least $\frac{1}{4}$. Indeed we iteratively add components until their union A weighs at least $\frac{1}{4}$. Considering that each component weighs less than $\frac{1}{4}$, the weight of A is less than $\frac{1}{2}$. Since the vertices of the path P have weight less than $\frac{1}{4}$, the remaining connected components have weight at least $\frac{1}{4}$, which gives B. □

Let us say that a graph G on n vertices is *sparse* if either its maximum degree is at most εn for some small ε, or it has no triangle. All the sparse graphs we will consider here satisfy the first hypothesis, but we choose that looser notion of sparsity in order to make Lemma 4 more general.

In a graph G, a *complete ℓ-bipartite graph* is a pair of disjoint subsets X, Y of vertices of G, both of size ℓ and inducing all edges between X and Y. We define similarly *empty ℓ-bipartite graph* when there is no edge between X and Y. Observe that we do not require any condition inside X or Y. A class of graphs $\mathcal{C}$ has the *strong Erdős-Hajnal property*, introduced in [11] if there exists a constant $c > 0$ such that every graph of $\mathcal{C}$ contains an empty cn-bipartite graph or a complete cn-bipartite graph. As we will see later, the strong Erdős-Hajnal property implies the Erdős-Hajnal property. The remaining of the proof consists in showing that the class of graphs with no hole and no anti-hole of length at least k has the strong Erdős-Hajnal property.

Lemma 4. *Let G be a sparse graph with no hole of length at least k, that admits a dominating induced path P. Then G contains an empty cn-bipartite graph. Here c depends of the coefficient ε of sparsity, and k.*

Proof. Let us consider a subpath I of P of length k. We assume that P is given in a left right order from one endpoint to the other. The vertices of $G \setminus P$ fall into three categories: the *left* of I denotes the vertices with all neighbors in P at the left of I, the *right* of I denotes the vertices with all neighbors in P at the right of I, and the *inside* of I denotes the other vertices of $G \setminus P$. Observe that if a vertex has both a neighbor at the left and a neighbor at the right of I, but no neighbor in I, then there is a hole of length at least k. Since P is a dominating set, a vertex inside of I that has no neighbor in I must have by definition both a neighbor at the left and a neighbor at the right of I, which provides a long hole. It follows that every vertex inside of I has a neighbor in I. Similarly, note that there is no edge between the left of I and the right of I. So the left of I and the right of I form an empty bipartite graph.
We claim that if G is sparse, then the inside vertices cannot be too many. This is straightforward if the degree is bounded by εn, since the inside vertices belong to the neighborhood of one of the k vertices of I. If there is no triangle in G, then the neighborhood of every vertex of I is a stable set, hence the neighborhood of I has chromatic number at most k. Consequently, if the inside of I is large, then there is a large stable set, and then empty bipartite graph in it. Since every sparse graph has maximum degree εn or has no triangle, we can assume that for every I, the inside of I is bounded in size by some small δn. Now take I to be the rightmost k-subpath of P that has more right vertices than left ones. Observe that both the left and the right of I contain close to $(\frac{1}{2} - \delta)n$ vertices, hence a large empty bipartite graph. □

A graph on n vertices is an *ε-stable set* if it has at most $\varepsilon\binom{n}{2}$ edges. The complement of an ε-stable set is an *ε-clique*. Fox and Sudakov proved the following in [10]. A stronger version of the following result was proved by Rödl [13].

Theorem 5. *For every positive integer k and every $\varepsilon > 0$, there exists $\delta > 0$ such that every graph G on n vertices satisfies one of the following:*

- *G induces all graphs on k vertices.*
- *G contains an ε-stable set of size at least δn.*
- *G contains an ε-clique of size at least δn.*

The proof of Rödl is based on the Szemerédi's regularity lemma. The proof of Fox and Sudakov provides a much better estimate with $\delta = 2^{-ck(\log 1/\epsilon)^2}$ with a rather different method. We now prove our main result:

Theorem 6. *For every k, the class of graphs with no hole nor anti-hole of size at least k has the strong Erdős-Hajnal property.*

Proof. Let us first prove that we can restrict the problem to sparse connected graphs without long holes. Indeed, since G contains no hole of size k, it does not induce all graphs on k vertices, and Theorem 5 ensures that G contains an ϵ-clique or an ϵ-stable set of linear size. If G contains an ϵ-stable set X, then we delete all the vertices of degree at least $2\epsilon|X|$. Since the average degree is at most ϵ, at most one half of the vertices are deleted. The remaining vertices have maximum degree at most 2ϵ, which provides a 4ϵ-sparse graph. If G contains an ϵ-clique of linear size, then $\overline{G}$, which also satisfies the theorem hypotheses, contains a linear-size ϵ-stable set. Thus $\overline{G}$ contains an empty or complete linear-size bipartite graph, and symmetrically, so does G. Finally, we can assume that G is connected: it suffices to apply the theorem on a large connected component if any, or to assemble the connected components in order to get a large empty bipartite graph.
Let G be a connected sparse graph with no long hole. We consider the tree T resulting from our algorithm with an arbitrary root. To every node v of T we associate a weight equal to the number of vertices x of G with $r(x) = v$. Note that the total weight equals n. By Lemma 3, we find in T a path with weight at least $\frac{n}{4}$ or two unrelated subsets of size at least $\frac{n}{4}$. In the first case, the graph G contains a subgraph of size $\frac{n}{4}$ which is dominated by an induced path, and we conclude using Lemma 4. The second case yields an empty $\frac{n}{4}$-bipartite graph, as Lemma 2 ensures that there is no edge between vertices in two unrelated sets. □

Finally, we can prove Theorem 1 using the following classical result due to Alon et al. [1] and Fox and Pach [11].

Theorem 7 ([1, 11])**.** *If $\mathcal{C}$ is a class of graphs that admits the strong Erdős-Hajnal property, then $\mathcal{C}$ has the Erdős-Hajnal property.*

Sketch of the proof. Let $c > 0$. Assume that every graph of the class $\mathcal{C}$ has a complete cn-bipartite graph or an empty cn-bipartite graph. Let $c' > 0$ such that $c^{c'} \geq \frac{1}{2}$. We prove by induction that every graph G of $\mathcal{C}$ induces a P_4-free graph of size $n^{c'}$. By our hypothesis on $\mathcal{C}$, there exists, say, a complete $c \cdot n$-bipartite graph X, Y in G. By applying the induction hypothesis independently on X and Y, we form a P_4-free graph on $2(c \cdot n)^{c'} \geq n^{c'}$ vertices. The Erdős-Hajnal property of $\mathcal{C}$ follows from the fact that every P_4-free $n^{c'}$-graph has a clique or a stable set of size at least $n^{\frac{c'}{2}}$. □

Combining Theorem 6 and 7 ensures that graphs with no long hole nor anti-hole satisfy the Erdős-Hajnal conjecture.

References

[1] N. Alon, J. Pach, R. Pinchasi, R. Radoičić and M. Sharir, Crossing patterns of semi-algebraic sets. *Journal of Combinatorial Theory Series A*, 111(2):310–326, 2005.

[2] N. Alon, J. Pach, and J. Solymosi, Ramsey-type theorems with forbidden subgraphs. *Combinatorica*, 21:155–170, 2001.

[3] N. Bousquet, A. Lagoutte, and S. Thomassé, The Erdős-Hajnal Conjecture for Paths and Antipaths. *http://arxiv.org/abs/1303.5205*, preprint, 2013.

[4] M. Chudnovsky and S. Safra, The Erdős-Hajnal Conjecture for bull-free graphs. *Journal of Combinatorial Theory, Series B* 98 (2008) 1301–1310.

[5] M. Chudnovsky, The Erdős-Hajnal Conjecture - A Survey. *Journal of Graph Theory*, 75(2):178–190, 2014.

[6] M. Chudnovsky, P. Maceli and I. Penev, Excluding four-edge path and their complements. *Submitted*

[7] M. Chudnovsky and P. Seymour, Excluding paths and antipaths. Preprint, 2012.

[8] M. Chudnovsky and Y. Zwols, Large cliques or stable sets in graphs with no four-edge path and no five-edge path in the complement. *Journal of Graph Theory*, 70:449–472, 2012.

[9] P. Erdős and A. Hajnal, Ramsey-type theorems. *Discrete Applied Mathematics*, 25:37–52, 1989.

[10] J. Fox and B. Sudakov, Induced Ramsey-type theorems. *Advances in Mathematics*, 219:1771–1800, 2008.

[11] J. Fox and J. Pach, Erdős-Hajnal-type results on intersection patterns of geometric objetcs *Horizon of Combinatorics* (G.O.H. Katona et al., eds.), Bolyai Society Studies in Mathematics, Springer, 79–103, 2008.

[12] A. Gyárfás, Problems from the world surrounding perfect graphs. *Zastos. Mat.*, 413–441, 1987.

[13] V. Rödl, On universality of graphs with uniformly distributed edges. *Discrete Mathematics*, 59:125–134, 1986.