# MATROID FRAGILITY AND RELAXATIONS OF CIRCUIT HYPERPLANES

JIM GEELEN AND FLORIAN HOERSCH

Abstract. We relate two conjectures that play a central role in the reported proof of Rota's Conjecture. Let $\mathbb{F}$ be a finite field. The first conjecture states that: the branch-width of any $\mathbb{F}$-representable $N$-fragile matroid is bounded by a function depending only upon $\mathbb{F}$ and $N$. The second conjecture states that: if a matroid $M_2$ is obtained from a matroid $M_1$ by relaxing a circuit-hyperplane and both $M_1$ and $M_2$ are $\mathbb{F}$-representable, then the branch-width of $M_1$ is bounded by a function depending only upon $\mathbb{F}$. Our main result is that the second conjecture implies the first.

## 1. Introduction

The purpose of this paper is to relate two concepts, $N$-fragile matroids and circuit-hyperplane relaxations, which both play a central role in the reported proof of Rota's Conjecture [1].

A matroid $M$ is *$N$-fragile* if $N$ is a minor of $M$, but there is no element $e \in E(M)$ such that $N$ is a minor of both $M\backslash e$ and $M/e$ or, equivalently, there is a unique partition $(C, D)$ of $E(M) - E(N)$ such that $N = M/C\backslash D$. Note that here we want $N$, itself, as a minor, not just an isomorphic copy of $N$.

For a finite field $\mathbb{F}$ of order $q$, we let $\mathbb{F}^k$ denote an extension field of $\mathbb{F}$ of order $q^k$. We prove the following result.

**Theorem 1.1.** *Let $\mathbb{F}$ be a finite field, let $N$ be a matroid with $k$ elements, let $B$ be a basis of $N$, and let $M$ be an $\mathbb{F}$-representable $N$-fragile matroid. Then there exist $\mathbb{F}^{2k^2}$-representable matroids $M_1$ and $M_2$ on the same ground set and elements $c, d \in E(M_1)$ such that $M_2$ is obtained from $M_1$ by relaxing a circuit-hyperplane and $M/B\backslash(E(N) - B) = M_1/c\backslash d$.*

*Date*: September 9, 2019.
1991 *Mathematics Subject Classification.* 05B35.
*Key words and phrases.* Rota's Conjecture, matroid fragility, circuit-hyperplane relaxation.
This research was partially supported by grants from the Office of Naval Research [N00014-10-1-0851] and NSERC [203110-2011].

The proof of Rota's Conjecture relies on the reported proofs of the following two conjectures by Geelen, Gerards, and Whittle.

**Conjecture 1.2.** *Let $\mathbb{F}$ be a finite field and let $N$ be a matroid. Then the branch-width of any $\mathbb{F}$-representable $N$-fragile matroid is bounded by a constant depending only upon $|\mathbb{F}|$ and $|N|$.*

For the definition of branch-width see Oxley [2]. For this paper it suffices to know that branch-width is a parameter associated with a matroid $M$, which we denote here by $\mathrm{bw}(M)$, and that for any minor $N$ of $M$ we have

$$\mathrm{bw}(M) - (|E(M)| - |E(N)|) \leq \mathrm{bw}(N) \leq \mathrm{bw}(M).$$

**Conjecture 1.3.** *Let $H$ be a circuit-hyperplane in a matroid $M_1$ and let $M_2$ be the matroid obtained by relaxing $H$. If $M_1$ and $M_2$ are both representable over a finite field $\mathbb{F}$, then the branch-width of $M_1$ is bounded by a constant depending only upon $|\mathbb{F}|$.*

Theorem 1.1 shows that Conjecture 1.3 implies Conjecture 1.2.

Our proof of Theorem 1.1 is via a sequence of results on matrices, but those results have interesting consequences for matroids, which we state below.

We call a matroid *isolated* if each of its components has only one element. Thus an isolated matroid consists only of loops and coloops; the set of coloops is the unique basis. The isolated matroid on ground set $E$ with basis $B$ is denoted $\mathrm{ISO}(B, E)$. For integers $r$ and $n$ with $0 \leq r \leq n$ we denote $\mathrm{ISO}(\{1, \ldots, r\}, \{1, \ldots, n\})$ by $\mathrm{ISO}(r, n)$.

The following result shows that, in order to prove Theorem 1.1, it suffices to consider the case that $N$ is an isolated matroid.

**Theorem 1.4.** *Let $\mathbb{F}$ be a finite field, let $B$ be a basis of a matroid $N$, and let $M$ be an $\mathbb{F}$-representable $N$-fragile matroid. Then there exists an $\mathbb{F}$-representable $\mathrm{ISO}(B, E(N))$-fragile matroid $M'$ such that $E(M') = E(M)$ and $M'/B = M/B$.*

The following result shows that, in order to prove Theorem 1.1, it suffices to consider the case that $N = \mathrm{ISO}(1, 2)$.

**Theorem 1.5.** *Let $\mathbb{F}$ be a finite field, let $X_1$ and $X_2$ be disjoint finite sets with $|X_1 \cup X_2| = k$, let $M$ be an $\mathbb{F}$-representable $\mathrm{ISO}(X_1, X_1 \cup X_2)$-fragile matroid, and let $c$ and $d$ be distinct elements not in $M$. Then there exists an $\mathbb{F}^{k^2}$-representable $\mathrm{ISO}(\{c\}, \{c, d\})$-fragile matroid $M'$ such that $E(M') = E(M) - (X_1 \cup X_2) \cup \{c, d\}$ and $M'/c\backslash d = M/X_1\backslash X_2$.*

The final result shows that an $\mathbb{F}$-representable $\mathrm{ISO}(1,2)$-fragile matroid has a circuit-hyperplane whose relaxation results in an $\mathbb{F}^2$-representable matroid.

**Theorem 1.6.** *Let $N = \mathrm{ISO}(\{c\},\{c,d\})$ where $c \neq d$, let $M$ be an $N$-fragile matroid representable over a finite field $\mathbb{F}$, and let $C$ and $D$ be disjoint subsets of $E(M)$ such that $N = M/C\backslash D$. Then $C \cup \{d\}$ is a circuit-hyperplane of $M$ and the matroid obtained from $M$ by relaxing $C \cup \{d\}$ is $\mathbb{F}^2$-representable.*

Observe that Theorem 1.1 is an immediate consequence of Theorems 1.4, 1.5, and 1.6.

We assume that the reader is familiar with elementary matroid theory; we use the terminology and notation of Oxley [2].

## 2. Fragile matrices

In this section we will give a matrix interpretation for minor-fragility in representable matroids. Towards this end, we develop convenient terminology for viewing a representable matroid with respect to a fixed basis.

For a basis $B$ of a matroid $M$ and a set $X \subseteq E(M)$ we denote the minor $M/(B-X)\backslash(E(M)-(B\cup X))$ of $M$ by $M[X,B]$. The following result is routine and well-known.

**Lemma 2.1.** *If $N$ is a minor of a matroid $M$, then there is a basis $B$ of $M$ such that $N = M[E(N),B]$.*

If $B$ is a basis of a matroid $M$ and $N = M[E(N),B]$, then we say that $B$ *displays* $N$.

When we refer to a matrix $A \in \mathbb{F}^{S_1\times S_2}$ we are implicitly defining $\mathbb{F}$ to be a field and $S_1$ and $S_2$ to be finite sets. Let $A \in \mathbb{F}^{S_1\times S_2}$ be a matrix where $S_1$ and $S_2$ are disjoint. We let $[I,A]$ denote the matrix obtained from $A$ by appending an $S_1 \times S_1$ identity matrix; thus $[I,A] \in \mathbb{F}^{S_1\times(S_1\cup S_2)}$. For $X \subseteq S_1 \cup S_2$, we let $A[X]$ denote the submatrix $A[X\cap S_1, X\cap S_2]$.

If $B$ is a basis of an $\mathbb{F}$-representable matroid $M$, then there is a matrix $A \in \mathbb{F}^{B\times E(M)-B}$ such that $M = M([I,A])$; we call $A$ a *standard representation* with respect to $B$. Note that, if $N$ is a minor of $M$ displayed by $B$ and $A$ is a standard representation of $M$ with respect to $B$, then $A[E(N)]$ is a standard representation of $N$ with respect to the basis $B \cap E(N)$.

For a finite set $X$, a matrix $A \in \mathbb{F}^{S_1\times S_2}$ is called $X$-*fragile* if

- $S_1$ and $S_2$ are disjoint,

- $X \subseteq S_1 \cup S_2$,
- $A[X] = 0$, and
- for each nonempty subset $Y$ of $(S_1 \cup S_2) - X$, we have $\text{rank}(A[X \cup Y]) > \text{rank}(A[Y])$.

Note that, if $A \in \mathbb{F}^{S_1 \times S_2}$ is an $X$-fragile matrix, then $M([I, A[X]]) = \text{ISO}(X \cap S_1, X)$.

The following result provides us with a matrix interpretation of minor-fragility for representable matroids.

**Lemma 2.2.** *Let $N$ be a matroid, let $M$ be an $\mathbb{F}$-representable $N$-fragile matroid, let $B$ be a basis of $M$ that displays $N$, and let $A$ be a standard representation of $M$ with respect to $B$. If $A'$ is the matrix obtained from $A$ by replacing each entry in the submatrix $A[E(N)]$ with $0$, then $A'$ is $E(N)$-fragile.*

*Proof.* Let $X = E(N)$. Suppose that $A'$ is not $X$-fragile. Then there is a non-empty set $Y \subseteq E(M) - X$ such that $\text{rank}(A'[X \cup Y]) = \text{rank}(A'[Y])$. By removing the other elements, we may assume that $E(M) = X \cup Y$. Let $C = B \cap Y$, $D = Y - B$, and let $B_N = B \cap E(N)$. Observe that $\text{rank}(A') = \text{rank}(A[C, D])$ by the choice of $Y$. We will obtain a contradiction to the fact that $M$ is $N$-fragile by showing that $N = M/D \backslash C$.

We start by constructing an isomorphic copy $A_0$ of $A'[B, X - B]$ by relabelling the columns so that the indices form a set $Z$ disjoint from $E(N)$. Now let $A_1 = [A, A_0]$ and $M_1 = M([I, A_1])$.

We claim that:

(i) $N = (M_1/Z)|X$, and
(ii) $B_N$ is independent in $M_1/(D \cup Z)$, and
(iii) $Z$ is a set of loops in $M_1/D$.

Note that $Z$ is a set of loops in $M_1/C$ and $N$ is a minor of $M_1/C$, so $M_1/Z$ contains $N$ as a minor. To show that $N$ is a restriction of $M_1/Z$ it suffices to show that $B_N$ spans $E(N)$ in $M_1/Z$, or, equivalently, that $B_N \cup Z$ spans $E(N)$ in $M_1$, which is clear from the construction. This proves $(i)$.

Note that $r_{M_1}(B_N \cup D \cup Z) = |B_N| + \text{rank}(A_1[C, D \cup Z]) = |B_N| + \text{rank}(A'[C, D \cup X]) = |B_N| + \text{rank}(A[C, D]) = |B_N| + \text{rank}(A[B, D])$, since $\text{rank}(A') = \text{rank}(A[C, D])$. Therefore $B_N$ is independent in in $M_1/(D \cup Z)$, proving $(ii)$.

Now $(iii)$ follows directly from the fact that $\text{rank}(A') = \text{rank}(A[C, D])$.

By $(iii)$, we have $M/D = (M_1/D) \backslash Z = (M_1/D)/Z$. By $(i)$, $N$ is a restriction of $M_1/Z$. By $(ii)$, the sets $B_N$ and $D$ are skew in $M_1/Z$

(that is, $r_{M_1/Z}(B_N \cup D) = r_{M_1/Z}(B_N) + r_{M_1/Z}(D)$), and hence $N$ is a restriction of $M_1/(D \cup Z)$. However $M/D = M_1/(D \cup Z)$, contradicting the fact that $M$ is $N$-fragile. □

The converse of Lemma 2.2 is not true in general, but the following result is a weak converse, and it implies Theorem 1.4.

**Lemma 2.3.** *If $A \in \mathbb{F}^{S_1 \times S_2}$ is an $X$-fragile matrix, where $X \subseteq S_1 \cup S_2$, then $M([I, A])$ is $\mathrm{ISO}(X \cap S_1, X)$-fragile.*

*Proof.* Let $M = M([I, A])$. Note that $M[X, S_1] = \mathrm{ISO}(X \cap S_1, X)$. Let $C$ and $D$ be a partition of $E(M) - X$ such that $C \neq S_1 - X$. We will prove that $M/C \backslash D \neq \mathrm{ISO}(X \cap S_1, X)$. By contracting $C \cap S_1$ and deleting $D - S_1$ we may assume that $D = S_1 - X$ and that $C = S_2 - X$.

Since $A$ is $X$-fragile, $\mathrm{rank}(A[D, C]) < \mathrm{rank}(A)$. Now either

(i) $\mathrm{rank}(A[D, C]) < \mathrm{rank}(A[S_1, C])$, or
(ii) $\mathrm{rank}(A[S_1, C]) < \mathrm{rank}(A)$.

In case (*i*), we have $r_{M/C}(S_1 \cap X) = r_M(C \cup (S_1 \cap X)) - r_M(C) = |S_1 \cap X| + \mathrm{rank}(A[D, C]) - \mathrm{rank}(A[S_1, C]) < |S_1 \cap X|$. So $S_1 \cap X$ is dependent in $M/C$ and hence $M/C \backslash D \neq \mathrm{ISO}(X \cap S_1, X)$, as required.

In case (*ii*), we have $r_{M/C}(X - S_1) = r_M((X - S_1) \cup C) - r_M(C) = \mathrm{rank}(A) - \mathrm{rank}(A[S_1, C]) > 0$, so $M/C \backslash D \neq \mathrm{ISO}(X \cap S_1, X)$, as required. □

## 3. Reduction to ISO(1, 2)-fragility

The results in this section prove Theorem 1.5.

Let $F$ be a flat of a matroid $M$. We say that a matroid $M'$ is obtained by *adding an element $e$ freely to $F$ in $M$* if $M'$ is a single-element extension by a new element $e$ in such a way that $F$ spans $e$ and that each flat of $M' \backslash e$ that spans $e$ contains $F$.

**Lemma 3.1.** *Let $M$ be an $\mathrm{ISO}(X_1, X_1 \cup X_2)$-fragile matroid, where $X_1$ and $X_2$ are disjoint finite sets, and let $M'$ be obtained from $M$ by adding a new element $d$ freely into the flat spanned by $X_2$. Then $M' \backslash X_2$ is $\mathrm{ISO}(X_1, X_1 \cup \{d\})$-fragile.*

*Proof.* Let $(C, D)$ be a partition of $E(M) - (X_1 \cup X_2)$. It suffices to show that $M/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup X_2)$ if and only if $(M' \backslash X_2)/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup \{d\})$. Note that $M'/C \backslash D$ is obtained from $M/C \backslash D$ by adding $d$ freely to the flat spanned by $X_2$. If $M/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup X_2)$, then $M'/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup X_2 \cup \{d\})$ and hence $(M' \backslash X_2)/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup \{d\})$. Conversely, if $(M' \backslash X_2)/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup \{d\})$, then $M'/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup X_2 \cup \{d\})$ and hence $(M' \backslash \{d\})/C \backslash D = \mathrm{ISO}(X_1, X_1 \cup X_2)$, as required. □

Note that, by Lemma 3.1, we can reduce an $\mathrm{ISO}(X_1, X_1 \cup X_2)$-fragile matroid to an $\mathrm{ISO}(X_1, X_1 \cup \{d\})$-fragile matroid. Repeating this in the dual we can further reduce to an $\mathrm{ISO}(\{c\}, \{c, d\})$-fragile matroid.

We can add an element freely into a flat in a represented matroid by going to a sufficiently large extension field; this is both routine and well-known.

**Lemma 3.2.** *Let $A \in \mathbb{F}^{S_1 \times S_2}$, let $M = M(A)$, let $X$ be a $k$-element subset of $S_2$, and let $M'$ be the matroid obtained from $M$ by adding a new element $e$ freely into the flat spanned by $X$. Then there is a vector $b \in (\mathbb{F}^k)^{S_1}$ such that $[A, b]$ is a representation of $M'$ over $\mathbb{F}^k$.*

*Proof.* Let $A_v$ denote the column of $A$ that is indexed by $v$. The elements of the field $\mathbb{F}^k$ form a vectorspace of dimension $k$ over $\mathbb{F}$; let $(\alpha_v : v \in X)$ be a basis of this vectorspace. Now let $b = \sum_{v \in X} \alpha_v A_v$ and let $M' = M([A, b])$. By construction, the new element $e$ of $M'$ is spanned by $X$. It remains to show that each flat of $M' \backslash e$ that spans $e$ also spans $X$. Consider an independent set $I \subseteq E(M)$ that does not span $X$ in $M$. We may apply elementary row-operations over $\mathbb{F}$ so that each column of $I$ contains exacly one non-zero entry. Let $R \subseteq S_1$ denote the set of rows containing non-zero entries in $A[S_1, I]$. Since $I$ does not span $X$, there exists $i \in S_1 - R$ such that $A[\{i\}, X]$ is not identically zero. However the entries of $A[\{i\}, X]$ are all in $\mathbb{F}$ and the values $(\alpha_v : v \in X)$ are linearly independent over $\mathbb{F}$, so $b_i = \sum_{v \in X} \alpha_v A_{i,v} \neq 0$. Hence $I$ does not span $e$ in $M'$, as required. □

## 4. Relaxing a circuit-hyperplane

The following result implies Theorem 1.6.

**Lemma 4.1.** *Let $\mathbb{F}$ be a field and $\mathbb{F}'$ be a field extension. Now let $A_1 \in \mathbb{F}^{S_1 \times S_2}$ be a $\{c, d\}$-fragile matrix where $c \in S_1$ and $d \in S_2$ and let $A_2$ be obtained from $A_1$ by replacing the $(c, d)$-entry with an element in $\mathbb{F}' - \mathbb{F}$. Then $(S_1 - \{c\}) \cup \{d\}$ is a circuit-hyperplane in $M([I, A_1])$ and $M([I, A_2])$ is the matroid obtained from $M([I, A_1])$ by relaxing $(S_1 - \{c\}) \cup \{d\}$.*

*Proof.* Let $M_1 = M([I, A_1])$, $M_2 = M([I, A_2])$, and $H = (S_1 - \{c\}) \cup \{d\}$. We claim that $H$ is a circuit of $M_1$; suppose otherwise. Note that $S_1$ is a basis, so $S_1 \cup \{d\}$ contains a unique circuit $C$. Since $A_1$ is $\{c, d\}$-fragile, we have $A[\{c\}, \{d\}] = 0$, and hence $c \notin C$. Since $H$ is not a circuit, there exists $e \in S_1 - \{c\}$ such that $e$ is a coloop of $M_1|(S_1 \cup \{d\})$. Then $(M_1|(S_1 \cup \{d\})) \backslash e = (M_1|(S_1 \cup \{d\}))/e$. But then $M_1$ is not $\mathrm{ISO}(\{c\}, \{c, d\})$-fragile, contrary to Lemma 2.3. Thus $H$ is a circuit as claimed.

Note that $M_1^* = M([A_1^T, I])$ and that $A_1^T$ is $\{c,d\}$-fragile. Then, by duality, $E(M_1) - H$ is a cocircuit and, hence, $H$ is a circuit-hyperplane.

To prove that $M_2$ is obtained from $M_1$ by relaxing $H$ it suffices to show, for each set $Z \subseteq S_1 \cup S_2$, that $\operatorname{rank} A_1[Z] \neq \operatorname{rank} A_2[Z]$ if and only if $Z = \{c,d\}$. Note that $\operatorname{rank} A_1[\{c,d\}] \neq \operatorname{rank} A_2[\{c,d\}]$. Consider a set $Z \subseteq S_1 \cup S_2$ such that $\operatorname{rank} A_1[Z] \neq \operatorname{rank} A_2[Z]$.

**Claim:** *We have* $\operatorname{rank} A_1[Z] < \operatorname{rank} A_2[Z]$.

*Proof of claim.* Suppose for a contradiction that $\operatorname{rank} A_1[Z] > \operatorname{rank} A_2[Z]$ and consider a minimal subset $X \subseteq Z$ such that $\operatorname{rank} A_1[X] > \operatorname{rank} A_2[X]$. Thus $A_1[X]$ is square and non-singular, $A_2[X]$ is singular, and $c, d \in X$. Let $B(x)$ denote the matrix obtained from $A_1[X]$ by replacing the $(c,d)$-entry with a variable $x$ and let $p(x) = \det(B(x))$. Note that $p(x) = \alpha x + \beta$ where $\alpha, \beta \in \mathbb{F}$. Since $A_1[X]$ is non-singular, we have $p(0) \neq 0$. Therefore $p(x)$ has at most one root and, since $\alpha, \beta \in \mathbb{F}$, if $p(x)$ has a root, that root is in $\mathbb{F}$. However, this contradicts the fact that $A_2[X]$ is singular. □

By construction, $c, d \in Z$ and we may assume that $Z \neq \{c,d\}$. Then, since $A_1$ is $\{c,d\}$-fragile,

$$\begin{aligned}
\operatorname{rank} A_1[Z - \{c,d\}] &\leq \operatorname{rank} A_1[Z] - 1 \\
&\leq \operatorname{rank} A_2[Z] - 2 \\
&\leq \operatorname{rank} A_2[Z - \{c,d\}] \\
&= \operatorname{rank} A_1[Z - \{c,d\}].
\end{aligned}$$

Hence $\operatorname{rank} A_1[Z] = \operatorname{rank} A_1[Z - \{c,d\}] + 1$ and $\operatorname{rank} A_2[Z] = \operatorname{rank} A_2[Z - \{c,d\}] + 2$. This second equation implies that $\operatorname{rank} A_2[Z - \{c\}] = \operatorname{rank} A_2[Z - \{c,d\}] + 1$. Therefore $\operatorname{rank} A_1[Z - \{c\}] = \operatorname{rank} A_1[Z - \{c,d\}] + 1$ and hence $\operatorname{rank} A_1[Z - \{c\}] = \operatorname{rank} A_1[Z]$. Thus the row $c$ of $A_1[Z]$ is a linear combination of the other rows. But then the row $c$ of $A_1[Z - \{d\}]$ is a linear combination of the other rows. So $\operatorname{rank} A_1[Z - \{d\}] = \operatorname{rank} A_1[Z - \{c,d\}]$ and, hence, $\operatorname{rank} A_2[Z - \{d\}] = \operatorname{rank} A_2[Z - \{c,d\}]$. However, this contradicts the fact that $\operatorname{rank} A_2[Z] = \operatorname{rank} A_2[Z - \{c,d\}] + 2$. □

## References

[1] J. Geelen, B. Gerards, G. Whittle, Solving Rota's Conjecture, Notices of the AMS 61 (2014), 736-743.

[2] J. Oxley, *Matroid Theory, second edition*, Oxford University Press, New York, (2011).

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada