REACHABILITY IN ARBORESCENCE PACKINGS*

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Abstract. Fortier et al. [4] proposed several research problems on packing arborescences. Some of them were settled in that paper and others were solved later by Matsuoka and Tanigawa [11] and Gao and Yang [8]. The last open problem will be settled in this paper. We show how to turn an inductive idea used in the last two articles into a simple proof technique that allows to relate previous results on arborescence packings.

We show how a strong version of Edmonds' theorem [3] on packing spanning arborescences implies Kamiyama, Katoh and Takizawa's result [9] on packing reachability arborescences and how Durand de Gevigney, Nguyen and Szigeti's theorem [2] on matroid-based packing of arborescences implies Király's result [10] on matroid-reachability-based packing of arborescences.

Finally, we deduce a new result on matroid-reachability-based packing of mixed hyperarborescences from a theorem on matroid-based packing of mixed hyperarborescences due to Fortier et al. [4].

All the proofs provide efficient algorithms to find a solution to the corresponding problems.

Key words. arborescence, packing, matroid

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1. Introduction.

This paper deals with the packing of arborescences. We focus on concluding characterizations of graphs admitting a packing of reachability-based arborescences from the corresponding theorems for spanning arborescences in several settings. We first give an overview of the results in this article. All technical terms which are not defined here will be explained in Section 3.

In 1973, Edmonds [3] characterized digraphs having a packing of k spanning r-arborescences for some $k \in \mathbb{Z}_+$ and for some vertex r. Since then, there have been numerous generalizations of this result. A first attempt is to allow different roots for the arborescences. A version with arbitrary, fixed roots can easily be derived from the theorem of Edmonds. This generalization has a significant deficiency occuring when some vertex is not reachable from some designated root. In this case, the only information it provides is that the desired packing does not exist. A concept to overcome this problem has been developed by Kamiyama, Katoh and Takizawa in [9]. Given a digraph D, can we find a packing of arborescences such that each of them spans all the vertices reachable from the root designated to it? They provide a characterization of these graphs. We reprove their theorem by a reduction from a stronger form of Edmonds' theorem.

Another way of generalizing the requirements on the packing of arborescences was introduced by Durand de Gevigney, Nguyen and Szigeti in [2]. Instead of requiring every vertex to be spanned by all arborescences, it is required to be spanned only by the arborescences which are associated to a basis of an arbitrary matroid where every arborescence is associated to an element of the matroid. Surprisingly, a characterization of graph-matroid pairs admitting such a packing of arborescences in this very general setting was found in [2]. A natural combination of the two aforementioned generalizations was introduced by Király [10]. He requires every vertex only to be spanned by a set of arborescences associated to a matroid basis of the set associated to the arborescences that could potentially reach the vertex. He provided a charac-

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terization of the graph-matroid pairs admitting such a packing of arborescences. We reprove this theorem by concluding it from the theorem in [2].

Finally, there are attempts to also generalize the objects considered from digraphs to more general objects like mixed graphs or dypergraphs. We consider a concept unifying all of these generalizations where we want to find a matroid-reachability based packing of mixed hyperarborescences in a matroid-rooted mixed hypergraph. We derive a characterization of these mixed hypergraph-matroid pairs from a characterization for the existence of a matroid-based packing of mixed hyperarborescences in a matroid-rooted mixed hypergraph by Fortier et al. in [4]. All our proofs are algorithmic.

In Section 3, we provide a more technical and detailed overview of the results considered. In Section 4, we give the reductions that yield our new proofs. Section 5 deals with the algorithmic impacts of our results.

2. Definitions.

In this section we provide the definitions and notation needed in the paper. For basic notions of matroid theory, we refer to [5], chapter 5.

2.1. Directed graphs.

We first provide some basic notation on *directed graphs (digraphs)*. Let $D = (V, A)$ be a digraph. For disjoint $X, Y \subseteq V$, we denote the set of arcs with tail in X and head in Y by $\boldsymbol{\rho_A(X, Y)}$ and $|\rho_A(X, Y)|$ by $\boldsymbol{d_A(X, Y)}$. We use $\boldsymbol{\rho^+_A(X)}$ for $\rho_A(X, V - X)$, $\boldsymbol{\rho^-_A(X)}$ for $\rho_A(V - X, X)$, $\boldsymbol{d^+_A(X)}$ for $|\rho^+_A(X)|$ and $\boldsymbol{d^-_A(X)}$ for $|\rho^-_A(X)|$. We denote by $\boldsymbol{N^+_D(X)}$ and $\boldsymbol{N^-_D(X)}$ the set of out-neighbors and in-neighbors of X, respectively. For a single vertex v, we abbreviate $\rho^+_A(\{v\})$ to $\boldsymbol{\rho^+_A(v)}$ etc.. We call v a *root* in D if $d^-_A(v) = 0$ and a *simple* root if additionally $d^+_A(v) \leq 1$.

An *arborescence* is a subgraph of D in which no circuit exists and every vertex except one has in-degree 1. Observe that every arborescence contains a unique root. An arborescence whose unique root is a vertex r is also called an r*-arborescence*. An arborescence B is said to *span* $V(B)$. A subgraph of D is called a *spanning arborescence* if it is an arborescence and it spans all the vertices of D. By a *packing of arborescences* or *arborescence packing* in D, we mean a set of arc-disjoint arborescences in D.

For $u, v \in V$, we say that v is *reachable* from u in D if there exists a directed path from u to v. For $X \subseteq V$, we denote by $\boldsymbol{U^D_X}$ the set of vertices which are reachable from at least one vertex in X, by $\boldsymbol{P^D_X}$ the set of vertices from which X is reachable and by $\boldsymbol{D[X]}$ the subgraph of D induced on X.

We define a *(simply) rooted digraph* as a digraph $D = (V \cup R, A)$ with $\boldsymbol{R}$ being a set of (simple) roots. A *(simply) matroid-rooted digraph* is a tuple $(D, \mathcal{M})$ where $\boldsymbol{D} = (V \cup R, A)$ is a (simply) rooted digraph and $\boldsymbol{\mathcal{M}} = (R, r_{\mathcal{M}})$ is a matroid with ground set R and rank function $\boldsymbol{r_{\mathcal{M}}}$. Note that a rooted digraph can be considered as a matroid-rooted digraph for the free matroid on R. Given a matroid-rooted digraph $(D = (V \cup R, A), \mathcal{M} = (R, r_{\mathcal{M}}))$, we call an arborescence packing $\{B_r\}_{r \in R}$ *matroid-based* (*matroid-reachability-based*) if for all $r \in R$, the unique root of B_r is r and for all $v \in V$, $\{r \in R : v \in V(B_r)\}$ is a basis of R (of $P^D_v \cap R$) in $\mathcal{M}$. We speak of a *spanning arborescence packing* and a *reachability arborescence packing*, respectively, if $\mathcal{M}$ is the free matroid on R.

2.2. Mixed hypergraphs.

We now turn our attention to the generalizations of the concept of arborescences from digraphs to more general objects, namely mixed hypergraphs.

A *mixed hypergraph* is a tuple $\mathcal{H} = (V, \mathcal{A} \cup \mathcal{E})$ where $\boldsymbol{V}$ is a set of vertices, $\boldsymbol{\mathcal{A}}$ is a set of directed hyperedges (dyperedges) and $\boldsymbol{\mathcal{E}}$ is a set of hyperedges. A *dyperedge* a is a tuple $(tail(a), head(a))$ where $\boldsymbol{head(a)}$ is a single vertex in V and $\boldsymbol{tail(a)}$ is a nonempty subset of $V - head(a)$ and a *hyperedge* is a subset of V of size at least two. A mixed hypergraph without hyperedges is called a *directed hypergraph (dypergraph)*. We say that $\mathcal{H}$ is a mixed graph if each dyperedge has a tail of size exactly one and each hyperedge has exactly two vertices.

Let $X \subseteq V$. We say that dyperedge $a \in \mathcal{A}$ *enters* X if $head(a) \in X$ and $tail(a) - X \neq \emptyset$ and a *leaves* X if a enters $V - X$. We denote by $\boldsymbol{\rho^-_{\mathcal{A}}(X)}$ the set of dyperedges entering X and by $\boldsymbol{\rho^+_{\mathcal{A}}(X)}$ the set of dyperedges leaving X. We use $\boldsymbol{d^-_{\mathcal{A}}(X)}$ for $|\rho^-_{\mathcal{A}}(X)|$ and $\boldsymbol{d^+_{\mathcal{A}}(X)}$ for $|\rho^+_{\mathcal{A}}(X)|$. We say that a hyperedge e *enters* or *leaves* X if e intersects both X and $V - X$ and denote by $\boldsymbol{d_{\mathcal{E}}(X)}$ the number of hyperedges entering X. We call a vertex r a *root* in $\mathcal{H}$ if $d^-_{\mathcal{A}}(r) = d_{\mathcal{E}}(r) = 0$ and $tail(a) = \{r\}$ for all $a \in \rho^+_{\mathcal{A}}(r)$ and a *simple* root if additionally $d^+_{\mathcal{A}}(r) \leq 1$. Given a subpartition $\{V_i\}_1^\ell$ of V, we denote by $\boldsymbol{e_{\mathcal{E}}(\{V_i\}_1^\ell)}$ the number of hyperedges in $\mathcal{E}$ entering some V_i $(i \in \{1, \ldots, \ell\})$.

Trimming a dyperedge a means that a is replaced by an arc uv with $v = head(a)$ and $u \in tail(a)$. *Trimming* a hyperedge e means that e is replaced by an arc uv for some $u \neq v \in e$. The mixed hypergraph $\mathcal{H}$ is called a *mixed hyperpath* (*mixed hyperarborescence*) if all the dyperedges and all the hyperedges can be trimmed to get a directed path (an arborescence). A *mixed r-hyperarborescence* for some $r \in V$ is a mixed hyperarborescence together with a vertex r where that arborescence can be chosen to be an r-arborescence.

For a vertex set $X \subseteq V$, we denote by $\boldsymbol{U^{\mathcal{H}}_X}$ the set of vertices which are reachable from the vertices in X by a mixed hyperpath in $\mathcal{H}$, by $\boldsymbol{P^{\mathcal{H}}_X}$ the set of vertices from which X is reachable by a mixed hyperpath in $\mathcal{H}$ and by $\boldsymbol{\mathcal{H}[X]}$ the mixed subhypergraph of $\mathcal{H}$ induced on X. A *strongly connected component* of a mixed hypergraph is a maximal set of vertices that can be pairwise reached from each other by a mixed hyperpath.

We define a *(simply) rooted mixed hypergraph* as a mixed hypergraph $\mathcal{H} = (V \cup R, \mathcal{A} \cup \mathcal{E})$ with $\boldsymbol{R}$ being a set of (simple) roots. A *(simply) matroid-rooted mixed hypergraph* is a tuple $(\mathcal{H}, \mathcal{M})$ where $\mathcal{H} = (V \cup R, \mathcal{A} \cup \mathcal{E})$ is a (simply) rooted mixed hypergraph and $\boldsymbol{\mathcal{M}} = (R, r_{\mathcal{M}})$ is a matroid with ground set R and rank function $\boldsymbol{r_{\mathcal{M}}}$. Note that a rooted mixed hypergraph can be considered as a matroid-rooted mixed hypergraph for the free matroid on R. A mixed hyperarborescence packing $\{\mathcal{B}_r\}_{r \in R}$ is called *matroid-based* if every $\mathcal{B}_r$ can be trimmed to an r-arborescence B_r such that $\{B_r\}_{r \in R}$ is a matroid-based arborescence packing. A mixed hyperarborescence packing $\{\mathcal{B}_r\}_{r \in R}$ is called *matroid-reachability-based* if every $\mathcal{B}_r$ can be trimmed to an r-arborescence B_r such that for all $v \in V$, $\{r \in R : v \in V(B_r)\}$ is a basis of $P^{\mathcal{H}}_v \cap R$ in $\mathcal{M}$. We speak of a *spanning mixed hyperarborescence packing* and a *reachability mixed hyperarborescence packing*, respectively, if $\mathcal{M}$ is the free matroid on R.

2.3. Bisets. Finally, we need to introduce some notation on bisets. Given some ground set V, a *biset* X consists of an *outer set* $X_O \subseteq V$ and an *inner set* $X_I \subseteq X_O$. We denote $X_O - X_I$ by $\boldsymbol{w(\mathsf{X})}$. For a vertex set $C \subseteq V$, a collection of bisets $\{\mathsf{X}^i\}_1^\ell$ is called a *biset subpartition* of C if $\{X_I^i\}_1^\ell$ is a subpartition of C and $w(\mathsf{X}^i) \subseteq V - C$ for $i = 1, \ldots, \ell$. In a mixed hypergraph $\mathcal{H} = (V, \mathcal{A} \cup \mathcal{E})$, we say that a dyperedge $a \in \mathcal{A}$ *enters* X (or $a \in \rho^-_{\mathcal{A}}(\mathsf{X})$) if $tail(a) - X_O \neq \emptyset$ and $head(a) \in X_I$.

3. Results. This section introduces all the results considered and shows how our contributions relate to the previous results.

3.1. Reachability in digraphs. The starting point of all studies on packing arborescences is the following theorem of Edmonds [3] mentioned in a simpler form in the introduction.

THEOREM 3.1. *([3]) Let $D = (V \cup R, A)$ be a simply rooted digraph. Then there exists a spanning arborescence packing $\{B_r\}_{r \in R}$ in D if and only if for all $X \subseteq V \cup R$ with $X - R \neq \emptyset$,*

$$d_A^-(X) \geq |R - X|. \tag{3.1}$$

We first mention a generalization of Theorem 3.1 omitting the simplicity condition that was found by Edmonds himself in [3]. Its proof is significantly more complicated than the one of Theorem 3.1.

THEOREM 3.2. *([3]) Let $D = (V \cup R, A)$ be a rooted digraph. Then there exists a spanning arborescence packing $\{B_r\}_{r \in R}$ in D if and only if for all $X \subseteq V \cup R$ with $X - R \neq \emptyset$,*

$$d_A^-(X) \geq |R - X|. \tag{3.2}$$

We now turn our attention to packing reachability arborescences. The following result of Kamiyama, Katoh and Takizawa [9] generalizes Theorem 3.2.

THEOREM 3.3. *([9]) Let $D = (V \cup R, A)$ be a rooted digraph. Then there exists a reachability arborescence packing $\{B_r\}_{r \in R}$ in D if and only if for all $X \subseteq V \cup R$ with $X - R \neq \emptyset$,*

$$d_A^-(X) \geq |P_X^D \cap R| - |X \cap R|. \tag{3.3}$$

Our first contribution is to show that surprisingly Theorem 3.2 implies Theorem 3.3. The very simple inductive proof can be found in Section 4.

3.2. Reachability and matroids. We now present another way of generalizing the concepts above, namely matroid-based packings and matroid-reachability-based packings.

The following result on matroid-based arborescence packing is due to Durand de Gevigney, Nguyen and Szigeti [2].

THEOREM 3.4. *([2]) Let $(D = (V \cup R, A), \mathcal{M} = (R, r_{\mathcal{M}}))$ be a simply matroid-rooted digraph. Then there exists a matroid-based arborescence packing in $(D, \mathcal{M})$ if and only if for all nonempty $X \subseteq V \cup R$ with $X \cap R = span_{\mathcal{M}}(N_D^-(X \cap V))$,*

$$d_A^-(X) \geq r_{\mathcal{M}}(R) - r_{\mathcal{M}}(X \cap R). \tag{3.4}$$

We now consider a reachability extension of Theorem 3.4. We first show that the simplicity condition in Theorem 3.4 can be omitted. This result might also be interesting for itself. It plays the same role for matroid-based packings as Theorem 3.2 played for basic packings. Interestingly, while the proof of Theorem 3.2 is self-contained and rather technical, the stronger matroid setting allows to directly derive Theorem 3.5 from Theorem 3.4.

THEOREM 3.5. *Let $(D = (V \cup R, A), \mathcal{M} = (R, r_{\mathcal{M}}))$ be a matroid-rooted digraph. Then there exists a matroid-based arborescence packing in $(D, \mathcal{M})$ if and only if for all nonempty $X \subseteq V \cup R$ with $X \cap R = span_{\mathcal{M}}(N_D^-(X \cap V) \cap R)$,*

$$d_A^-(X) \geq r_{\mathcal{M}}(R) - r_{\mathcal{M}}(X \cap R). \tag{3.5}$$

A reachability extension of Theorem 3.4 was obtained by Király [10]. We deduce the following stronger version of it from Theorem 3.5 in Section 4.

THEOREM 3.6. *([10]) Let $(D = (V \cup R, A), \mathcal{M} = (R, r_{\mathcal{M}}))$ be a matroid-rooted digraph. Then there exists a matroid-reachability-based arborescence packing in $(D, \mathcal{M})$ if and only if for all $X \subseteq V \cup R$ with $X - R \neq \emptyset$,*

$$d_A^-(X) \geq r_{\mathcal{M}}(P_X^D \cap R) - r_{\mathcal{M}}(X \cap R). \tag{3.6}$$

3.3. Generalizations. This part deals with another way of generalizing Theorem 3.1: rather than changing the requirements on the packing, one can consider changing the basic objects of consideration from digraphs to more general objects. One such generalization was suggested by Frank, Király and Király [7]. They considered dypergraphs instead of digraphs and they generalized Theorem 3.1 to dypergraphs. A result where the concepts of reachability and dypergraphs were combined was obtained by Bérczi and Frank in [1]. Yet another class Theorem 3.1 can be generalized to was considered by Frank in [6]: mixed graphs. He gave a characterization of mixed graphs admitting a mixed spanning arborescence packing.

A natural question now is whether several of the aforementioned generalizations can be combined into a single one. In [4], the authors surveyed all possible combinations of these generalizations and gave an overview of all existing results. A significant amount of cases was covered by Fortier et al [4]. They first prove a characterization combining the concepts of dypergraphs, matroids and reachability. They further prove a theorem that combines the concepts of matroids, hypergraphs and mixed graphs. We make use of the following characterization for the last result in this article.

THEOREM 3.7. *([4]) Let $(\mathcal{H} = (V \cup R, \mathcal{A} \cup \mathcal{E}), \mathcal{M} = (R, r_{\mathcal{M}}))$ be a simply matroid-rooted mixed hypergraph. Then there exists a matroid-based mixed hyperarborescence packing in $(\mathcal{H}, \mathcal{M})$ if and only if for every biset subpartition $\{\mathsf{X}^i\}_1^\ell$ of V with $w(\mathsf{X}^i) = span_{\mathcal{M}}(\{r \in R : N_{\mathcal{H}}^+(r) \cap X_I^i \neq \emptyset\})$ for $i = 1, \dots, \ell$,*

$$e_{\mathcal{E}}(\{X_I^i\}_1^\ell) + \sum_{i=1}^{\ell} d_{\mathcal{A}}^-(\mathsf{X}^i) \geq \sum_{i=1}^{\ell} (r_{\mathcal{M}}(R) - r_{\mathcal{M}}(w(\mathsf{X}^i))). \tag{3.7}$$

3.4. Reachability and mixed graphs. Theorem 3.7 had a lot of corollaries generalizing Theorem 3.1, however, the cases of combinations including both reachability and mixed graphs remained open. They seemed hard to deal with as all natural generalizations failed. Indeed, it turned out that the remaining cases required a deeper concept, namely the use of bisets. While the use of bisets in our statement of Theorem 3.7 is only for convenience, it is essential in the following theorems.

The following theorem is equivalent to the result of Matsuoka and Tanigawa [11] on reachability mixed arborescence packing, as it was shown in [8].

THEOREM 3.8. *([11]) Let $F = (V \cup R, A \cup E)$ be a rooted mixed graph. Then there exists a reachability mixed arborescence packing $\{B_r\}_{r \in R}$ in F if and only if for every biset subpartition $\{\mathsf{X}^i\}_1^\ell$ of a strongly connected component C of F such that $w(\mathsf{X}^i) = P_{w(\mathsf{X}^i)}^F$ for all $i = 1, \dots, \ell$,*

$$e_E(\{X_I^i\}_1^\ell) + \sum_{i=1}^{\ell} d_A^-(\mathsf{X}^i) \geq \sum_{i=1}^{\ell} (|P_C^F \cap R| - |X_O^i \cap R|). \tag{3.8}$$

The next step was made by Gao and Yang who managed to generalize Theorem 3.8 to the matroidal case by proving the following result [8].

THEOREM 3.9. *([8]) Let $(F=(V\cup R, A\cup E), \mathcal{M}=(R, r_{\mathcal{M}}))$ be a matroid-rooted mixed graph. Then there exists a matroid-reachability-based mixed arborescence packing in $(F,\mathcal{M})$ if and only if for every biset subpartition $\{\mathsf{X}^i\}_1^\ell$ of a strongly connected component C of $F-R$ such that $w(\mathsf{X}^i)=P^F_{w(\mathsf{X}^i)}$ for all $i=1,\dots,\ell$,*

$$e_E(\{X_I^i\}_1^\ell)+\sum_{i=1}^{\ell} d_A^-(\mathsf{X}^i)\geq \sum_{i=1}^{\ell}(r_{\mathcal{M}}(P_C^F\cap R)-r_{\mathcal{M}}(X_O^i\cap R)). \tag{3.9}$$

3.5. New results. The remaining open problems were the generalizations of Theorems 3.8 and 3.9 to mixed hypergraphs. Proving such generalizations is the last contribution of this article. While such a result can be obtained by the proof technique used by Gao and Yang for Theorem 3.9, we follow a different approach: we derive such a characterization from Theorem 3.7. Again, we first show that the simplicity condition in Theorem 3.7 can be omitted.

THEOREM 3.10. *Let $(\mathcal{H}=(V\cup R,\mathcal{A}\cup\mathcal{E}), \mathcal{M}=(R, r_{\mathcal{M}}))$ be a matroid-rooted mixed hypergraph. Then there exists a matroid-based mixed hyperarborescence packing in $(\mathcal{H},\mathcal{M})$ if and only if for every biset subpartition $\{\mathsf{X}^i\}_1^\ell$ of V with $w(\mathsf{X}^i)=span_{\mathcal{M}}(\{r\in R: N^+_{\mathcal{H}}(r)\cap X_I^i\neq\emptyset\})$ for $i=1,\dots,\ell$,*

$$e_{\mathcal{E}}(\{X_I^i\}_1^\ell)+\sum_{i=1}^{\ell} d_{\mathcal{A}}^-(\mathsf{X}^i)\geq \sum_{i=1}^{\ell}(r_{\mathcal{M}}(R)-r_{\mathcal{M}}(w(\mathsf{X}^i))). \tag{3.10}$$

Theorem 3.10 allows us to derive the following new theorem. Observe that this is a common generalization of all the theorems mentioned before in this article. It includes all the theorems surveyed in [4].

THEOREM 3.11. *Let $(\mathcal{H}=(V\cup R,\mathcal{A}\cup\mathcal{E}), \mathcal{M}=(R, r_{\mathcal{M}}))$ be a matroid-rooted mixed hypergraph. Then there exists a matroid-reachability-based mixed hyperarborescence packing in $(\mathcal{H},\mathcal{M})$ if and only if for every biset subpartition $\{\mathsf{X}^i\}_1^\ell$ of a strongly connected component C of $\mathcal{H}-R$ such that $w(\mathsf{X}^i)=P^{\mathcal{H}}_{w(\mathsf{X}^i)}$ for all $i=1,\dots,\ell$,*

$$e_{\mathcal{E}}(\{X_I^i\}_1^\ell)+\sum_{i=1}^{\ell} d_{\mathcal{A}}^-(\mathsf{X}^i)\geq \sum_{i=1}^{\ell}(r_{\mathcal{M}}(P_C^{\mathcal{H}}\cap R)-r_{\mathcal{M}}(X_O^i\cap R)). \tag{3.11}$$

We obtain the only remaining case, a generalization of Theorem 3.8 to mixed hypergraphs as a corollary by applying Theorem 3.11 to the free matroid.

COROLLARY 3.12. *Let $\mathcal{H}=(V\cup R,\mathcal{A}\cup\mathcal{E})$ be a rooted mixed hypergraph. Then there exists a reachability mixed hyperarborescence packing $\{\mathcal{B}_r\}_{r\in R}$ in $\mathcal{H}$ if and only if for every biset subpartition $\{\mathsf{X}^i\}_1^\ell$ of a strongly connected component C of $\mathcal{H}-R$ such that $w(\mathsf{X}^i)=P^{\mathcal{H}}_{w(\mathsf{X}^i)}$ for all $i=1,\dots,\ell$,*

$$e_{\mathcal{E}}(\{X_I^i\}_1^\ell)+\sum_{i=1}^{\ell} d_{\mathcal{A}}^-(\mathsf{X}^i)\geq \sum_{i=1}^{\ell}(|P_C^{\mathcal{H}}\cap R|-|X_O^i\cap R|). \tag{3.12}$$

4. Reductions. This section contains the proofs of the old and new theorems that we mentioned before. All the proofs work by reductions from the spanning versions to the reachability versions.

4.1. Proof of Theorem 3.3. The proof uses Theorem 3.2 and is self-contained otherwise.

Proof. (of **Theorem 3.3**) Necessity is evident.

For sufficiency, let $\boldsymbol{D} = (V \cup R, A)$ be a minimum counterexample. Obviously, $V \neq \emptyset$.

Let $\boldsymbol{C} \subseteq V$ be the vertex set of a strongly connected component of D that has no arc leaving. Since each $r \in R$ is a root, C exists. Note that each vertex of C is reachable in D from the same set of roots since $D[C]$ is strongly connected. We can hence divide the problem into two subproblems, a smaller one on reachability arborescence packing and one on spanning arborescence packing.

Let $\boldsymbol{D_1} = (\boldsymbol{V_1} \cup R, \boldsymbol{A_1}) = D - C$. Note that D_1 is a rooted digraph.

LEMMA 4.1. *D_1 has a reachability arborescence packing $\{\boldsymbol{B^1_r}\}_{r\in R}$.*

Proof. By $d^+_A(C) = 0$, we have $d^-_{A_1}(X) = d^-_A(X)$ and $P^{D_1}_X = P^D_X$ for all $X \subseteq V_1 \cup R$. Then, since D satisfies (3.3), so does D_1. Hence, by the minimality of D, the desired packing exists in D_1. ◻

Let $\boldsymbol{D_2} = (V_2 \cup R_2, A_2)$ be the rooted digraph where $\boldsymbol{V_2} = C \cup T$, $\boldsymbol{T} = \{$new vertices $\boldsymbol{t_{uv}}$: $uv \in \rho^-_A(C)\}$, $\boldsymbol{R_2} = P^D_C \cap R$ and $\boldsymbol{A_2} = A(D[C]) \cup \{rt_{uv} : r \in R_2, u \in U^D_r, t_{uv} \in T\} \cup \{t_{uv}v, |R_2| * vt_{uv} : t_{uv} \in T\}$.

LEMMA 4.2. *D_2 has a spanning arborescence packing $\{\boldsymbol{B^2_r}\}_{r\in R}$.*

Proof. We show in the following claim that D_2 satisfies (3.2).

CLAIM 4.3. *$d^-_{A_2}(X) \geq |R_2 - X|$ for all $X \subseteq V_2 \cup R_2$ with $X - R_2 \neq \emptyset$.*

Proof. If $X \cap C = \emptyset$, then $d^-_{A_2}(X) \geq d_{A_2}(v, t_{uv}) = |R_2| \geq |R_2 - X|$ for some $t_{uv} \in X - R_2$. If $X \cap C \neq \emptyset$, then, since $D[C]$ is strongly connected, $R_2 = P^D_C \cap R = P^D_{X\cap C} \cap R$. Let $\boldsymbol{Y} = (V \cup R) - U^D_{R_2 - X}$, $\boldsymbol{Z} = (X \cap C) \cup Y$ and $\boldsymbol{uv} \in \rho^-_A(Z)$. Since $\rho^-_A(Y) = \emptyset$, $v \in X \cap C$. If $u \in C$, then $uv \in \rho^-_{A_2}(X)$. If $u \notin C$, then $u \in U^D_r$ for some $\boldsymbol{r} \in R_2 - X$ and $t_{uv} \in T$, so $rt_{uv}, t_{uv}v \in A_2$. Since $v \in X$ and $r \notin X$, rt_{uv} or $t_{uv}v \in \rho^-_{A_2}(X)$. Thus, by (3.3), $d^-_{A_2}(X) \geq d^-_A(Z) \geq |(P^D_Z - Z) \cap R| = |R_2 - X|$. ◻

By Claim 4.3 and Theorem 3.2, the desired packing exists in D_2. This completes the proof of Lemma 4.2. ◻

With the help of the packings $\{B^1_r\}_{r\in R}$ in D_1 and $\{B^2_r\}_{r\in R_2}$ in D_2 obtained in Lemmas 4.1 and 4.2, a packing in D can be constructed yielding a contradiction.

LEMMA 4.4. *D has a reachability arborescence packing.*

Proof. For all $r \in R - R_2$, let $\boldsymbol{B_r} = B^1_r$ and for all $r \in R_2$, let $\boldsymbol{B_r}$ be obtained from the union of B^1_r and $B^2_r - (R_2 \cup T)$ by adding the arc uv for all $t_{uv}v \in A(B^2_r)$. Since $\{B^1_r\}_{r\in R}$ and $\{B^2_r\}_{r\in R_2}$ are packings, so is $\{B_r\}_{r\in R}$. For $r \in R - R_2$, $B_r = B^1_r$ is an r-arborescence and it spans $U^{D_1}_r = U^D_r$. Let now $r \in R_2$. Since B^1_r and B^2_r do not contain circuits, neither does B_r. Since for all $v \in V(B^1_r) - r$, $d^-_{A(B^1_r)}(v) = 1$, for all $v \in C$, $d^-_{A(B^2_r)}(v) = 1$ and when $t_{uv}v \in A(B^2_r)$ is replaced by $uv \in A(B_r)$ then $u \in U^D_r$, we have for all $v \in V(B_r) - r$, $d^-_{A(B_r)}(v) = 1$. It follows that B_r is an r-arborescence. Since B^1_r spans $U^{D_1}_r$ and B^2_r spans $V_2 \cup r$, B_r spans $U^{D_1}_r \cup C = U^D_r$. It follows that $\{B_r\}_{r\in R}$ has the desired properties. This completes the proof of Lemma 4.4. ◻

Lemma 4.4 contradicts the fact that D is a counterexample and hence the proof of Theorem 3.3 is complete. ◻

4.2. Proof of Theorems 3.5 and 3.6. In this section, the generalization to matroids is considered.

We first derive Theorem 3.5 from Theorem 3.4. The strong matroid setting allows for a rather simple proof.

Proof. (of **Theorem 3.5**) Necessity is evident.

For sufficiency, let $(\boldsymbol{D'} = (\boldsymbol{V} \cup \boldsymbol{R'}, \boldsymbol{A'}), \boldsymbol{\mathcal{M}'} = (\boldsymbol{R'}, \boldsymbol{r_{\mathcal{M}'}}))$ be the simply matroid-rooted digraph obtained from $(D, \mathcal{M})$ by replacing every root $r \in R$ by a set $\boldsymbol{Q_r}$ of $|N_D^+(r)|$ simple roots in the digraph such that $N_{D'}^+(Q_r) = N_D^+(r)$ and by $|Q_r|$ parallel copies of r in the matroid.

Now let $\boldsymbol{X'} \subseteq V \cup R'$ with $X' \cap R' = span_{\mathcal{M}'}(N_{D'}^-(X' \cap V) \cap R')$. Observe that for every $r \in R$, either $Q_r \subseteq X'$ or $Q_r \cap X' = \emptyset$. Let $\boldsymbol{X} = (X' \cap V) \cup \{r \in R : Q_r \subseteq X'\}$. Observe that $X \cap R = span_{\mathcal{M}}(N_D^-(X \cap V) \cap R)$. Further, we have $d_A^-(X) = d_{A'}^-(X')$, $r_{\mathcal{M}}(R) = r_{\mathcal{M}'}(R')$ and $r_{\mathcal{M}}(X \cap R) = r_{\mathcal{M}'}(X' \cap R')$. Then, by (3.5), we obtain $d_{A'}^-(X') = d_A^-(X) \geq r_{\mathcal{M}}(R) - r_{\mathcal{M}}(X \cap R) = r_{\mathcal{M}'}(R') - r_{\mathcal{M}'}(X' \cap R')$, that is $(D', \mathcal{M}')$ satisfies (3.4). We can now apply Theorem 3.4 to obtain in $(D', \mathcal{M}')$ a matroid-based arborescence packing $\{\boldsymbol{B'_{r'}}\}_{r' \in R'}$.

For all $r \in R$, let $\boldsymbol{B_r}$ be obtained from $\{B'_{r'}\}_{r' \in Q_r}$ by contracting Q_r into r. Since $\{B'_{r'}\}_{r' \in R'}$ is a packing, so is $\{B_r\}_{r \in R}$. Let $r \in R$. Since $\{r' \in R' : v \in V(B'_{r'})\}$ is independent in $\mathcal{M}'$ for all $v \in V$ and Q_r is a set of parallel elements in $\mathcal{M}'$, $\{B'_{r'}\}_{r' \in Q_r}$ is a set of vertex-disjoint r'-arborescences in D' and hence B_r is an r-arborescence in D. Moreover, for all $v \in V$, $r_{\mathcal{M}}(\{r \in R : v \in V(B_r)\}) = r_{\mathcal{M}'}(\{r' \in R' : v \in V(B'_{r'})\}) = r_{\mathcal{M}'}(R') = r_{\mathcal{M}}(R)$. Thus the packing $\{B_r\}_{r \in R}$ of arborescences has the desired properties. ◻

We are now ready to derive Theorem 3.6 from Theorem 3.5. The role of Theorem 3.5 in the proof is similar to the role of Theorem 3.2 in the proof of Theorem 3.3. While the proof contains similar ideas to the ones in the proof of Theorem 3.3, it is somewhat more technical.

Proof. (of **Theorem 3.6**) Necessity is evident.

For sufficiency, let $(\boldsymbol{D} = (V \cup R, A), \boldsymbol{\mathcal{M}} = (R, r_{\mathcal{M}}))$ be a minimum counterexample. Obviously $V \neq \emptyset$. Let $\boldsymbol{C} \subseteq V$ be the vertex set of a strongly connected component of D that has no arc leaving. Since each $r \in R$ is a root, C exists.

Let $\boldsymbol{D_1} = (\boldsymbol{V_1} \cup R, \boldsymbol{A_1}) = D - C$. Note that $(D_1, \mathcal{M})$ is a matroid-rooted digraph.

LEMMA 4.5. $(D_1, \mathcal{M})$ *contains a matroid-reachability-based arborescence packing* $\{B_r^1\}_{r \in R}$ *and* $P_v^{D_1} = P_v^D$ *for all* $v \in V_1$.

Proof. By $d_A^+(C) = 0$, we have $d_{A_1}^-(X) = d_A^-(X)$ and $P_X^{D_1} = P_X^D$ for all $X \subseteq V_1 \cup R$. Then, since D satisfies (3.6), so does D_1. Hence, by the minimality of D and $P_v^{D_1} = P_v^D$ for all $v \in V_1$, the desired packing exists in D_1. ◻

By Lemma 4.5, $(D_1, \mathcal{M})$ has a matroid-reachability-based arborescence packing $\{\boldsymbol{B_r^1}\}_{r \in R}$. We now define a matroid-rooted digraph $(D_2, \mathcal{M}_2)$ which depends on the arborescences. Let $\boldsymbol{R_2} = P_C^D \cap R$, $\boldsymbol{\mathcal{M}_2}$ the restriction of $\mathcal{M}$ to R_2 and $\boldsymbol{D_2} = (V_2 \cup R_2, A_2)$ with $\boldsymbol{V_2} = C \cup T$, $\boldsymbol{T} = \{\text{new vertices } \boldsymbol{t_{uv}} : uv \in \rho_A^-(C)\}$, $\boldsymbol{A_2} = A(D[C]) \cup \{r t_{uv} : r \in R_2, u \in V(B_r^1), t_{uv} \in T\} \cup \{t_{uv} v, r_{\mathcal{M}_2}(R_2) * v t_{uv} : t_{uv} \in T\}$.

LEMMA 4.6. $(D_2, \mathcal{M}_2)$ *has a matroid-based arborescence packing* $\{B_r^2\}_{r \in R_2}$.

Proof. We show in the following claim that $(D_2, \mathcal{M}_2)$ satisfies (3.5). Let $\boldsymbol{X} \subseteq V_2 \cup R_2$ with $X \cap R_2 = span_{\mathcal{M}_2}(N_{D_2}^-(X \cap V) \cap R_2)$.

CLAIM 4.7. $d_{A_2}^-(X) \geq r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(X \cap R_2)$.

Proof. If $X \cap C = \emptyset$, then $d^-_{A_2}(X) \geq d_{A_2}(v, t_{uv}) = r_{\mathcal{M}_2}(R_2) \geq r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(X \cap R_2)$ for some $t_{uv} \in X - R_2$. If $X \cap C \neq \emptyset$, then, since $D[C]$ is strongly connected, we have $R_2 = P^D_C \cap R = P^D_{X \cap C} \cap R$. Let $\boldsymbol{Y} = (V \cup R) - U^D_{R-X}$ and $\boldsymbol{Z} = (X \cap C) \cup Y$. Then $P^D_Z \cap R = R_2$ and $Z \cap R = X \cap R_2$.

Proposition 4.8. $d^-_{A_2}(X) \geq d^-_A(Z)$.

Proof. Let $\boldsymbol{uv} \in \rho^-_A(Z)$. Since $\rho^-_A(Y) = \emptyset$, $v \in X \cap C$. If $u \in C$, then $uv \in \rho^-_{A_2}(X)$. Otherwise, $u \in U^D_{\bar{r}}$ for some $\bar{\boldsymbol{r}} \in R - X$ and $t_{uv} \in T$. Then, by $uv \in A$, we have $\bar{r} \in P^D_u \cap R \subseteq P^D_{X \cap C} \cap R = R_2$. Note that $\{r \in R : \bar{r} \in V(B^1_r)\} = \{\bar{r}\} = P^D_{\bar{r}}$. If $t_{uv} \in X$, then, since $\{r \in R : u \in V(B^1_r)\}$ is a basis of $P^D_u \cap R$ in $\mathcal{M}$, we have

$$
\begin{aligned}
\bar{r} \notin X \cap R_2 &= span_{\mathcal{M}_2}(N^-_{D_2}(X - R_2) \cap R_2) \supseteq span_{\mathcal{M}_2}(N^-_{D_2}(t_{uv}) \cap R_2) \\
&= span_{\mathcal{M}_2}(\{r \in R_2 : u \in V(B^1_r)\}) = span_{\mathcal{M}}(\{r \in R : u \in V(B^1_r)\}) \cap R_2 \\
&\supseteq P^D_u \cap R_2 \supseteq \{\bar{r}\},
\end{aligned}
$$

a contradiction. Thus $t_{uv} \notin X$ and so $t_{uv}v \in \rho^-_{A_2}(X)$. ∎

By Proposition 4.8 and (3.6), we have $d^-_{A_2}(X) \geq d^-_A(Z) \geq r_{\mathcal{M}}(P^D_Z \cap R) - r_{\mathcal{M}}(Z \cap R) = r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(X \cap R_2)$ and the proof of Claim 4.7 is complete. ∎

By Claim 4.7 and Theorem 3.5, the desired packing exists in D_2. This completes the proof of Lemma 4.6. ∎

By Lemma 4.6, $(D_2, \mathcal{M}_2)$ has a matroid-based arborescence packing $\{\boldsymbol{B^2_r}\}_{r \in R_2}$. With the help of the packings $\{B^1_r\}_{r \in R}$ and $\{B^2_r\}_{r \in R_2}$, a packing in $(D, \mathcal{M})$ can be constructed yielding a contradiction.

Lemma 4.9. *$(D, \mathcal{M})$ has a matroid-reachability-based arborescence packing.*

Proof. For all $r \in R - R_2$, let $\boldsymbol{B_r} = B^1_r$ and for all $r \in R_2$, let $\boldsymbol{B_r}$ be obtained from the union of B^1_r and $B^2_r - (R_2 \cup T)$ by adding the arc uv for all $t_{uv}v \in A(B^2_r)$. Since $\{B^1_r\}_{r \in R}$ and $\{B^2_r\}_{r \in R_2}$ are packings, so is $\{B_r\}_{r \in R}$. Since B^1_r and B^2_r are arborescences, for all $r \in R$ and $v \in V$, we have $d^-_{A(B_r)}(v) \leq 1$ and $d^+_{A(B_r)}(v) \geq 1$ implies $d^-_{A(B_r)}(v) = 1$ or $v = r$. It follows that B_r is an r-arborescence indeed. For $v \in V - C$, we have $\{r \in R : v \in V(B_r)\} = \{r \in R : v \in V(B^1_r)\}$ which is a basis of $P^D_v \cap R$ in $\mathcal{M}$ by Lemma 4.5. For $v \in C$, we have $\{r \in R : v \in V(B_r)\} = \{r \in R_2 : v \in V(B^2_r)\}$ which is a basis of $\mathcal{M}_2$, so a basis of $R_2 = P^D_v \cap R$ in $\mathcal{M}$. It follows that $\{B_r\}_{r \in R}$ has indeed the desired properties. ∎

Lemma 4.9 contradicts the fact that $(D, \mathcal{M})$ is a counterexample and hence completes the proof of Theorem 3.6. ∎

4.3. Proof of Theorems 3.10 and 3.11. In an analogous structure as before, we first derive Theorem 3.10 from Theorem 3.7.

Proof. (of **Theorem 3.10**) Necessity is evident.

For sufficiency, we define a simply matroid-rooted mixed hypergraph $\boldsymbol{(\mathcal{H}' = (V \cup R', \mathcal{A}' \cup \mathcal{E}'), \mathcal{M}' = (R', r_{\mathcal{M}'}))}$ obtained from $(\mathcal{H}, \mathcal{M})$ by replacing every root $r \in R$ by a set $\boldsymbol{Q_r}$ of $|N^+_{\mathcal{H}}(r)|$ simple roots such that $N^+_{\mathcal{H}'}(Q_r) = N^+_{\mathcal{H}}(r)$ in the mixed hypergraph and by $|Q_r|$ parallel copies of r in the matroid.

Now let $\boldsymbol{\{\mathsf{X}^i\}^\ell_1}$ be a biset subpartition of V with $w(\mathsf{X}^i) = span_{\mathcal{M}'}(\{r \in R' : N^+_{\mathcal{H}'}(r) \cap X^i_I \neq \emptyset\})$ for $i = 1, \dots, \ell$. Let $\boldsymbol{i} \in \{1, \dots, \ell\}$. Note that for all $r \in R$, either $Q_r \subseteq w(\mathsf{X}^i)$ or $Q_r \cap w(\mathsf{X}^i) = \emptyset$. Let $\boldsymbol{\mathsf{Y}^i} = (X^i_I \cup \{r \in R : Q_r \subseteq w(\mathsf{X}^i)\}, X^i_I)$. Observe that $w(\mathsf{Y}^i) = span_{\mathcal{M}}(\{r \in R : N^+_{\mathcal{H}}(r) \cap X^i_I \neq \emptyset\})$, $d^-_{\mathcal{A}}(\mathsf{Y}^i) = d^-_{\mathcal{A}'}(\mathsf{X}^i)$, $r_{\mathcal{M}}(R) = r_{\mathcal{M}'}(R')$ and $r_{\mathcal{M}}(w(\mathsf{Y}^i)) = r_{\mathcal{M}'}(w(\mathsf{X}^i))$. Then, by (3.10), we obtain $e_{\mathcal{E}}(\{X^i_I\}^\ell_1) \geq \sum_{i=1}^{\ell}(r_{\mathcal{M}}(R) - r_{\mathcal{M}}(w(\mathsf{Y}^i)) - d^-_A(\mathsf{Y}^i)) = \sum_{i=1}^{\ell}(r_{\mathcal{M}'}(R') - r_{\mathcal{M}'}(w(\mathsf{X}^i)) - d^-_{A'}(\mathsf{X}^i))$, that is $(\mathcal{H}', \mathcal{M}')$ satisfies (3.7).

We now apply Theorem 3.7 to obtain in $(\mathcal{H}', \mathcal{M}')$ a matroid-based mixed hyperarborescences packing $\{\boldsymbol{\mathcal{B}'_{r'}}\}_{r' \in R'}$ with arborescences $\{\boldsymbol{B'_{r'}}\}_{r' \in R'}$ as trimmings. For all $r \in R$, let $\boldsymbol{B_r}$ be obtained from $\{B'_{r'}\}_{r' \in Q_r}$ by contracting Q_r into r. As in the proof of Theorem 3.5, we can see that $\{B_r\}_{r \in R}$ is a matroid-based arborescence packing. Finally, for all $r \in R$, let $\boldsymbol{\mathcal{B}_r}$ be obtained from $\{\mathcal{B}'_{r'}\}_{r' \in Q_r}$ by contracting Q_r into r. As B_r is a trimming of $\mathcal{B}_r$ for all $r \in R$, $\{\mathcal{B}_r\}_{r \in R}$ is a packing of mixed hyperarborescences with the desired properties. ∎

We are now ready to derive Theorem 3.11 from Theorem 3.10. Again, the proof has certain similarities to the previous ones.

Proof. (of **Theorem 3.11**) We first prove necessity. Suppose that there exists a matroid-reachability-based mixed hyperarborescence packing $\{\boldsymbol{\mathcal{B}_r}\}_{r \in R}$. By definition, for every $r \in R$, there is an r-arborescence B_r that is a trimming of $\mathcal{B}_r$ with $\{r \in R : v \in V(B_r)\}$ being a basis of $P^{\mathcal{H}}_v \cap R$ in $\mathcal{M}$ for all $v \in V$. Let $\boldsymbol{\{\mathsf{X}^i\}^\ell_1}$ be a biset subpartition of a strongly connected component $\boldsymbol{C}$ of $\mathcal{H} - R$ such that $w(\mathsf{X}^i) = P^{\mathcal{H}}_{w(\mathsf{X}^i)}$ for all $i = 1, \ldots, \ell$.

Let $\boldsymbol{i} \in \{1, \ldots, \ell\}$, $\boldsymbol{R_i} = \{r \in R - X^i_O : V(B_r) \cap X^i_I \neq \emptyset\}$ and $\boldsymbol{v} \in X^i_I$. Then we have

$$r_{\mathcal{M}}(R_i \cup (X^i_O \cap R)) \geq r_{\mathcal{M}}(\{r \in R : v \in V(B_r)\}) = r_{\mathcal{M}}(P^{\mathcal{H}}_v \cap R) = r_{\mathcal{M}}(P^{\mathcal{H}}_C \cap R).$$

Thus, by the subcardinality and the submodularity of $r_{\mathcal{M}}$, we have

$$|R_i| \geq r_{\mathcal{M}}(R_i) \geq r_{\mathcal{M}}(R_i \cup (X^i_O \cap R)) - r_{\mathcal{M}}(X^i_O \cap R) \geq r_{\mathcal{M}}(P^{\mathcal{H}}_C \cap R) - r_{\mathcal{M}}(X^i_O \cap R).$$

Since $w(\mathsf{X}^i) = P^{\mathcal{H}}_{w(\mathsf{X}^i)}$, no dyperedge and no hyperedge enters $w(\mathsf{X}^i)$ in $\mathcal{H}$. Then, by $v \in X^i_I$, every B_r with $r \in R_i$ has an arc that enters X^i, that is $\mathcal{B}_r$ contains either a dyperedge in $\mathcal{A}$ entering X^i or a hyperedge in $\mathcal{E}$ entering X^i_I. Thus, since $\{\mathcal{B}_r\}_{r \in R}$ is a packing, we have

$$e_{\mathcal{E}}(\{X^i_I\}^\ell_1) + \sum_{i=1}^{\ell} d^-_{\mathcal{A}}(\mathsf{X}^i) \geq \sum_{i=1}^{\ell} |R_i| \geq \sum_{i=1}^{\ell}(r_{\mathcal{M}}(P^{\mathcal{H}}_C \cap R) - r_{\mathcal{M}}(X^i_O \cap R)).$$

For sufficiency, let $(\boldsymbol{\mathcal{H}} = (V \cup R, \mathcal{A} \cup \mathcal{E}), \boldsymbol{\mathcal{M}} = (R, r_{\mathcal{M}}))$ be a minimum counterexample. Obviously, $V \neq \emptyset$. Let $\boldsymbol{C} \subseteq V$ be the vertex set of a strongly connected component of $\mathcal{H}$ that has no dyperedge and hyperedge leaving. Since each $r \in R$ is a root, C exists.

Let $\boldsymbol{\mathcal{H}_1} = (\boldsymbol{V_1} \cup R, \boldsymbol{\mathcal{A}_1} \cup \boldsymbol{\mathcal{E}_1}) = \mathcal{H} - C$. Note that $(\mathcal{H}_1, \mathcal{M})$ is a matroid-rooted mixed hypergraph.

Lemma 4.10. *$(\mathcal{H}_1, \mathcal{M})$ has a matroid-reachability-based mixed hyperarborescence packing $\{\mathcal{B}^1_r\}_{r \in R}$ and $P^{\mathcal{H}_1}_v = P^{\mathcal{H}}_v$ for all $v \in V_1$.*

Proof. The fact that $d^+_{\mathcal{A}}(C) = d_{\mathcal{E}}(C) = 0$ implies that for all $X \subseteq V_1 \cup R$, we have $P^{\mathcal{H}_1}_X = P^{\mathcal{H}}_X$, for every subpartition $\mathcal{P}$ of $V \cup R_1$, we have $e_{\mathcal{E}}(\mathcal{P}) = e_{\mathcal{E}_1}(\mathcal{P})$, and for every biset X, $d^-_{\mathcal{A}_1}(\mathsf{X}) = d^-_{\mathcal{A}}(\mathsf{X})$. Then, since $\mathcal{H}$ satisfies (3.11), so does $\mathcal{H}_1$. Hence, by the minimality of $\mathcal{H}$ and $P^{\mathcal{H}_1}_v = P^{\mathcal{H}}_v$ for all $v \in V_1$, the desired packing exists. ∎

By Lemma 4.10, $(\mathcal{H}_1, \mathcal{M})$ has a matroid-reachability-based mixed hyperarborescence packing $\{\boldsymbol{\mathcal{B}^1_r}\}_{r \in R}$. By definition, $\mathcal{B}^1_r$ can be trimmed to an r-arborescence B^1_r

for all $r \in R$ such that $\{r \in R : v \in V(B_r^1)\}$ is a basis of $P_v^{\mathcal{H}_1} = P_v^{\mathcal{H}}$ in $\mathcal{M}$ for all $v \in V_1$. We now define a matroid-rooted mixed hypergraph $(\mathcal{H}_2, \mathcal{M}_2)$ which depends on the arborescences $\{B_r^1\}_{r\in R}$. Let $\boldsymbol{R_2} = P_C^{\mathcal{H}} \cap R$, $\boldsymbol{\mathcal{M}_2}$ the restriction of $\mathcal{M}$ to R_2 and let $\boldsymbol{\mathcal{H}_2} = (V_2 \cup R_2, \mathcal{A}_2 \cup \mathcal{E}_2)$ be obtained from $\mathcal{H}[C]$ by adding a set $\boldsymbol{T}$ of new vertices $\boldsymbol{t_a}$ for all $a \in \rho^-_{\mathcal{A}}(C)$ and the vertex set R_2 and by adding dypereddges $\boldsymbol{a'}$ $= ((tail(a) \cap C) \cup t_a, head(a))$ for all $t_a \in T$, the arcs rt_a for all $r \in R_2, t_a \in T$ with $tail(a) \cap V(B_r^1) \neq \emptyset$ and $r_{\mathcal{M}_2}(R_2)$ parallel arcs $head(a)t_a$ for all $t_a \in T$.

LEMMA 4.11. $(\mathcal{H}_2, \mathcal{M}_2)$ *contains a matroid-based mixed hyperarborescence packing* $\{\mathcal{B}_r^2\}_{r\in R_2}$.

Proof. We show in the following claim that $(\mathcal{H}_2, \mathcal{M}_2)$ satisfies (3.10). Let $\boldsymbol{\{\mathsf{X}^i\}_1^\ell}$ be a biset subpartition of $V_2 = C \cup T$ with $w(\mathsf{X}^i) = span_{\mathcal{M}_2}(\{r \in R_2 : N^+_{\mathcal{H}_2}(r) \cap X_I^i \neq \emptyset\})$ for all $i = 1, \dots, \ell$.

CLAIM 4.12. $e_{\mathcal{E}_2}(\{X_I^i\}_1^\ell) \geq \sum_{i=1}^\ell (r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(w(\mathsf{X}^i)) - d^-_{\mathcal{A}_2}(\mathsf{X}^i))$.

Proof. Suppose that $X_I^i \cap C \neq \emptyset$ for all $i \in \{1, \dots, j\}$ and $X_I^i \cap C = \emptyset$ for all $i \in \{j+1, \dots, \ell\}$. For $i \in \{j+1, \dots, \ell\}$, $d^-_{\mathcal{A}_2}(\mathsf{X}^i) \geq d^-_{\mathcal{A}_2}(head(a), t_a) \geq r_{\mathcal{M}_2}(R_2)$ for some $t_a \in X_I^i$, thus $0 \geq r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(w(\mathsf{X}^i)) - d^-_{\mathcal{A}_2}(\mathsf{X}^i)$.

Let now $\boldsymbol{i} \in \{1, \dots, j\}$. Since $\mathcal{H}[C]$ is strongly connected, we have $R_2 = P_C^{\mathcal{H}} \cap R = P^{\mathcal{H}}_{X_I^i \cap C} \cap R$. Let $\boldsymbol{Y^i} = (V \cup R) - (U^{\mathcal{H}}_{R - w(\mathsf{X}^i)} \cup C)$ and $\boldsymbol{\mathsf{Z}^i} = ((X_I^i \cap C) \cup Y^i, X_I^i \cap C)$. Note that $Z_I^i = X_I^i \cap C$ and $Z_O^i \cap R = Y^i \cap R = R - (R - w(\mathsf{X}^i)) = w(\mathsf{X}^i)$, so $r_{\mathcal{M}}(Z_O^i \cap R) = r_{\mathcal{M}_2}(w(\mathsf{X}^i))$.

PROPOSITION 4.13. $d^-_{\mathcal{A}_2}(\mathsf{X}^i) \geq d^-_{\mathcal{A}}(\mathsf{Z}^i)$.

Proof. Let $\boldsymbol{a} \in \rho^-_{\mathcal{A}}(\mathsf{Z}^i)$. If $a \notin \rho^-_{\mathcal{A}}(C)$, then $a \in \rho^-_{\mathcal{A}_2}(\mathsf{X}^i)$. Otherwise, let $\boldsymbol{u}$ $\in tail(a) - Z_O^i - C$. Then $u \in U^{\mathcal{H}}_{\bar{r}}$ for some $\boldsymbol{\bar{r}} \in R - w(\mathsf{X}^i)$ and $t_a \in T$. Thus, by $a \in \mathcal{A}$, we have $\bar{r} \in P_u^{\mathcal{H}} \cap R \subseteq P^{\mathcal{H}}_{X_I^i \cap C} \cap R = R_2$. Note that $\{r \in R : \bar{r} \in V(B_r^1)\} = \{\bar{r}\} = P^{\mathcal{H}}_{\bar{r}}$. If $t_a \in X_I^i$, then, since $\{r \in R : u \in V(B_r^1)\}$ is a basis of $P_u^{\mathcal{H}} \cap R$ in $\mathcal{M}$, we obtain

$$\begin{aligned}
\bar{r} \notin w(\mathsf{X}^i) &= span_{\mathcal{M}_2}(\{r \in R_2 : N^+_{\mathcal{H}_2}(r) \cap X_I^i \neq \emptyset\}) \\
&\supseteq span_{\mathcal{M}_2}(\{r \in R_2 : t_a \in N^+_{\mathcal{H}_2}(r)\}) \\
&= span_{\mathcal{M}_2}(\{r \in R_2 : tail(a) \cap V(B_r^1) \neq \emptyset\}) \\
&\supseteq span_{\mathcal{M}}(\{r \in R : u \in V(B_r^1)\}) \cap R_2 \\
&\supseteq P_u^{\mathcal{H}} \cap R_2 \supseteq \{\bar{r}\},
\end{aligned}$$

a contradiction. It follows that $a' \in \rho^-_{\mathcal{A}_2}(\mathsf{X}^i)$. ∎

Since $w(\mathsf{Z}^i) \cap C = \emptyset$, $\{Z_I^i\}_1^j$ is a biset subpartition of C. Moreover, no dyperedge and no hyperedge leaves $U^{\mathcal{H}}_{R-w(\mathsf{X}^i)} \cup C$, so $w(\mathsf{Z}^i) = Y^i = P^{\mathcal{H}}_{Y^i} = P^{\mathcal{H}}_{w(\mathsf{Z}^i)}$. Then, by (3.11) and Proposition 4.13, we have $e_{\mathcal{E}_2}(\{X_I^i\}_1^\ell) = e_{\mathcal{E}_2}(\{X_I^i\}_1^j) = e_{\mathcal{E}}(\{Z_I^i\}_1^j) \geq \sum_{i=1}^j (r_{\mathcal{M}}(P_C^{\mathcal{H}} \cap R) - r_{\mathcal{M}}(Z_O^i \cap R) - d^-_{\mathcal{A}}(\mathsf{Z}^i)) \geq \sum_{i=1}^j (r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(w(\mathsf{X}^i)) - d^-_{\mathcal{A}_2}(\mathsf{X}^i)) \geq \sum_{i=1}^\ell (r_{\mathcal{M}_2}(R_2) - r_{\mathcal{M}_2}(w(\mathsf{X}^i)) - d^-_{\mathcal{A}_2}(\mathsf{X}^i))$, that completes the proof of Claim 4.12. ∎

By Claim 4.12 and Theorem 3.10, the desired packing exists in $\mathcal{H}_2$. ∎

By Lemma 4.11, $(\mathcal{H}_2, \mathcal{M}_2)$ has a matroid-reachability-based mixed hyperarborescence packing $\{\boldsymbol{\mathcal{B}_r^2}\}_{r\in R_2}$ with r-arborescences $\{\boldsymbol{B_r^2}\}_{r\in R_2}$ as trimmings. With the help of the packings $\{\mathcal{B}_r^1\}_{r\in R}$ and $\{\mathcal{B}_r^2\}_{r\in R_2}$, a packing of $(\mathcal{H}, \mathcal{M})$ can be constructed yielding a contradiction.

LEMMA 4.14. $(\mathcal{H}, \mathcal{M})$ *has a matroid-reachability-based mixed hyperarborescence packing.*

Proof. For $r \in R - R_2$, let $\boldsymbol{B_r} = B_r^1$ and for $r \in R_2$, let $\boldsymbol{B_r}$ be obtained from the union of B_r^1 and $B_r^2 - R_2 - T$ by adding an arc uv for all $t_a v \in \mathcal{A}(B_r^2)$ for some $u \in tail(a) \cap V(B_r^1)$. As in the proof of Theorem 3.6, we can see that $\{B_r\}_{r \in R}$ is a packing of arborescences such that the root of B_r is r for all $r \in R$ and $\{r \in R : v \in V(B_r)\}$ is a basis of $P_v^{\mathcal{H}} \cap R$ in $\mathcal{M}$ for all $v \in V$.

Finally, for $r \in R - R_2$, let $\boldsymbol{\mathcal{B}_r} = \mathcal{B}_r^1$ and for $r \in R_2$, let $\boldsymbol{\mathcal{B}_r}$ be obtained from $\mathcal{B}_r^1$ and $\mathcal{B}_r^2 - R_2 - T$ by adding the dyperedge $a \in \mathcal{A}$ for all $a' \in \mathcal{A}(\mathcal{B}_r^2)$. The above argument shows that this is a packing of mixed hyperarborescences in $\mathcal{H}$ (with arborescences $\{B_r\}_{r \in R}$ as trimmings) with the desired properties. ◻

Lemma 4.14 contradicts the fact that $(\mathcal{H}, \mathcal{M})$ is a counterexample and hence the proof of Theorem 3.11 is complete. ◻

5. Algorithmic aspects. This section deals with the algorithmic consequences of our proofs.

For the basic case, we show that our proof of Theorem 3.3 yields a polynomial time algorithm. We acknowledge that so is the original proof in [9]. We first mention that the packings in Theorem 3.2 can be found in polynomial time, following either the proof of Edmonds in [3] or the proof of Frank (Theorem 10.2.1 in [5]). Using this, we can turn our proof of Theorem 3.3 into a polynomial time algorithm for finding the desired packing of arborescences. We first find the arborescences B_r^1 in the smaller instance $D - C$. As the size of D_2 is polynomial in the size of D, we can apply the algorithm mentioned above to obtain the arborescences B_r^2 in polynomial time. The obtained arborescences can be merged efficiently to obtain the B_r.

For the matroidal case, we show that our proof of Theorem 3.6 is algorithmic if an independence oracle for $\mathcal{M}$ is given. We acknowledge that so is the original proof in [10]. We first recall that the packings in Theorem 3.4 can be found in polynomial time as mentioned in [2]. It is easy to see that the proof of Theorem 3.5 yields a polynomial time algorithm if a matroid oracle is given. By similar arguments as before and the fact that an independence oracle for $\mathcal{M}$ yields independence oracles for all matroids considered, we obtain that the proof of Theorem 3.6 can be turned into a polynomial time algorithm if an independence oracle for $\mathcal{M}$ is given.

For the more general case, using the fact that the proof of Theorem 3.7 is algorithmic if a matroid oracle is given ([4]), we obtain that also Theorems 3.10 and 3.11 yield polynomial time algorithms given independence oracles. In particular, the arborescences in Corollary 3.12 can be found in polynomial time.

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