Checking the admissibility of odd-vertex pairings is hard

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Abstract

Nash-Williams proved that every graph has a well-balanced orientation. A key ingredient in his proof is admissible odd-vertex pairings. We show that for two slightly different definitions of admissible odd-vertex pairings, deciding whether a given odd-vertex pairing is admissible is co-NP-complete. This resolves a question of Frank. We also show that deciding whether a given graph has an orientation that satisfies arbitrary local arc-connectivity requirements is NP-complete.

1 Introduction

This article proves some negative results which are related to the strong orientation theorem of Nash-Williams.

Our graphs are undirected unless specified otherwise. Let $\boldsymbol{G = (V, E)}$ be a graph. For some disjoint $X, Y \subseteq V$, we use $\boldsymbol{d_G(X, Y)}$ for the number of edges that are incident to one vertex in X and one vertex in Y. We use $\boldsymbol{d_G(X)}$ for $d_G(X, V - X)$. For some integer k, we say that X is *k-edge-connected* if $d_G(X) \geq k$ for all nonempty $X \subset V$. We abbreviate 1-edge-connected to *connected*. A *connected component* of G is a maximal connected subgraph. We denote by $\boldsymbol{G[X]}$ the subgraph of G induced by X. For a single vertex v, we use $\boldsymbol{d_G(v)}$ for $d_G(\{v\})$ and call this number the *degree* of v. We call G *eulerian* if the degree of every vertex in V is even. For $s, t \in V$ and $X \subseteq V$, we say that X is an *$s\bar{t}$-set* if $s \in X$ and $t \in V - X$. We use $\boldsymbol{\lambda_G(s, t)}$ for the minimum of $d_G(X)$ over all $s\bar{t}$-sets X. By the undirected edge version of Menger's theorem [6], this is the same as the maximum size of a set of edge-disjoint st-paths in G. For some nonempty $X \subset V$, we use $\boldsymbol{R_G(X)}$ for $\max\{2\lfloor\frac{\lambda_G(s,t)}{2}\rfloor : X \text{ is an } s\bar{t}\text{-set}\}$. We define $R_G(\emptyset) = R_G(V) = 0$. For two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same vertex set V, we use $\boldsymbol{G_1 + G_2}$ for $(V, E_1 \cup E_2)$.

Let $\boldsymbol{D = (V, A)}$ be a directed graph. For some $X \subseteq V$, we use $\boldsymbol{d_D^-(X)}$ for the number of arcs in A entering X and $\boldsymbol{d_D^+(X)}$ for $d_D^-(V - X)$. For a single vertex v, we use $\boldsymbol{d_D^-(v)}$ and $\boldsymbol{d_D^+(v)}$ for $d_D^-(\{v\})$ and $d_D^+(\{v\})$, respectively. We call D *eulerian* if $d_D^-(v) = d_D^+(v)$ for all $v \in V$. We use $\boldsymbol{\lambda_D(s, t)}$ for the minimum of $d_D^+(X)$ over all $s\bar{t}$-sets X. By the directed arc version of Menger's theorem [6], this is the same as the maximum size of a set of arc-disjoint st-paths in D. For two directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ on the same vertex set V, we use $\boldsymbol{D_1 + D_2}$ for $(V, A_1 \cup A_2)$. A directed graph $\vec{G}$ that is obtained from a graph $G = (V, E)$ by choosing an orientation for each of its edges is called an *orientation* of G. The orientation $\vec{G}$ is called *well-balanced* if $\lambda_{\vec{G}}(s, t) \geq \lfloor \frac{\lambda_G(s,t)}{2} \rfloor$ for all $s, t \in V$.

In 1960, Nash-Williams proved the following celebrated theorem on well-balanced orientations [7].

Theorem 1. *Every graph has a well-balanced orientation.*

The key ingredient in the proof of Theorem 1 is the consideration of a new graph F on V such that F is a perfect matching on the vertices in V that are of odd degree in G. We call such a graph an *odd-vertex pairing* of G. Observe that if F is an odd-vertex pairing of G, then $G + F$ is eulerian. Nash-Williams proves the existence of an odd-vertex pairing F such that for every eulerian orientation $\vec{G} + \vec{F}$ of $G + F$, the restricted orientation $\vec{G}$ is a well-balanced orientation of G. We call an odd-vertex pairing F with this property *orientation-admissible.*

Actually, Nash-Williams proves the existence of an odd-vertex pairing with a somewhat stronger property: the odd-vertex pairings he finds satisfy the cut condition $d_G(X) - d_F(X) \geq R_G(X)$ for all $X \subseteq V$. We call such an odd-vertex pairing *cut-admissible.* It is easy to prove that every cut-admissible odd-vertex pairing is orientation-admissible. On the other hand, not every orientation-admissible odd-vertex pairing is cut-admissible. An example can be found in Figure 1.

The main difficulty in the proof of Theorem 1 is to show that for every graph, there is a cut-admissible odd-vertex pairing. This part of the proof is quite involved.

Király and Szigeti use the existence of an orientation-admissible pairing to prove the existence of well-balanced orientations with some extra properties [5]. Nevertheless, most algorithmic considerations related to well-balanced orientations remain hard to deal with due to the difficulty of the proof of Theorem 1. In [1], Bernáth et al. provide a collection of negative results for questions concerning well-balanced orientations with extra properties.

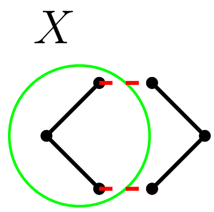

Figure 1: The edges of G are marked in solid and those of F are marked in dashed. The set X shows that F is not cut-admissible but F is trivially orientation-admissible.

This naturally raises the following question which is asked by Frank in [2] as Research Problem 9.8.1. For a given odd-vertex pairing, can its admissibility properties be checked efficiently? The purpose of this work is to give a negative answer to this question. More formally, we consider the following two problems:

CUT-ADMISSIBILITY (**CA**):

Instance: A graph G and an odd-vertex pairing F of G.

Question: Is F cut-admissible in G?

ORIENTATION-ADMISSIBILITY (**OA**):

Instance: A graph G and an odd-vertex pairing F of G.

Question: Is F orientation-admissible in G?

While it is not clear whether CA and OA are in NP, they can easily be seen to be in co-NP. As our main results, we prove the following two theorems.

Theorem 2. *CA is co-NP-complete.*

Theorem 3. *OA is co-NP-complete.*

In the last part of this article, we consider another problem on graph orientation. Given a graph G, we aim to find an orientation of G that meets arbitrary local arc-connectivity requirements. Formally, we consider the following problem:

LOCAL ARC-CONNECTIVITY ORIENTATION (**LACO**):

Instance: A graph G and a requirement function $r : V^2 \to \mathbb{Z}_{\geq 0}$.

Question: Is there an orientation $\vec{G}$ of G such that $\lambda_{\vec{G}}(u, v) \geq r(u, v)$ for all $u, v \in V^2$?

We were surprised not to find any previous work on the algorithmic tractability of this problem. By a reduction using one of the negative results in [1], we fill this gap.

Theorem 4. *LACO is NP-complete.*

While the proof of Theorems 2 and 3 is slightly involved, the proof of Theorem 4 is quite simple.

In Section 2, we give some preparatory results for the proof of Theorems 2 and 3. In Section 3, we give a reduction that serves as a proof for both Theorem 2 and Theorem 3. Finally, in Section 4, we prove Theorem 4.

2 Preliminaries

In this section, we collect some preliminary results we need in our reduction.

2.1 A modified MAXCUT problem

The unweighted MAXCUT problem can be formulated as follows:

MAXCUT:
Instance: A graph $H = (V, E)$ and a positive integer k.
Question: Is there some $X \subseteq V$ such that $d_H(X) > k$?

A proof of the following theorem can be found in [4].

Theorem 5. *MAXCUT is NP-hard.*

For our reduction in Section 3, we need a slightly adapted version of MAXCUT.

ADAPTED MAXCUT(**AMAXCUT**):
Instance: A graph $H = (V, E)$ such that $|E| \geq 6$ is even and $d_H(v)$ is even for all $v \in V$ and an even integer k.
Question: Is there some $X \subseteq V$ such that $d_H(X) > k$?

Lemma 1. *AMAXCUT is NP-hard.*

Proof. We show this by a reduction from MAXCUT. Let $(H = (V, E), k)$ be an instance of MAXCUT. We may obviously suppose that $|E| \geq 3$. Let $H' = (V, E')$ be the graph which is obtained from H by replacing every edge of E by 2 parallel copies of itself. Observe that $|E'| = 2|E| \geq 6$ is even and

$d_{H'}(v) = 2d_H(v)$ is even for all $v \in V$. Further, for every $X \subseteq V$, we have $d_{H'}(X) = 2d_H(X)$. This yields that (H, k) is a positive instance of MAXCUT if and only if $(H', 2k)$ is a positive instance of AMAXCUT.

□

2.2 Augmented (α, β)-grids

In this subsection, we introduce a class of grid-like graphs which will be used as a gadget in our reduction. A *grid* is a graph on ground set $\{1, \ldots, \mu\} \times \{1, \ldots, \nu\}$ for some positive integers μ, ν where two vertices (i_1, j_1) and (i_2, j_2) are adjacent if $|i_1 - i_2| + |j_1 - j_2| = 1$. For some $i \in \{1, \ldots, \mu\}$, we call $\{(i, 1), \ldots, (i, \nu)\}$ the *row* i. Similarly, for some $j \in \{1, \ldots, \nu\}$, we call $\{(1, j), \ldots, (\mu, j)\}$ the *column* j.

In order to define augmented (α, β)-grids for an odd integer $\boldsymbol{\alpha} \geq 3$ and an integer $\boldsymbol{\beta} \geq 2$, we first consider a grid with $\alpha\beta$ rows and $\frac{\alpha+1}{2}$ columns. Now, for some $1 \leq \gamma \leq \beta$, let $\boldsymbol{L_\gamma} = \{\boldsymbol{l_1, \ldots, l_\gamma}\} = \{(\alpha, 1), (2\alpha, 1), \ldots, (\gamma\alpha, 1)\}$ and $\boldsymbol{P_\gamma} = \{\boldsymbol{p_1, \ldots, p_\gamma}\} = \{(\alpha, \frac{\alpha+1}{2}), (2\alpha, \frac{\alpha+1}{2}), \ldots, (\gamma\alpha, \frac{\alpha+1}{2})\}$. We use $\boldsymbol{L}$ for L_β and $\boldsymbol{P}$ for P_β. We now create the *augmented* (α, β)*-grid* $\boldsymbol{W}$ by adding an edge from $(1, j)$ to $(\alpha\beta, j)$ for all $j = 1, \ldots, \frac{\alpha+1}{2}$ and by adding parallel edges in the columns 1 and $\frac{\alpha+1}{2}$ in a way that none of them is incident to a vertex in $L \cup P$ and that every vertex in $V(W) - (L \cup P)$ has degree 4 in W. Observe that this is possible because both $\alpha - 1$ and $\alpha + 1$ are even. An example can be found in Figure 2.

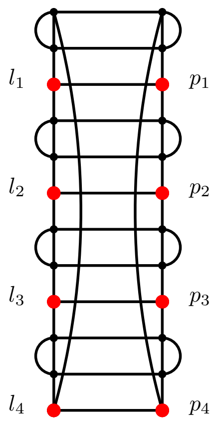

Figure 2: An augmented $(3, 4)$-grid.

Later, when W is not clear from the context, we use $\boldsymbol{L(W)}$ for the set L etc. We now collect some properties of augmented (α, β)-grids.

Lemma 2. *Let $W = (V, E)$ be an augmented (α, β)-grid for some odd integer $\alpha \geq 3$ and some integer $\beta \geq 2$. Then W is 3-edge-connected and if $d_W(X) = 3$ for some nonempty $X \subset V$, then $X = \{v\}$ or $X = V - \{v\}$ for some $v \in L(W) \cup P(W)$.*

Proof. Let $\emptyset \subset X \subset V$ such that $d_W(X) \leq 3$. Observe that every row that intersects both X and $V - X$ contributes at least 1 to $d_W(X)$ and every column that intersects both X and $V - X$ contributes at least 2 to $d_W(X)$. It follows that one of X or $V - X$ is contained in one row and one column. We obtain that $|X| = 1$ or $|V - X| = 1$ and so the statement follows by construction. □

Lemma 3. *Let $W = (V, E)$ be an augmented (α, β)-grid for some odd integer $\alpha \geq 3$ and some integer $\beta \geq 2$. Further, let $X \subseteq V$ such that both $W[X]$ and $W[V - X]$ have a connected component containing at least two vertices of $L(W) \cup P(W)$. Then $d_W(X) > \alpha$.*

Proof. Suppose for the sake of a contradiction that there is some $X \subseteq V$ such that both $W[X]$ and $W[V - X]$ have a connected component containing at least two vertices of $L(W) \cup P(W)$ and $d_W(X) \leq \alpha$. We choose X so that the total number of connected components of $W[X]$ and $W[V - X]$ is minimized. First suppose that $W[X]$ is disconnected. It follows from the assumption that $W[X]$ has a connected component C such that $W[X] - C$ has a connected component containing at least two vertices in $L(W) \cup P(W)$. Let $X' = X - V(C)$. We obtain $d_W(X') \leq d_W(X) \leq \alpha$, a contradiction to the minimal choice of X. It follows that $W[X]$ is connected. Similarly, $W[V - X]$ is connected.

If every column contains an element of X and an element of $V - X$, each column contributes 2 to $d_W(X)$ and so $d_W(X) \geq 2\frac{\alpha+1}{2} > \alpha$. We may hence suppose by symmetry that there is a column that is completely contained in X and that there are two vertices $l_{i_1}, l_{i_2} \in (V - X) \cap L$. Observe that every path from l_{i_1} to l_{i_2} intersects at least $|i_1 - i_2|\alpha + 1 > \alpha$ rows. Each of these rows contributes 1 to $d_W(X)$, so $d_W(X) > \alpha$.

□

2.3 Eulerian orientations

For the proof of the co-NP completeness of OA, we need the following result on eulerian orientations which can be found in [3].

Theorem 6. *Let G, F be graphs on the same vertex set V such that $G + F$ is an eulerian graph and let $\vec{F}$ be an orientation of F. Then there is an orientation $\vec{G}$ of G such that $\vec{G} + \vec{F}$ is eulerian if and only if $d_G(X) \geq d^+_{\vec{F}}(X) - d^-_{\vec{F}}(X)$ for all $X \subseteq V$.*

3 The reduction for admissibility

This section is dedicated to giving a reduction proving that CA and OA are co-NP-complete. In a first step, we reduce AMAXCUT to a problem which is somewhat similar to CA but has a more local cut condition. Next, we modify this construction to obtain a reduction for CA. Finally we show that the obtained instance is positive for OA if and only if it is positive for CA.

3.1 The intermediate cut problem

Let $\boldsymbol{(H = (V_H, E_H), k)}$ be an instance of AMAXCUT. We abbreviate $|V_H|$ and $|E_H|$ to $\boldsymbol{n}$ and $\boldsymbol{m}$, respectively. Let $\boldsymbol{M} = mn - k$. We now create a graph $\boldsymbol{G_1 = (V_1, E_1)}$ with $V_1 = V_H \cup \{q, s, t\}$ where $\boldsymbol{q, s}$ and $\boldsymbol{t}$ are 3 new vertices. Let E_1 consist of M edges from q to s, m edges from s to every $v \in V_H$ and m edges from t to every $v \in V_H$. A schematic drawing of G_1 can be found in Figure 3.

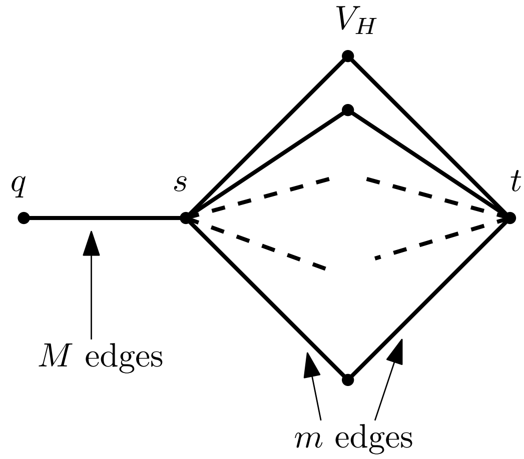

Figure 3: A schematic drawing of G_1.

Lemma 4. *There is some $q\bar{t}$-set $X \subseteq V_1$ such that $d_{G_1}(X) - d_H(X \cap V_H) < M$ if and only if (H, k) is a positive instance of AMAXCUT.*

Proof. First suppose that (H, k) is a positive instance of AMAXCUT, so there is some $X \subseteq V_H$ such that $d_H(X) > k$. Let $X' = \{q, s\} \cup X$. Observe that X' is a $q\bar{t}$-set and $d_{G_1}(X') = mn$. This yields $d_{G_1}(X') - d_H(X' \cap V_H) = d_{G_1}(X') - d_H(X) < M$.

Now suppose that there is some $q\bar{t}$-set $X \subseteq V_1$ such that $d_{G_1}(X) - d_H(X \cap V_H) < M$.

Claim 1. *$s \in X$.*

Proof. Suppose otherwise. If $X = \{q\}$, then $d_{G_1}(X) - d_H(X \cap V_H) = M - 0 \not< M$, a contradiction. We may hence suppose that X contains some $v \in V_H$. It follows from $d_H(X \cap V_H) \leq m$ and construction that $d_{G_1}(X) - d_H(X \cap V_H) \geq d_{G_1}(q, s) + d_{G_1}(v, t) - m = M + m - m \not< M$, a contradiction. □

By Claim 1 and construction, we obtain $d_{G_1}(X) = mn$. This yields $d_H(X \cap V_H) > d_{G_1}(X) - M = mn - M = k$, so (H, k) is a positive instance of AMAXCUT. □

3.2 The main construction

We now construct an instance (G_2, F) of CA. The graph $\boldsymbol{G_2 = (V_2, E_2)}$ is obtained from G_1 by replacing all vertices in $V_1 - \{q, t\}$ by certain gadgets.

For every $v \in V_H$, G_2 contains an augmented $(M + m + 1, m + \frac{d_H(v)}{2})$-grid W^v. Further, G_2 contains an augmented $(M+m+1, M+\frac{k}{2})$-grid W^s. Observe that W^v for all $v \in V_H$ and W^s are well-defined because m, k, M and $d_H(v)$ for all $v \in V_H$ are even. Let $V_2 = \cup_{v \in V_H} V(W^v) \cup V(W^s) \cup \{q, t\}$. We now add an edge from q to each vertex in $L_M(W^s)$. We next add a perfect matching between $(L(W^s) - L_M(W^s)) \cup P(W^s)$ and $\cup_{v \in V_H} L_m(W^v)$. Observe that this is possible because $|(L(W^s) - L_M(W^s)) \cup P(W^s)| = \frac{k}{2} + M + \frac{k}{2} = mn = |\cup_{v \in V_H} L_m(W^v)|$. Finally, we add an edge from every vertex in $\cup_{v \in V_H} P_m(W^v)$ to t. Observe that G_1 can be obtained from G_2 by contracting each W^v and W^s into single vertices.

We now prove an important property of G_2.

Lemma 5. *For any $\emptyset \subset X \subset V_2$, we have*

$$R_{G_2}(X) = 2\lfloor \frac{\min\{\max\{d_{G_2}(v) : v \in X\}, \max\{d_{G_2}(v) : v \in V_2 - X\}\}}{2} \rfloor.$$

Proof. As G_1 is 4-edge-connected and Lemma 2 applied to W^s and W^v for all $v \in V_H$, we obtain that $\lambda_{G_2}(u, v) = \min\{d_{G_2}(u), d_{G_2}(v)\}$ for all $u, v \in V_2$ with $\{u, v\} \neq \{q, t\}$. This shows the statement for all $\emptyset \subset X \subset V_2$ such that $\{q, t\} \subseteq X$ or $\{q, t\} \subseteq V_2 - X$. On the other hand, if X is a $q\bar{t}$-set or a $t\bar{q}$-set, we have $\min\{\max\{d_{G_2}(v) : v \in X\}, \max\{d_{G_2}(v) : v \in V_2 - X\}\} = M$. As M is even, it hence suffices to prove that $\lambda_{G_2}(q, t) = M$.

We have $\lambda_{G_2}(q, t) \leq d_{G_2}(q) = M$. Next, there is an edge between q and $l_{j_1}(W^s)$ for all $j_1 = 1, \dots, M$ which can be concatenated to a path from $l_{j_1}(W^s)$ to $p_{j_1}(W^s)$ using only vertices of a single row of W^s. Now there is an edge from $p_{j_1}(W^s)$ to a vertex $l_{j_2}(W^v)$ for some $j_2 \in \{1, \dots, m\}$ and some

$v \in V_H$. Finally, there is a path from $l_{j_2}(W^v)$ to $p_{j_2}(W^v)$ and an edge from $p_{j_2}(W^v)$ to t. This yields a set of M edge-disjoint qt-paths, so $\lambda_{G_2}(q,t) \geq M$. □

For some $v \in V_H$, let $\boldsymbol{B_v}$ denote $(L(W^v) - L_m(W^v)) \cup (P(W^v) - P_m(W^v))$. Now we define $\boldsymbol{F}$ to be an odd-vertex pairing of G_2 in the following way: For every $uv \in E_H$, F contains an edge between B_u and B_v. This is possible because for every $v \in V_H$, the set of vertices in $V(W^v)$ which are of odd degree in G_2 is exactly B_v and $|B_v| = d_H(v)$.

3.3 Reduction for CA

This subsection is dedicated to proving the following lemma which gives a relation of the cut sizes in G_1 and G_2.

Lemma 6. *(G_2, F) is a negative instance of CA if and only if there is some $q\bar{t}$-set $X \subseteq V_1$ such that $d_{G_1}(X) - d_H(X \cap V_H) < M$.*

Proof. First suppose that there is some $q\bar{t}$-set $X \subseteq V_1$ such that $d_{G_1}(X) - d_H(X \cap V_H) < M$. Let $X' \subseteq V_2$ be the set that contains $q \cup \cup_{v \in X} V(W^v)$ and that contains $V(W^s)$ if X contains s. Then Lemma 5 yields $d_{G_2}(X') - d_F(X') = d_{G_1}(X) - d_H(X \cap V_H) < M = R_{G_2}(X')$, so (G_2, F) is a negative instance of AC.

Now suppose that (G_2, F) is a negative instance of CA, so there is some $X \subset V_2$ such that $d_{G_2}(X) - d_F(X) < R_{G_2}(X)$. We choose $\boldsymbol{X}$ among all such sets such that $d_{G_2}(X)$ is minimal.

Claim 2. *Let $W \in W^s \cup \{W^v : v \in V_H\}$. Then each connected component of $W[X]$ or $W[V_2 - X]$ contains at least two vertices of $L(W) \cup P(W)$.*

Proof. By symmetry and as $d_{G_2}(X) = d_{G_2}(V_2 - X)$, it suffices to prove the statement for $W[X]$. For the sake of a contradiction, suppose that for the vertex set C of a connected component of $W[X]$, we have $|C \cap (L(W) \cup P(W))| \leq 1$.

First suppose that $X = C$. If X consists of a single vertex v with $d_F(v) = 1$, Lemma 5 yields $d_{G_2}(X) - d_F(X) = 3 - 1 = 2 = R_{G_2}(X)$, a contradiction. Otherwise, Lemma 2 yields $d_{G_2}(X) \geq 4$ and so, as $d_F(X) \leq 1$ and $G + F$ is eulerian, we obtain by Lemma 5 that $d_{G_2}(X) - d_F(X) \geq 4 = R_{G_2}(X)$, a contradiction.

We may hence suppose that $\boldsymbol{X'} = X - C$ is nonempty, so, by Lemma 5 and as $q, t \notin V(W)$, we have $R_{G_2}(X) - R_{G_2}(X') \leq 4 - 2 = 2$. If C consists of a single vertex v with $d_F(v) = 0$, we obtain $d_{G_2}(X') - d_F(X') \leq d_{G_2}(X) -$

$2 - d_F(X) \leq R_{G_2}(X) - 2 \leq R_{G_2}(X')$, a contradiction to the minimality of X. Otherwise, Lemma 2 yields $d_{G_2}(X) - d_{G_2}(X') \leq d_W(X) - 1 \leq 4 - 1 = 3$ and $d_F(X') - d_F(X) \leq 1$. This yields $d_{G_2}(X') - d_F(X') = (d_{G_2}(X) - 3) - (d_F(X) - 1) \leq R_{G_2}(X) - 2 \leq R_{G_2}(X')$, a contradiction to the minimality of X.

□

We are now ready to show that $V(W) \subseteq X$ or $V(W) \cap X \neq \emptyset$ for every $W \in W^s \cup \{W^v : v \in V_H\}$. Suppose otherwise, then by Claim 2, both $W[X]$ and $W[V_2 - X]$ have a connected component each containing at least two vertices of $L(W) \cup P(W)$. By Lemmas 3 and 5, this yields $d_{G_2}(X) - d_F(X) \geq M + m + 1 - m > M \geq R_{G'}(X)$, a contradiction.

Now let $X^* \subseteq V_1$ be the set of vertices that contains v whenever $V(W^v) \subseteq X$ and s if $V(W^s) \subseteq X$. Observe that $d_{G_2}(X) = d_{G_1}(X^*) \geq 2m$ by construction. Also, observe that $d_F(X) = d_H(X^* \cap V_H)$. By symmetry, we may suppose that $q \in X$. If X is not a $q\bar{t}$-set, Lemma 5 yields $d_{G_2}(X) - d_F(X) \geq d_{G_1}(X^*) - m \geq 2m - m = m > 4 \geq R_{G_2}(X)$, a contradiction. If X^* is a $q\bar{t}$-set, by Lemma 5, we obtain $d_{G_1}(X^*) - d_H(X^* \cap V_H) = d_{G_2}(X) - d_F(X) < R_{G_2}(X) = M$. □

3.4 Reduction for OA

The following result can be obtained by analogous methods to the proof of Lemma 6. Several arguments simplify.

Lemma 7. *There is no $X \subseteq V_2$ such that $d_{G_2}(X) < d_F(X)$.*

We here prove the following result that allows for a reduction for OA. While this proof does not require any new arguments apart from Lemma 7, we include it here for the sake of selfcontainment. The first implication is part of the proof of Nash-Williams of Theorem 1 in [7] while the second implication can be found in a similar form in [5].

Lemma 8. *(G_2, F) is a negative instance of OA if and only if (G_2, F) is a negative instance of CA.*

Proof. First suppose that (G_2, F) is a negative instance of OA. Then there is an eulerian orientation $\vec{G}_2 + \vec{F}$ of $G_2 + F$ such that $\vec{G}_2$ is not well-balanced. This means that there are some $u, v \in V_2$ such that $\lambda_{\vec{G}_2}(u, v) < \lfloor \frac{\lambda_{G_2}(u,v)}{2} \rfloor$. Therefore there is some $u\bar{v}$-set $X \subset V_2$ such that $d^+_{\vec{G}_2}(X) < \lfloor \frac{\lambda_{G_2}(u,v)}{2} \rfloor$. As $G_2 + F$ is eulerian, we obtain that $d_F(X) \geq d^-_{\vec{G}_2}(X) - d^+_{\vec{G}_2}(X) = d_{G_2}(X) -$

$2d^+_{\vec{G_2}}(X) > d_{G_2}(X) - 2\lfloor\frac{\lambda_{G_2}(s,t)}{2}\rfloor \geq d_{G_2}(X) - R_{G_2}(X)$, so (G_2, F) is a negative instance of CA.

For the other direction, suppose that (G_2, F) is a negative instance of CA, so there is some $X \subset V_2$ such that $d_{G_2}(X) - d_F(X) < R_{G_2}(X)$. Let $u \in X$ and $v \in V_2 - X$ such that $R_{G_2}(X) = 2\lfloor\frac{\lambda_{G_2}(u,v)}{2}\rfloor$. Let $\vec{F}$ be an orientation of F such that all the edges with exactly one endvertex in X are directed away from X. By Lemma 7 and Theorem 6, there is an orientation $\vec{G_2}$ of G_2 such that $\vec{G_2} + \vec{F}$ is eulerian. This yields $\lambda_{\vec{G_2}}(u,v) \leq d^+_{\vec{G_2}}(X) = \frac{1}{2}(d_{G_2}(X) + d_F(X)) - d^+_{\vec{F}}(X) = \frac{1}{2}(d_{G_2}(X) + d_F(X)) - d_F(X) = \frac{1}{2}(d_{G_2}(X) - d_F(X)) < \frac{1}{2}R_{G_2}(X) = \lfloor\frac{\lambda_{G_2}(u,v)}{2}\rfloor$. We obtain that $\vec{G_2}$ is not well-balanced, so (G_2, F) is a negative instance of OA. □

3.5 Conclusion

By Lemmas 4 and 6, we obtain that (G_2, F) is a negative instance of CA if and only if (H, k) is a positive instance of AMAXCUT. By Lemma 1 and as the size of (G_2, F) is polynomial in the size of (H, k), we obtain Theorem 2.

By Lemmas 4, 6 and 8, we obtain that (G_2, F) is a negative instance of OA if and only if (H, k) is a positive instance of AMAXCUT. By Lemma 1 and as the size of (G_2, F) is polynomial in the size of (H, k), we obtain Theorem 3.

4 Local arc-connectivity orientation

This section is dedicated to proving Theorem 4. We need to consider the following algorithmic problem.

Bounded well-balanced orientation (BWBO)

Instance A graph $G = (V, E)$ and two functions $l^+, l^- : V \to \mathbb{Z}_{\geq 0}$.

Question Is there a well-balanced orientation $\vec{G}$ of G such that $d^+_{\vec{G}}(v) \geq l^+(v)$ and $d^-_{\vec{G}}(v) \geq l^-(v)$ for all $v \in V$?

The following result is proven in [1].

Lemma 9. *BWBO is NP-hard.*

We are now ready to give the reduction for Theorem 4.

Proof. (of Theorem 4)

We prove this by a reduction from BWBO. Let $(\boldsymbol{G = (V, E), l^+, l^-})$ be an instance of BWBO. We add two vertices x and y and for every $v \in V$, we add $d_G(v)$ edges between v and each of x and y. We denote this graph by $\boldsymbol{G' = (V', E')}$. Observe that $|V'| = |V| + 2$ and $|E'| = 5|E|$, so the size of G' is polynomial in the size of G. We now define $\boldsymbol{r}$: $(V')^2 \to \mathbb{Z}_{\geq 0}$ by $r(u,v) = \lfloor \frac{\lambda_G(u,v)}{2} \rfloor$ for all $u, v \in V^2$, $r(x,v) = d_G(v) + l^-(v), r(v,x) = 0, r(y,v) = 0$ and $r(v,y) = d_G(v) + l^+(v)$ for all $v \in V$ and $r(x,y) = 2|E|$.

We prove that (G', r) is a positive instance of LACO if and only if (G, l^+, l^-) is a positive instance of BWBO. First suppose that (G', r) is a positive instance of LACO, so there is an orientation $\vec{G'}$ of G' such that $\lambda_{\vec{G'}}(u,v) \geq r(u,v)$ for all $u, v \in (V')^2$. Observe that $d_{G'}(x) = r(x,y) = d_{G'}(y)$, so x is a source and y is a sink in $\vec{G'}$. We show that $\vec{G}$, the restriction of $\vec{G'}$ to G is a well-balanced orientation $\vec{G}$ of G such that $d^+_{\vec{G}}(v) \geq l^+(v)$ and $d^-_{\vec{G}}(v) \geq l^-(v)$ for all $v \in V$. As x is a source and y is a sink in $\vec{G'}$, for any $u, v \in V^2$, we have $\lambda_{\vec{G}}(u,v) = \lambda_{\vec{G'}}(u,v) \geq r(u,v) = \lfloor \frac{\lambda_G(u,v)}{2} \rfloor$, so $\vec{G}$ is well-balanced. Further, for any $v \in V$, we have $d^-_{\vec{G}}(v) = d^-_{\vec{G'}}(v) - d_{\vec{G'}}(x,v) \geq \lambda_{\vec{G'}}(x,v) - d_{\vec{G'}}(x,v) \geq r(x,v) - d_{\vec{G'}}(x,v) = d_G(v) + l^-(v) - d_G(v) = l^-(v)$. Similarly, $d^+_{\vec{G}}(v) \geq l^+(v)$, so (G, l^+, l^-) is a positive instance of $BWBO$.

Now suppose that (G, l^+, l^-) is a positive instance of $BWBO$, so there is a well-balanced orientation $\vec{G}$ of G such that $d^+_{\vec{G}}(v) \geq l^+(v)$ and $d^-_{\vec{G}}(v) \geq l^-(v)$ for all $v \in V$. We complete this to an orientation $\vec{G'}$ of G' by orienting all edges incident to x away from x and all edges incident to y toward y. As $\vec{G}$ is well-balanced, we have $\lambda_{\vec{G'}}(u,v) = \lambda_{\vec{G}}(u,v) \geq \lfloor \frac{\lambda_G(u,v)}{2} \rfloor = r(u,v)$ for all $u, v \in V^2$. By construction, we have $\lambda_{\vec{G'}}(x,y) = \sum_{v \in V} d_G(V) = 2|E| = r(x,y)$. For any $v \in V$, we have $d_G(v)$ arc-disjoint xv-paths of length 1. Further, for every arc uv entering v in $\vec{G}$, we have a path xuv. As all these paths can be chosen to be arc-disjoint, we obtain that $\lambda_{\vec{G'}}(x,v) \geq d_{\vec{G'}}(x,v) + d^-_{\vec{G}}(v) \geq d_G(v) + l^-(v) = r(x,v)$. Similarly, $\lambda_{\vec{G'}}(v,y) \geq r(v,y)$, so (G', r) is a positive instance of LACO. □

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