Connectivity of orientations of 3-edge-connected graphs

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Abstract

We attempt to generalize a theorem of Nash-Williams stating that a graph has a k-arc-connected orientation if and only if it is $2k$-edge-connected. In a strongly connected digraph we call an arc *deletable* if its deletion leaves a strongly connected digraph. Given a 3-edge-connected graph G, we define its Frank number $f(G)$ to be the minimum number k such that there exist k orientations of G with the property that every edge becomes a deletable arc in at least one of these orientations. We are interested in finding a good upper bound for the Frank number. We prove that $f(G) \leq 7$ for every 3-edge-connected graph. On the other hand, we show that a Frank number of 3 is attained by the Petersen graph. Further, we prove better upper bounds for more restricted classes of graphs and establish a connection to the Berge-Fulkerson conjecture. We also show that deciding whether all edges of a given subset can become deletable in one orientation is NP-complete.

1. Introduction

This paper deals with ways of orienting undirected graphs so that the obtained directed graph has certain connectivity properties. Our goal is to generalize classical results of Robbins and Nash-Williams.

Let $G = (V, E)$ be an undirected graph. For some $F \subseteq E$, $\boldsymbol{G(F)} = (V, F)$ denotes the subgraph induced by F. For a set $X \subseteq V$, the subgraph induced by X is denoted by $\boldsymbol{G[X]}$. We use $\boldsymbol{\delta_G(X)}$ to denote the set of edges between X and $V - X$ and $\boldsymbol{d_G(X)}$ for $|\delta_G(X)|$. For some vertex $v \in V$, we call $d(\{v\})$ the *degree* of v. The graph G is called *cubic* if $d_G(\{v\}) = 3$ for all $v \in V$. We say that G is *k-edge-connected* if $d_G(X) \geq k$ for all nonempty, proper subset X of V. We call G *Eulerian* if every vertex of G is of even degree. For some $e \in E$, we denote by $\boldsymbol{G/e}$ the graph obtained from G by contracting e, that is deleting e and identifying its two endvertices. For some $F = \{e_1, \dots, e_t\} \subseteq E$, we denote $G/e_1/\dots/e_t$ by $\boldsymbol{G/F}$. For some subgraph H of G, we abbreviate $G/E(H)$ to $\boldsymbol{G/H}$. An *orientation* of G is a directed graph $D = (V, A)$ such that each edge $uv \in E$ is replaced by exactly one of the arcs uv or vu. Given some $e \in E(G)$, we use $\vec{e}$ to denote the associated arc in the orientation. We say that G or a subset of V is *trivial* if it contains only one vertex. We call G *essentially (k+1)-edge-connected* if G is k-edge-connected and for all edge-cuts of size k one

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side is trivial. A *cycle* is a connected graph each vertex of which is of degree 2. A *path* is a connected graph in which two vertices are of degree 1 and all other vertices are of degree 2. A *cycle packing* is a collection of vertex-disjoint cycles of G. We say that a vertex or an edge is in the cycle packing if it is contained in one of the cycles of the packing.

An edge set M of G is called a *matching* if each vertex of G is incident to at most one edge of M. A matching M is *perfect* if each vertex of G is incident to exactly one edge of M. We say that G is *k-edge-colorable* if the edge set of G can be partitioned into k matchings.

Let $D = (V, A)$ be a directed graph. For some $F \subseteq A$, let $\boldsymbol{D(F)} = (V, F)$ denote the subgraph induced on F. The subgraph induced by some $X \subseteq V$ is denoted by $\boldsymbol{D[X]}$. We use $\boldsymbol{\delta^+_D(X)}$ to denote the set of arcs from X to $V - X$ and $\boldsymbol{\delta^-_D(X)}$ for $\delta^+_D(V - X)$. For a vertex $v \in V$, we call $|\delta^+_D(\{v\})|$ the *out-degree* and $|\delta^-_D(\{v\})|$ the *in-degree* of v. The graph that is obtained from D by replacing each arc by an edge between the same two vertices is called the *underlying graph* of D. We call D *weakly connected* if its underlying graph is connected. We call D *strongly connected* if $|\delta^+_D(X)| \geq 1$ for every nonempty, proper subset X of V. More generally, we say that D is *k-arc-connected* if $|\delta^+_D(X)| \geq k$ for every nonempty, proper subset X of V. We call D *Eulerian* if $|\delta^+_D(\{v\})| = |\delta^-_D(\{v\})|$ for every $v \in V$. For some $a \in A$, we denote by $\boldsymbol{D/a}$ the directed graph obtained from D by contracting a, that is deleting a and identifying its head and its tail. For some $F = \{a_1, \dots, a_t\} \subseteq A$, we denote $D/a_1/\dots/a_t$ by $\boldsymbol{D/F}$. For some subgraph H of D, we abbreviate $D/A(H)$ to $\boldsymbol{D/H}$. Let $\overleftarrow{\boldsymbol{D}}$ denote the orientation that arises from D by reversing the orientation of all arcs. A *circuit* is a strongly connected orientation of a cycle. A *directed path* is an orientation of a path such that at most one arc enters and at most one arc leaves each vertex. Subscripts may be omitted when the graph or directed graph is clear from the context. We also use basic notions of complexity theory which can be found in Chapter 15 of [5].

As one of the first important results in the theory of graph orientations, Robbins proved in 1939 that a graph has a strongly connected orientation if and only if it is 2-edge-connected [10]. This was later generalized by Nash-Williams [7] who proved that for any positive integer k, a graph has a k-arc-connected orientation if and only if it is $2k$-edge-connected. This naturally raises the question whether odd edge-connectivity also yields distinctive orientability properties. Our approach to this consists in relaxing the goal to obtain exactly one orientation of the graph to allowing several of them. We say that an arc is *deletable* in a k-arc-connected orientation of a $(2k+1)$-edge-connected graph if its deletion leaves it k-arc-connected. We ask how many orientations are necessary for each edge of the original graph to become a deletable arc in at least one of the orientations. Surprisingly, the number of necessary orientations is bounded by a constant depending only upon k. This is a consequence of a theorem of DeVos, Johnson and Seymour [1]. We focus on the case $k = 1$, meaning we want to find orientations of a 3-edge-connected graph such that for every edge of the graph, the deletion of the associated arc leaves a strongly connected graph in at least one of the orientations. In honor of András Frank who proposed this problem and had an immense impact on the development of the theory of graph orientations, we call the minimum number of necessary orientations for a graph G its *Frank number* $\boldsymbol{f(G)}$. Observe that the Frank number of any 4-edge-connected graph is 1 as it has a 2-arc-connected orientation by the theorem of Nash-Williams. On the other hand, any graph G containing a 3-edge-cut has Frank number at least 2. This follows directly from the fact that in any strongly connected orientation of G, there is one arc of the 3-edge-cut that is oriented differently than the other two arcs. This arc cannot be deletable in this orientation, so at least one more orientation is needed. It is an interesting question to find upper bounds for the Frank number of graphs. A first constant bound can easily be obtained by the following theorem of DeVos, Johnson and Seymour [1]:

Theorem 1. *Let $G = (V, E)$ be a 3-edge-connected graph. Then there is a partition $\{E_1, \ldots, E_9\}$ of E such that $G - E_i$ is 2-edge-connected for all $i = 1, \ldots, 9$.*

This implies the following:

Corollary 1. *Every 3-edge-connected graph G satisfies $f(G) \leq 9$.*

Indeed, by Robbins' Theorem, for all $i = 1, \ldots, 9$, there is a strongly connected orientation of $G - E_i$. Giving an arbitrary orientation to the edges of E_i yields an orientation in which the arcs of $\vec{E_i}$ are deletable.

The main contribution of this paper is to further narrow down the values attained by the Frank number. We first show a better upper bound.

Theorem 2. *Every 3-edge-connected graph G satisfies $f(G) \leq 7$.*

In attempt to improve on this, we also establish a relationship between our problem and a well-known conjecture about matchings in cubic graphs, the conjecture of Berge-Fulkerson mentioned in Section 2.

Theorem 3. *Every 3-edge-connected graph G satisfies $f(G) \leq 5$ unless the conjecture of Berge-Fulkerson fails.*

Further, we prove a stronger bound for two more restricted classes of 3-edge-connected graphs.

Theorem 4. *Every 3-edge-connected 3-edge-colorable graph G satisfies $f(G) \leq 3$.*

Theorem 5. *Every essentially 4-edge-connected graph Gsatisfies $f(G) \leq 3$.*

For the lower bound, we show that there are graphs whose Frank number is strictly bigger than 2, more precisely:

Theorem 6. *The Frank number of the Petersen graph is 3.*

A drawing of the Petersen graph can be found in Figure 1.

Given a directed graph D, we call a set $F \subseteq A(D)$ *deletable* if $D - f$ is strongly connected for all $f \in F$. Given a graph G, we call a set $F \subseteq E(G)$ *deletable* if there exists an orientation $\vec{G}$ of G such that $\vec{F}$ is deletable in $\vec{G}$.

One of the main difficulties in improving the upper bound on the Frank number consists in finding a useful class of deletable sets. We consider the

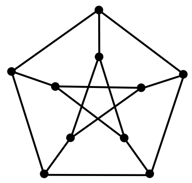

Figure 1: The Petersen graph

problem of testing algorithmically whether a set is deletable. More formally, we define the following problem:

DELETABILITY

Instance: A graph $G = (V, E)$ and a set $S \subseteq E$.

Question: Is there an orientation D of G such that $D - \vec{s}$ is strongly connected for all $s \in S$?

The following result shows that an efficient algorithm for DELETABILITY seems out of reach. This implies that a good characterization of deletable sets is hard to obtain.

Theorem 7. *DELETABILITY is NP-complete for cubic 3-edge-connected graphs.*

In Section 2, we present several classical results we will make use of and make some preparatory observations. Also, we introduce an auxiliary graph that will help to reduce the problems to cubic graphs later. In Section 3, we deal with the general case of 3-edge-connected graphs proving Theorems 2 and 3. Section 4 is concerned with essentially 4-edge-connected graphs, in particular the proof of Theorem 5. In Section 5, we prove Theorem 6. Theorem 7 is proven in Section 6. Finally, in Section 7 we conclude our work and give directions for further research on this topic.

2. Preliminaries

In the first part of this section, we give classical results we will make use of later. In the second part, we make some easy preparatory observations which will prove useful later. In the third part, we introduce a way to construct a cubic graph from an arbitrary graph of minimum degree at least 3 and give some basic properties of it.

2.1. Previous results

The following result was proven by Nash-Williams [7] and is the starting point of our work as it characterizes the graphs admitting a k-arc-connected orientation.

Theorem 8. *A graph has a k-arc-connected orientation if and only if it is $2k$-edge-connected.*

In fact, there is an immense strengthening of this theorem. For any two vertices u, v of a graph G, let $\boldsymbol{\lambda_G(u,v)}$ be the maximum number of edge-disjoint paths between u and v. An orientation D of G is called *well-balanced* if for any $u, v \in V(G)$, there exist at least $\lfloor \frac{\lambda_G(u,v)}{2} \rfloor$ directed paths from u to v and also from v to u in D. An *odd-vertex pairing* of G is a perfect matching in the complete graph whose vertex set is the set of vertices of odd degree of G. An odd-vertex pairing P is called *admissible* if the restriction of any Eulerian orientation of $G + P$ to $E(G)$ yields a well-balanced orientation of G. Nash-Williams [7] proved the following:

Theorem 9. *Every graph has an admissible odd-vertex pairing.*

Observe that this implies that every graph has a well-balanced orientation. Also, as well-balanced orientations of $2k$-edge-connected graphs are k-arc-connected, Theorem 9 implies Theorem 8.

There are several other well-known results we make use of in this article. The first one is about packing spanning trees and is also due to Nash-Williams [8].

Theorem 10. *Every $2k$-edge-connected graph has k edge-disjoint spanning trees.*

The next one concerns matching theory and was proven by Petersen [9].

Theorem 11. *Every cubic 2-edge-connected graph has a perfect matching.*

The next one concerns minimum edge cuts and can for example be found as Theorem 7.1.2 in [3].

Proposition 1. *Let G be a 3-edge-connected graph and let $\delta(X)$ and $\delta(X')$ be 3-edge-cuts of G. Then $\delta(X)$ and $\delta(X')$ are not crossing, i.e. one of $X - X'$, $X' - X$, $X \cap X'$ or $V(G) - (X \cup X')$ is empty.*

Let $G = (V, E)$ be a graph and $T \subseteq V$. A *T-join* is defined to be a set $F \subseteq E$ such that the set of odd degree vertices of $G(F)$ is T. We use the following characterization of the existence of T-joins that can be found as Proposition 12.7 in [5].

Proposition 2. *Let $G = (V, E)$ be a graph and $T \subseteq V$. Then G contains a T-join if and only if every connected component of G contains an even number of elements of T.*

The following theorem is due to Menger [6] and is a fundamental characterization of k-edge-connected graphs.

Theorem 12. *A graph $G = (V, E)$ is k-edge-connected if and only if $\lambda_G(u, v) \geq k$ for all $u, v \in V$.*

Further, we mention an intensively studied conjecture which was proposed independently by Berge and Fulkerson [12].

Conjecture 1. *Every cubic 2-edge-connected graph has a set of six perfect matchings such that every edge is contained in exactly two of them.*

We also consider the following algorithmic problem which is well-known in the literature:

Monotone Not-all-equal-3SAT(MNAE3SAT)

Instance: A set X of boolean variables, a formula consisting of a set $\mathcal{C}$ of clauses each containing 3 distinct variables, none of which are negated.

Question: Is there a truth assignment to the variables of X such that every clause in $\mathcal{C}$ contains at least one true and at least one false literal?

This problem will be used in the reduction in Section 6 which is justified by the following result due to Schaefer [11].

Theorem 13. *MNAE3SAT is NP-complete.*

2.2. Preparatory results

The following two results show that connectivity properties are maintained when contracting or blowing up sufficiently connected subgraphs. As they are of basic nature, they are given without proof.

Proposition 3. *Let G be a graph.*
(a) If G is k-edge-connected, then so is any contraction of G.
(b) If G is essentially k-edge-connected, then so is any contraction of G.

Proposition 4. *For a subgraph Q of a directed graph D,*
(a) if D is strongly connected, then so is D/Q,
(b) if D/Q and Q are strongly connected, then so is D.

The following observation concerns Eulerian orientations and follows easily from Proposition 5 of [4]. For the sake of completeness we provide an easy proof for it.

Proposition 5. *Let $G = (V, E)$ be an Eulerian graph and $\{e_v, f_v\}$ two edges incident to v for all $v \in V' \subseteq V$. Then there is an Eulerian orientation of G such that exactly one of e_v and f_v enters v for all $v \in V'$.*

Proof Let G' be the graph obtained from G by detaching each vertex $v \in V'$ into two vertices u_v and w_v such that u_v is incident to $\{e_v, f_v\}$ and w_v is incident to $\delta_G(\{v\}) - \{e_v, f_v\}$ in G'. As G is Eulerian, so is G'. Hence there exists an Eulerian orientation D' of G'. By identifying u_v and w_v in D' for all $v \in V'$, we obtain the required orientation. ■

The following result is a direct consequence of the definition of strongly connected directed graphs.

Proposition 6. *Given a directed graph $D = (V, A)$, a set $F \subseteq A$ is deletable if and only if $\delta^-_D(X)$ contains either at least one arc of $A - F$ or at least two arcs for every nonempty, proper subset X of V.*

Consequently, given a graph G, a subset $F \subseteq E(G)$ is deletable if and only if there is an orientation satisfying the above properties.

Finally, we show one more result about strongly connected orientations of 3-edge-connected graphs which we need in the proof of Theorem 5.

Lemma 1. *Let D be a strongly connected orientation of a 3-edge-connected graph G and C a circuit of D. Then C contains an arc a such that $D - a$ is strongly connected.*

Proof Let (G, D, C) be a counterexample that minimizes the number of vertices of D. Let e be an edge of G that is incident to a vertex of C and that does not belong to C. By the 3-edge-connectivity of G, e exists. Since D is strongly connected, $\vec{e}$ belongs to a directed path P whose end-vertices belong to C but whose internal vertices do not. Then P can be extended by a possibly trivial directed subpath of C to form a circuit C^*. Let (G', D', C') be obtained from (G, D, C) by contracting C^*. Then, by Propositions 3(a) and 4(a), the assumptions of the lemma are satisfied for (G', D', C'). By the minimality of (G, D, C), C' contains an arc a' such that $D' - a'$ is strongly connected. Let a be the arc of C in D that corresponds to a'. Since $D' - a'$ and C^* are strongly connected, by Proposition 4(b), so is $D - a$. ■

2.3. Cubic extensions

We introduce for any graph $G = (V, E)$ of minimum degree at least 3 an auxiliary graph $\boldsymbol{H_G}$ that is cubic. For each vertex $v \in V$ of degree at least 4, H_G contains a set S_v of $d_G(\{v\})$ vertices. For each vertex of degree 3, let $S_v = \{v\}$. Next, for each $v \in V$ of degree at least 4, we add a cycle C_v whose vertex set is S_v. Finally, for each edge $uv \in E$, we add an edge between S_u and S_v to H_G. We do this in a way so that H_G becomes cubic. We call H_G a *cubic extension* of G. Note that H_G is not unique. This ambiguity has no consequences though.

Proposition 7. *Let $G = (V, E)$ be a graph of minimum degree at least 3 and H_G be a cubic extension of G.*

(a) If G is 3-edge-connected and $G - v$ is connected for all $v \in V$, then H_G is 3-edge-connected.

(b) If G is essentially 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in V$, then H_G is essentially 4-edge-connected.

Proof (a) Assume for a contradiction that $d_{H_G}(X) \leq 2$ for some nonempty, proper subset X of $V(H_G)$. Since G is 3-edge-connected, there is at least one $v \in V$ such that $S_v \cap X$ and $S_v - X$ are nonempty. It follows that $2 \leq d_{C_v}(X) \leq d_{H_G}(X) \leq 2$. This yields that for every $u \in V - v$ we have $S_u \subseteq X$ or $S_u \subseteq V(H_G) - X$ and for all $uw \in E$ with $v \notin \{u, w\}$ we have $S_u \cup S_w \subseteq X$ or $S_u \cup S_w \subseteq V(H_G) - X$. If there are vertices $u, w \in V$ such that $S_u \subseteq X$ and $S_w \subseteq V(H_G) - X$, it follows that $G - v$ is not connected, contradicting the assumption. Therefore, by symmetry we may assume that X is a nonempty, proper subset of S_v. We then have $d_{C_v}(X) \geq 2$ and there is at least one additional edge between X and $V(H_G) - S_v$, a contradiction to $d_{H_G}(X) \leq 2$.

(b) By (a), H_G is 3-edge-connected. For the sake of a contradiction, suppose that there is some non-trivial, proper subset X of $V(H_G)$ such that $d_{H_G}(X) = 3$. If there are two vertices $u, v \in V$ such that $S_u \cap X$, $S_u - X$, $S_v \cap X$ and $S_v - X$ are nonempty, we have $2 + 2 \leq d_{C_u}(X) + d_{C_v}(X) \leq d_{H_G}(X) \leq 3$, a contradiction.

Now consider the case that there is exactly one $v \in V$ such that $S_v \cap X$ and $S_v - X$ are nonempty. We have that $d_{H_G}(X) - d_{C_v}(X) \leq 1$. It follows that in H_G there is at most one edge between $X - S_v$ and $V(H_G) - X - S_v$. If $X - S_v$ and $V(H_G) - X - S_v$ are nonempty, then $G - v$ is not 2-edge-connected, a contradiction to the assumption. By symmetry, we may therefore assume that $X \subset S_v$. We have that $d_{C_v}(X) = 2$ and there are $|X|$ edges between X and $V(H_G) - S_v$. It follows that $|X| = 1$, which is a contradiction.

Finally assume that $S_v \subseteq X$ or $S_v \cap X = \emptyset$ for all $v \in V$. Let $X' = \{v \in V : S_v \subseteq X\}$. As G is essentially 4-edge-connected, we may assume by symmetry that $X' = \{v\}$ for some vertex v of degree 3. This yields that $|X| = |S_v| = 1$, which is a contradiction. ■

3. 3-edge-connected graphs

This section is dedicated to proving Theorems 2, 3 and 4. In the first part, we show that a certain class of edge sets is deletable. After, we show how to cover cubic 3-edge-connected graphs with such sets. Next, we use this to conclude cubic versions of Theorems 2 and 3 and to prove Theorem 4. Finally, we extend this to obtain the general versions of Theorems 2 and 3.

3.1. A class of deletable edge sets

Given a packing $\mathcal{C}$ of cycles in a 3-edge-connected graph G, the *special set* of $\mathcal{C}$ is defined to be the set of edges in $E(G) - E(\mathcal{C})$ that belong to no 3-edge-cut of $G/\mathcal{C}$.

Lemma 2. *Let M be the special set of a cycle packing $\mathcal{C}$ of a 3-edge-connected graph G. Then M is deletable.*

Proof Let $G' = G/\mathcal{C}$. Since G is 3-edge-connected, so is G' by Proposition 3(a). Consider a well-balanced orientation D' of G' which exists by Theorem 9. Then D' is strongly connected. Let D be the orientation of G obtained from D' by orienting all cycles of $\mathcal{C}$ as a circuit.

We have to show that $D - \vec{f}$ is strongly connected for all $f \in M$. By Proposition 4(b), it is enough to show that $D' - \vec{f}$ is strongly connected for all $f \in M$. Let $\vec{f} = uv$ for some $f \in M$ and suppose that there exists some non-empty, proper subset X of $V(D')$ with $|\delta^+_{D'-\vec{f}}(X)| = 0$. Obviously $u \in X$ and $v \in V(D') - X$. Since G' is 3-edge-connected and f belongs to no 3-edge-cut in G', Theorem 12 guarantees that $\lambda_{G'}(u, v) \geq 4$. As D' is well-balanced, it follows that $0 = |\delta^+_{D'-\vec{f}}(X)| = |\delta^+_{D'}(X)| - 1 \geq \lfloor \frac{\lambda_{G'}(u,v)}{2} \rfloor - 1 \geq 2 - 1 = 1$, a contradiction. ■

3.2. Covering cubic graphs with special sets

In the following we show that any cubic 3-edge-connected graph can be covered by 7 special sets. For technical reasons, we will need the following slight strengthening.

Lemma 3. *For every cubic 3-edge-connected graph, there exist 7 cycle packings satisfying the following conditions:*

(a) Every edge is in the special set of at least one cycle packing.

(b) Every edge is in exactly 4 of the cycle packings.

Proof For the sake of a contradiction, let $G = (V, E)$ be a counterexample to the lemma that minimizes $|V|$.

Claim 1. *G is essentially 4-edge-connected.*

Proof For the sake of a contradiction, let $\{A_1, A_2\}$ be a partition of $V(G)$ such that $|A_i| \geq 2$ and a 3-edge-cut $F := \{e_1, e_2, e_3\}$ exists between A_1 and A_2. Construct the graphs G_i from G by contracting A_{3-i} to v_i. As G_i is cubic, 3-edge-connected by Proposition 3(a) and smaller than G, there exists a set of cycle packings $\mathbb{C}^i = \{\mathcal{C}_1^i, \ldots, \mathcal{C}_7^i\}$ of G_i satisfying (a) and (b).

Observe that since G_i is cubic, (b) implies that for $j \in \{1, 2, 3\}$, there are exactly two cycle packings in $\mathbb{C}^i$ that contain $\{e_1, e_2, e_3\} - \{e_j\}$. It follows that v_i is in exactly 6 cycle packings of $\mathbb{C}^i$. By relabeling if needed, we may assume that $\mathcal{C}_1^i$ is the cycle packing that does not contain v_i and $\{e_1, e_2, e_3\} - \{e_j\}$ is contained in $\mathcal{C}_{2j}^i$ and $\mathcal{C}_{2j+1}^i$. We may also assume, by (a), that e_j is in the special set of $\mathcal{C}_{2j}^i$.

We construct $\mathbb{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_7\}$ so that $E(\mathcal{C}_k) = E(\mathcal{C}_k^1) \cup E(\mathcal{C}_k^2)$ for $k = 1, \ldots, 7$. Observe that this is a set of seven cycle packings. We finish the proof by showing that $\mathbb{C}$ satisfies (a) and (b).

First observe that (b) follows directly from the construction and the fact that an edge is in $\mathcal{C}_k$ if and only if it is in $\mathcal{C}_k^1$ or $\mathcal{C}_k^2$. For (a), let first e be an edge in $G[A_i]$. By (a), there exists a $k \in \{1, \ldots, 7\}$ such that e is in the special set of $\mathcal{C}_k^i$. First observe that e is in $E(G) - E(\mathcal{C}_k)$. If e is in a 3-edge-cut F' of $G/\mathcal{C}_k$, since F' is not a 3-edge-cut of $G_i/\mathcal{C}_k^i$, F' contains an edge of $G[A_{3-i}]$. This yields that F and F' are crossing 3-edge-cuts of G, a contradiction to Proposition 1.

Now consider the edge e_j for some $j \in \{1, 2, 3\}$. As $e_j \in E(G_i) - E(\mathcal{C}_{2j}^i)$, we have $e_j \in E(G) - E(\mathcal{C}_{2j})$. Again, assume that e_j is in a 3-edge-cut F' of $G/\mathcal{C}_{2j}$. As F' is not a 3-edge-cut in $G_1/\mathcal{C}_{2j}^1$ and $G_2/\mathcal{C}_{2j}^2$, we obtain that F' and F are crossing in G contradicting Proposition 1. This finishes the proof of the claim. ■

By Theorem 11, G contains a perfect matching $\boldsymbol{M}$. Since G is cubic, the connected components of $G - M$ form a cycle packing $\boldsymbol{\mathcal{C}_1}$. Now consider the graph $\boldsymbol{G'} := G/\mathcal{C}_1$ (including arising loops) and let $\boldsymbol{T}$ be its set of odd-degree vertices.

Claim 2. *The edge set of G' can be partitioned into three T-joins $\boldsymbol{F_1}$, $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$.*

Proof As G is essentially 4-edge-connected by Claim 1, G' is essentially 4-edge-connected by Proposition 3(b). Every vertex v of G' corresponds to a cycle $C \in \mathcal{C}_1$. It follows that $d_{G'}(v) \geq d_G(C) \geq 4$, so G' is 4-edge-connected. By Theorem 10, there exist two edge-disjoint spanning trees F_1', F_2' of G'. By Proposition 2, each of them contains a T-join F_i, $i = 1, 2$. As $F_1 \cup F_2$ is Eulerian, $F_3 = E(G') - F_1 - F_2$ is also a T-join. ■

Claim 3. *For $i = 1, 2, 3$, there exist V-joins S_{2i} and S_{2i+1} of G such that $S_{2i} \cap S_{2i+1} = F_i$ and $S_{2i} \cup S_{2i+1} = (E - M) \cup F_i$.*

Proof For $i = 1, 2, 3$, let $\boldsymbol{T_i}$ be the set of vertices in V not incident to an edge in F_i. Let $C \in \mathcal{C}_1$ and let v_C be the associated vertex in $V(G')$. Observe that, as G is cubic and $F_i \subseteq M$ is a matching in G, we obtain $|V(C)| \equiv d_G(V(C)) = d_{G'}(v_c)$ and $|V(C) \cap V(F_i)| \equiv d_{F_i}(V(C)) = d_{F_i}(v_c)$. As F_i is a T-join in G', this yields $|T_i \cap V(C)| = |V(C)| - |V(C) \cap V(F_i)| \equiv d_{G'}(v_C) - d_{F_i}(v_C) \equiv 0$, so $|T_i \cap V(C)|$ is even. Hence, by Proposition 2, we obtain that $G - M$ contains a T_i-join $\boldsymbol{N_i}$. Let $\boldsymbol{S_{2i}} := F_i \cup N_i$ and $\boldsymbol{S_{2i+1}} := F_i \cup (E - M - N_i)$. By construction, we have that S_{2i} and S_{2i+1} are V-joins in G such that $S_{2i} \cap S_{2i+1} = F_i$ and $S_{2i} \cup S_{2i+1} = (E - M) \cup F_i$. ■

For $j = 2, \ldots, 7$, we define $\boldsymbol{\mathcal{C}_j}$ to be the set of nontrivial connected components of $G - S_j$. Observe that all of them are cycles as S_j is a V-join and G is cubic.

Claim 4. *$\mathcal{C}_1, \ldots, \mathcal{C}_7$ satisfy (a) and (b).*

Proof (a) For $e \in M$, since G is essentially 4-edge-connected, e is in the special set of $\mathcal{C}_1$.

For $e \in E - M$, let f and g be the two edges of M adjacent to e. Since F_1, F_2 and F_3 are disjoint, there is an F_i that contains neither f nor g. Then, since G is cubic and by Claim 3, one of the V-joins S_{2i} and S_{2i+1}, say S_j, contains e but none of the edges adjacent to e. It follows that both endvertices of e in G are in cycles of $\mathcal{C}_j$. As G is essentially 4-edge-connected, it follows that both endvertices of e in $G/\mathcal{C}_j$ are of degree at least 4. As G is essentially 4-connected, so is $G/\mathcal{C}_j$ by Proposition 3(b). This yields that e is in no 3-edge-cut of $G/\mathcal{C}_j$ and so e is in the special set of $\mathcal{C}_j$.

(b) For $e \in M$, by Claim 2, e is in exactly one F_i, say F_1. Then, by Claim 3, e is in $\mathcal{C}_4, \ldots, \mathcal{C}_7$ and not in $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$.

For $e \in E - M$, e is in $\mathcal{C}_1$ and, by Claim 3, in exactly one of $\mathcal{C}_{2i}$ and $\mathcal{C}_{2i+1}$ for $i = 1, 2, 3$. ■

Claim 4 finishes the proof of Lemma 3. ■

3.3. Cubic case

We first show how to conclude a cubic version of Theorem 2.

Theorem 14. *Let G be a cubic 3-edge-connected graph. Then $f(G) \le 7$.*

Proof Lemma 3 yields that $E(G)$ can be covered by 7 special sets $S_1, \ldots, S_7$. By Lemma 2, there exist orientations $D_1, \ldots, D_7$ of G such that S_i is deletable in D_i for $i = 1, \ldots, 7$. It follows that the Frank number of G is at most 7. ■

Next, we use Lemma 2 to show that perfect matchings with a certain additional property are deletable. As corollaries, we obtain Theorem 4 and a cubic version of Theorem 3.

Lemma 4. *Let M be a perfect matching of a cubic 3-edge-connected graph G intersecting every 3-edge-cut of G in exactly one edge. Then M is deletable.*

Proof As G is cubic and M is a perfect matching of G, the connected components of $G - M$ form a packing $\mathcal{C}$ of cycles. We show that $G/\mathcal{C}$ is 4-edge-connected. By Proposition 4(a) and since G is 3-edge-connected, so is $G/\mathcal{C}$. A 3-edge-cut of $G/\mathcal{C}$ would provide a 3-edge-cut of G intersecting M in 3 edges contradicting the assumption. It follows that M is the special set of $\mathcal{C}$ and therefore deletable by Lemma 2. ■

We first show how to conclude Theorem 4 from Lemma 4.

Proof (of Theorem 4) Let G be a 3-edge-colorable 3-edge-connected graph. Then G is cubic and has 3 disjoint perfect matchings M_1, M_2, M_3 covering the edge set of G. Let $\delta(X)$ be a 3-edge-cut of G. Since G is cubic and $d(X) = 3$, we obtain that $|X|$ is odd. Then, since M_i is a perfect matching, we obtain that $\delta(X)$ intersects each M_i. As $d(X) = 3$ and the matchings are disjoint, we obtain that $\delta(X)$ intersects each of M_1, M_2, M_3 exactly once. It follows by Lemma 4 that each of M_1, M_2, M_3 is deletable, so $f(G) \leq 3$. ■

Next, we prove in a similar way the following cubic version of Theorem 3.

Theorem 15. *Let G be a cubic 3-edge-connected graph that satisfies Conjecture 1. Then $f(G) \leq 5$.*

Proof By assumption, there exist 6 perfect matchings $M_1, \ldots, M_6$ of G covering each edge of G exactly twice.

Let $\delta(X)$ be a 3-edge-cut of G. Since G is cubic and $d(X) = 3$, we obtain that $|X|$ is odd. Then, since M_i is a perfect matching, $\delta(X)$ intersects each M_i. Since each of the 3 edges of $\delta(X)$ belongs to exactly 2 M_i's, $\delta(X)$ intersects each of $M_1, \ldots, M_6$ exactly once. It follows by Lemma 4 that each of $M_1, \ldots, M_6$ is deletable. As every edge of G is covered by at least one of $M_1, \ldots, M_5$, it follows that $f(G) \leq 5$. ■

3.4. Non-cubic case

We first show how to prove the general case of Theorem 2.

Proof (of Theorem 2) Let G be a counterexample minimizing $|V(G)|$.

Claim 5. *G is 2-vertex-connected.*

Proof For the sake of a contradiction, assume that G has a cut vertex v. So G has two non-trivial subgraphs G_1 and G_2 such that $G_1 = G/G_2$ and $G_2 = G/G_1$. As G is 3-edge-connected, so is G_i by Proposition 3(a). Since G_i is smaller than G, G_i has Frank number at most 7. So there exist 7 orientations D_j^i of G_i such that for each edge e of G_i, one of $D_j^i - \vec{e}$ is strongly connected. We can now construct the 7 orientations D_j of G by giving each edge in G_i its orientation in D_j^i also in D_j. Now consider an edge e of G_i and let $D_j^i - \vec{e}$ be strongly connected. Since $D_j^i - \vec{e} = (D_j - \vec{e})/D_j^{3-i}$ and D_j^{3-i} are strongly connected, Proposition 4(b) implies that so is $D_j - \vec{e}$. It follows that G has Frank number at most 7, a contradiction. ■

Let H_G be a cubic extension of G as defined in Section 2.3. By Claim 5 and Proposition 7(a), H_G is 3-edge-connected. Then, by Theorem 14, the Frank number of H_G is at most 7, that is there exist 7 orientations D'_i of H_G such that for each edge e of H_G, one of $D'_i - \vec{e}$ is strongly connected. Let D_i be the orientation of G obtained from D'_i by contracting the subgraphs C_v for all $v \in V(G)$. For any $e \in E(G) \subset E(H_G)$, one of $D'_i - \vec{e}$ is strongly connected, therefore, by Proposition 4(a), so is $D_i - \vec{e}$. It follows that the Frank number of G is at most 7, a contradiction. ■

The same reduction and Theorem 15 show Theorem 3.

4. Essentially 4-edge-connected graphs

This section is dedicated to proving Theorem 5. Again, first we prove the result for cubic graphs and then we show how it implies the non-cubic case.

4.1. Cubic case

In the case of essentially 4-edge-connected graphs, we can show that every matching is deletable. We prove the following slightly stronger statement.

Lemma 5. *Let G be an essentially 4-edge-connected graph, M a matching of G and $\mathcal{C}$ a cycle packing of $G - M$. Then there exists an orientation of G in which $\vec{M}$ is deletable and each cycle of $\mathcal{C}$ is oriented as a circuit.*

Proof Let $\boldsymbol{\mathcal{F}}$ be the set of maximal 2-edge-connected subgraphs of $G - M$. Let $\boldsymbol{G' = (V', E' \dot\cup M)}$ be the graph obtained from G by contracting each graph of $\mathcal{F}$. Note that $G' - M$ is a forest. Since G is essentially 4-edge-connected, by Proposition 3(b), so is G' and every vertex of degree 3 in G' is an original vertex of G. Then, since M is a matching of G, every vertex v of degree 3 in G' is incident to at least 2 edges e_v^1, e_v^2 in E'.

By Theorem 9, there exists an admissible pairing P of G'. As $G' + P$ is Eulerian, Proposition 5 yields that $G' + P$ has an Eulerian orientation $\vec{G'} + \vec{P}$ such that for each vertex v of degree 3 in G', one of $\vec{e}_v^1, \vec{e}_v^2$ enters v and the other one leaves v. By the definition of admissible pairings, $\vec{G'}$ is a well-balanced orientation of G'.

For all $F \in \mathcal{F}$, by Proposition 3(a), Theorem 8 and Proposition 4(b), there exists a strongly connected orientation $\vec{F}$ of F such that each cycle of $\mathcal{C}$ contained in F is oriented as a circuit.

Let $\vec{G}$ be the orientation of G obtained by combining $\vec{G'}$ and $\vec{F}$ for all $F \in \mathcal{F}$. Proposition 4(b) yields that $\vec{G}$ is strongly connected. Since each cycle C of $\mathcal{C}$ belongs to some $F \in \mathcal{F}$, C is oriented as a circuit in $\vec{G}$.

We will finish the proof by showing that $\vec{G} - \vec{e}$ is strongly connected for all $e \in M$. Since $\vec{F}$ is strongly connected for all $F \in \mathcal{F}$ and $\bigcup_{F \in \mathcal{F}} E(F)$ contains no edge in M, it suffices to prove, by Proposition 4(b), that $\vec{G'} - \vec{e}$ is strongly connected for all $e \in M$. Let X be a subset of V'. By Proposition 6, it is enough to prove that either at least two arcs or at least one arc of $\vec{E'}$ leave X.

If there are $x \in X$ and $y \in V' - X$ of degree at least 4, then, since G' is essentially 4-edge-connected, there is no 3-edge-cut separating x and y in G' and therefore, as $\vec{F'}$ is well-balanced, there are 2 arcs leaving X, and we are done.

Hence, by considering $V' - X$ and $\overleftarrow{G'}$ if necessary, we may assume without loss of generality that X only contains vertices of degree 3 and there is no arc of $\vec{E}'$ leaving X. By construction, every vertex v of X has at least one arc $\vec{e}_v^1$ or $\vec{e}_v^2$ of $\vec{E}'$ leaving v. As there is no arc of $\vec{E}'$ leaving X, we obtain that $\vec{G}'[X]$ contains a circuit $\vec{C}$ of arcs in $\vec{E}'$. This cycle C provides a contradiction since $G' - M$ is a forest. ∎

We are now ready to prove a cubic version of Theorem 5.

Theorem 16. *Let G be a cubic essentially 4-edge-connected graph. Then $f(G) \leq 3$.*

Proof Since G is cubic and 2-edge-connected, by Theorem 11, G has a perfect matching M_1 and the connected components of $G - M_1$ form a packing $\mathcal{C}$ of cycles. By Lemma 5, there exists an orientation D_1 of G such that each cycle of $\mathcal{C}$ is oriented as a circuit and M_1 is deletable in D_1. By Lemma 1, each $C_i \in \mathcal{C}$ contains a deletable arc $\vec{e}_i$ in D_1. Note that the connected components of $G - M_1 - \cup\{e_i : C_i \in \mathcal{C}\}$ form a packing of paths which is the union of two matchings M_2 and M_3. By Lemma 5, there exist orientations D_2 and D_3 of G such that M_2 is deletable in D_2 and M_3 is deletable in D_3. Since $E(G) = M_1 \cup M_2 \cup M_3 \cup \{e_i : C_i \in \mathcal{C}\}$, Theorem 16 follows. ∎

4.2. Non-cubic case

We now generalize the results of the previous part to arbitrary essentially 4-edge-connected graphs.

Proof (of Theorem 5). Let $G = (V, E)$ be a counterexample minimizing $|V|$.

Claim 6. *$G - v$ is 2-edge-connected for all $v \in V$.*

Proof For the sake of a contradiction, assume that $G - v$ is not 2-edge-connected for some $v \in V$. If $G - v$ is disconnected, we obtain a contradiction using the same argument as in the proof of Claim 5. We therefore have a partition $A_1 \cup A_2$ of $V - \{v\}$ such that A_1 and A_2 are only connected by a single edge e_0 in $G - v$. Let us denote the end-vertices of e_0 by $u_i \in A_i$. Consider the graph G_i that arises from G by contracting $A_{3-i} \cup \{v\}$ into a vertex v_i. Note that $E(G_1) \cap E(G_2) = \{e_0\}$. Since G is essentially 4-edge-connected, so is G_i. Moreover, G_i is smaller than G. It follows that there exist 3 orientations D_j^i of G_i such that one of $D_j^i - \vec{e}$ is strongly connected for all $e \in E(G_i)$. We may suppose that $D_1^1 - \vec{e}_0$ and $D_1^2 - \vec{e}_0$ are strongly connected. Reversing the arcs in D_j^i if needed, we may assume that e_0 has the same orientation in D_j^1 and D_j^2. We can construct the 3 orientations D_j of G by merging D_j^1 and D_j^2. We will finish the proof by showing that for all $e \in E$, there exists a j such that $D_j - \vec{e}$ is strongly connected. Let $e \in E$ and $j \in \{1, 2, 3\}$ such that both $D_j^1 - \vec{e}$ and $D_j^2 - \vec{e}$ are strongly connected. Observe that if $e \neq e_0$, then either $D_j^1 - \vec{e} = D_j^1$ or $D_j^2 - \vec{e} = D_j^2$. Assume that there is a nonempty, proper subset X of V that has no arc leaving in $D_j - \vec{e}$. Without loss of generality, we may assume that $v \in X$. As $(X \cap A_i) \cup \{v_i\}$ has an arc leaving in $D_j^i - \vec{e}$, e_0 must be directed away from v_i in D_j^i for $i = 1, 2$. This is a contradiction as D_j^1 and D_j^2 were chosen to both have the same orientation of e_0. ∎

Let H_G be a cubic extension of G as defined in Section 2.3. By Claim 6 and Proposition 7(b), H_G is a cubic essentially 4-edge-connected graph. Then, by Theorem 16, the Frank number of H_G is at most 3. There exist therefore 3 orientations D'_j of H_G such that for each edge $e \in E(H_G)$, there is some $j \in \{1,2,3\}$ such that $D'_j - \vec{e}$ is strongly connected. Consider now the 3 orientations D_j of G which arise from D'_j by contracting the subgraphs C_v for all $v \in V$. By Proposition 4(a), if $D'_j - \vec{e}$ is strongly connected for an edge $e \in E$, so is $D_j - \vec{e}$. It follows that the Frank number of G is at most 3, a contradiction. ■

5. The Petersen graph

In this section, we show that there are graphs of Frank number higher than two, more precisely we prove Theorem 6. While this result can also be established computationally, we prefer to give a proof by hand.

Proof (of Theorem 6) Let $\boldsymbol{G} = (V,E)$ be the Petersen graph, see Figure 1. We frequently make use of the symmetry properties of G. By Theorem 5 and since G is essentially 4-edge-connected, but not 4-edge-connected, it suffices to prove that its Frank number is different from 2. Suppose that G has Frank number 2 and let $\boldsymbol{D_1} = (V, A_1)$ and $\boldsymbol{D_2} = (V, A_2)$ be two orientations of G such that

$$D_1 - \vec{e} \text{ or } D_2 - \vec{e} \text{ is strongly connected for each edge } e \text{ of } G. \qquad (*)$$

We say that an arc of D_1 is *stable* if the same arc exists in D_2, otherwise it is *changing*. Let $\boldsymbol{S}$ and $\boldsymbol{C}$ be the set of stable and changing arcs, respectively. Note that D_1 and $\overleftarrow{D}_2$ also satisfy $(*)$ and stable and changing arcs are exchanged. Hence, whatever is proved for stable arcs is also true for changing arcs.

We first show that S and C induce a 2-edge-coloring of G with certain properties and then that no such 2-edge-coloring exists. Observe that none of the considered colorings are required to be proper. For a 2-edge-coloring R, B of G, we define an auxiliary graph $\boldsymbol{H^{R,B}} := (V, F)$ where $uv \in F$ if there exists a 3-path $tuvw$ in $G(R)$ or in $G(B)$ or there exists a (u,v)-path that is a connected component of $G(R)$ or of $G(B)$.

Lemma 6. *G has a 2-edge-coloring R, B such that*

$$\text{no monochromatic 3-star exists,} \qquad (1)$$

$$H^{R,B} \text{ is bipartite.} \qquad (2)$$

Proof We show that the 2-edge-coloring induced by S and C satisfies (1) and (2). To show (1) we need the following claim.

Claim 7. *Each vertex is incident to at least one stable arc.*

Proof Suppose that a vertex v is incident only to changing arcs. Since G is cubic and D_1 is strongly connected, either the in-degree or the out-degree of v is 1, say $\vec{e}$ is the only arc entering v. Then $\overleftarrow{e}$ is the only arc leaving v in D_2. Then, $D_1 - \vec{e}$ and $D_2 - \overleftarrow{e}$ are not strongly connected, which is a contradiction. ■

To show (2) we need the following claims.

Claim 8. *The weakly connected components of $D_1(S)$ are directed paths or circuits.*

Proof By Claim 7 applied for stable arcs and then for changing arcs, the connected components of $D_1(S)$ are paths or cycles. If two stable arcs are incident to a vertex v then one of them enters and the other one leaves v. Otherwise, let e be the third arc incident to v. Then, $D_1 - \vec{e}$ and $D_2 - \vec{e}$ are not strongly connected, which is a contradiction. Now the claim follows. ■

Claim 9. *Let P be a weakly connected component of $D_1(S)$ that is a directed (u, v)-path. Then the in-degrees of u and v in D_1 are of different parity.*

Proof Since G is cubic and u and v are incident to exactly one stable arc in D_1, u and v are incident to exactly two changing arcs in D_1. Then, by Claim 8 applied for $D_1(C)$, exactly one changing arc enters both u and v in D_1. Since P is a directed path between u and v, the claim follows. ■

Claim 10. *Let $tuvw$ be a 3-path in $D_1(S)$. Then the in-degrees of u and v are of different parity in D_1.*

Proof By Claim 8, exactly one stable arc enters both u and v in D_1. By Claim 7, the two other arcs incident to u and v are changing. If both are entering or leaving then $D_1 - uv$ and $D_2 - uv$ are not strongly connected, which is a contradiction. Now the claim follows. ■

Claim 11. *$H^{S,C}$ is a bipartite graph.*

Proof Since G is cubic and D_1 and D_2 are strongly connected, each vertex is of in-degree 1 or 2. By Claims 9 and 10, each edge of $H^{S,C}$ is between a vertex of in-degree 1 and a vertex of in-degree 2, so $H^{S,C}$ is bipartite. ■

By Claim 7 applied for $R := S$ and $B := C$ and by Claim 11, Lemma 6 follows. ■

We show that G does not admit any 2-edge-coloring satisfying (1) and (2) and obtain a contradiction to Lemma 6.

The following result yields a strong property such a coloring would have to satisfy.

Lemma 7. *Let R, B be a 2-edge-coloring satisfying (1) and (2). Then G has a 5-cycle that contains a monochromatic 4-path whose end-vertices are incident to 2 edges of the other color.*

Proof We first show two weaker statements which are useful in the proof later on.

Claim 12. *G has a monochromatic 3-path.*

Proof Suppose not. Since G is cubic, there are two adjacent edges of the same color, without loss of generality $ab, ae \in R$. Then, by the assumption for $deab, eabc, eabi$ and $heab$, we obtain that $de, bc, bi, eh \in B$. Thus, by the assumption for $cbih, cbij$ and $dehg$, we obtain that $jihg$ forms a monochromatic 3-path, contradicting the assumption. See Figure 2a. ■

This result is helpful in proving a strengthening of itself.

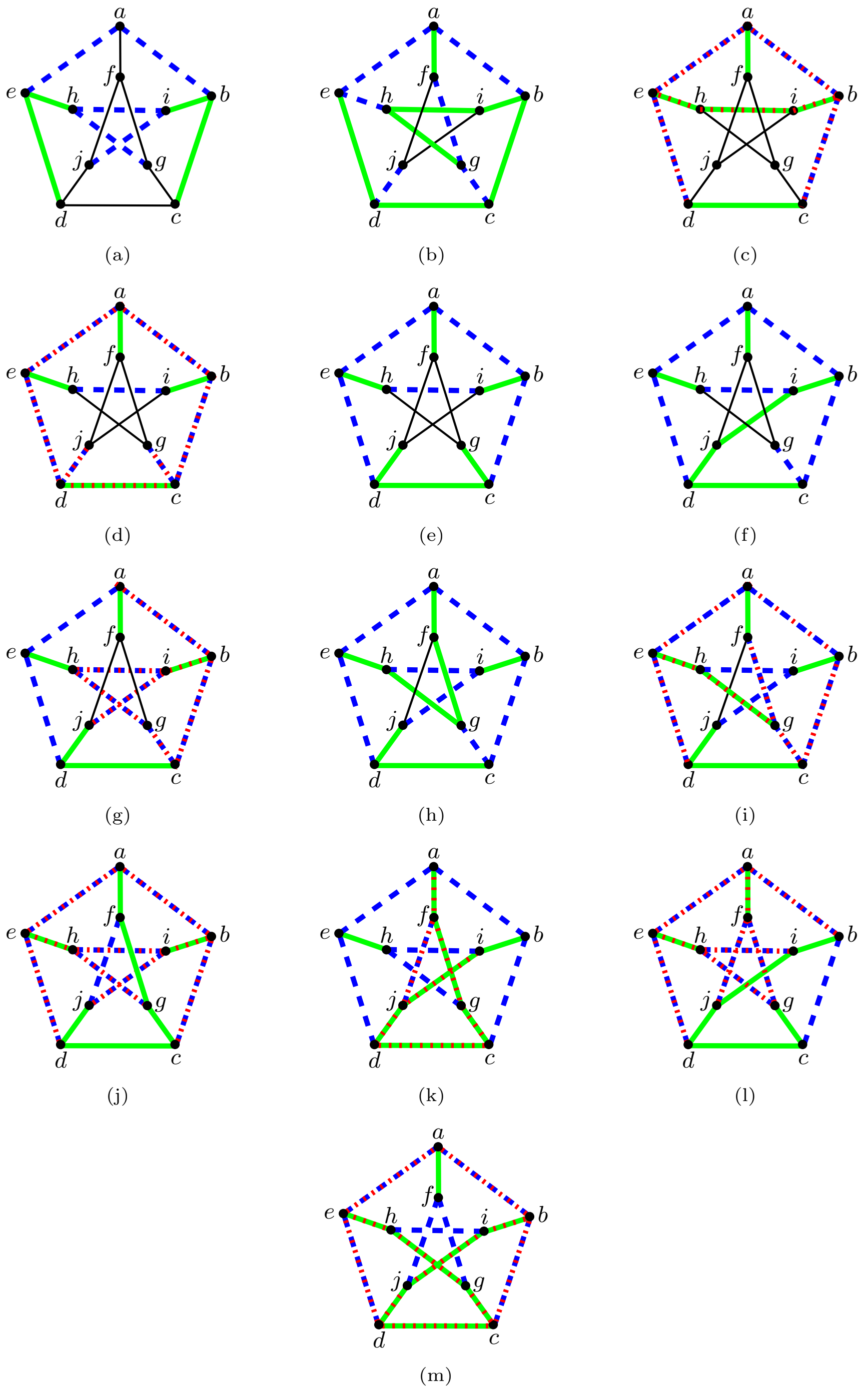

Figure 2

Claim 13. *G has a 5-cycle that contains a monochromatic 4-path.*

Proof Suppose not. By Claim 12, without loss of generality $bc, cd, de \in B$. Then, by the assumption for $abcde$, we obtain that $ab, ae \in R$. By (1) for a, c and d, we obtain that $af \in B$ and $cg, dj \in R$. By (1) for f, one of fg and fj is in R. By symmetry, without loss of generality $fg \in R$. Then, by (1) for g, $gh \in B$. So, by the assumption for $cdehg$, we obtain that $eh \in R$. Then, by the assumption for $abihe$, we obtain that $hi, bi \in B$. Thus $cbihg$ forms a monochromatic 4-path in the 5-cycle $cbihg$, that contradicts the assumption. See Figure 2b. ∎

By Claim 13, without loss of generality $ab, bc, de, ea \in R$. Then, by (2) for $abcde$, we obtain that $cd \in B$. By (1) for a, b and e, we obtain that $af, bi, eh \in B$. If $hi \in B$, then, by the 3-paths of $deabc$ and by $ehib$, we obtain that $H^{R,B}$ contains the 3-cycle abe that contradicts (2). See Figure 2c. Hence, $hi \in R$. If $cg, dj \in R$, then, by the 3-paths of $jdeabcg$ and by cd, we obtain that $H^{R,B}$ contains the 5-cycle $abcde$ that contradicts (2). See Figure 2d. Hence, by symmetry, we may suppose that $dj \in B$.

Now suppose for the sake of a contradiction that G does not contain a 5-cycle that contains a monochromatic 4-path whose end-vertices are incident to 2 edges of the other color. If $cg \in B$, then $abcde$ contradicts the assumption. See Figure 2e. Hence $cg \in R$. If $ij \in B$, then $bcdji$ contradicts the assumption. See Figure 2f. Hence $ij \in R$. If $gh \in R$, then, by the 3-paths of $abcghij$ and by bi, we obtain that $H^{R,B}$ contains the 5-cycle $bcghi$ that contradicts (2). See Figure 2g. Hence $gh \in B$. If $fg \in B$, then $afghe$ contradicts the assumption. See Figure 2h. Hence $fg \in R$. Then, by the 3-paths of $deabcgf$ and by ehg, we obtain that $H^{R,B}$ contains the 5-cycle $abcge$ that contradicts (2). See Figure 2i. This finishes the proof of Lemma 7. ∎

Lemma 7 yields that G has a 5-cycle, without loss of generality $abcde$, that contains a monochromatic 4-path whose end-vertices are incident to 2 edges of the other color. By similar arguments as before, we obtain the partial coloring of Figure 2e. By (1) for f, one of fg and fj is in R. By symmetry, without loss of generality $fj \in R$.

Suppose that $fg \in B$. Then, by (1) for g, $gh \in R$. If $ij \in R$, then, by the 3-paths of $deabc$ and for $ghij$, and by eh and ib, $H^{R,B}$ contains the 5-cycle $eabih$ contradicting (2). See Figure 2j. If $ij \in B$, then, by the 3-paths of $afgcdji$ and by fj, we obtain that $H^{R,B}$ contains the 5-cycle $fgcdj$ contradicting (2). See Figure 2k.

Hence $fg \in R$. Then, by (2) for $fghij$, one of hg and ij is in B. By symmetry, we may suppose that $ij \in B$. If $hg \in R$, then, by the 3-paths of $jfghi$ and for $deab$, and by eh and af, we obtain that $H^{R,B}$ contains the 5-cycle $fghea$ contradicting (2). See Figure 2l. If $hg \in B$, then, by the 3-paths of $deabc$ and by $bijdcghe$, we obtain that $H^{R,B}$ contains the 3-cycle eab contradicting (2). See Figure 2m.

In all cases we obtain a contradiction which implies that G has Frank number different from 2. This finishes the proof of Theorem 6. ∎

6. Algorithmic aspects

This section is dedicated to proving Theorem 7.

Our reduction is from a slightly stronger variation of MNAE3SAT. In the first part, we introduce this problem and show that it is NP-complete by a reduction from MNAE3SAT. Next, we introduce our construction and show that the constructed graph is cubic and 3-edge-connected. The last two parts are dedicated to showing that the reduction works indeed.

6.1. Boolean formulas

Given a MNAE3SAT formula $F = (X, \mathcal{C})$, we call a truth assignment to the variables of X *feasible* if every clause of $\mathcal{C}$ contains at least one true and at least one false literal. We define the formula graph $\boldsymbol{G_F}$ by $V(G_F) = X \cup \mathcal{C}$ and there is an edge between the vertices corresponding to a variable x_i and a clause C_j if x_i is contained in C_j. We call a formula F *connected* if G_F is connected. We show that MNAE3SAT stays NP-complete with this additional assumption.

Connected Monotone Not-all-equal-3SAT(CMNAE3SAT)

Instance: A set X of boolean variables, a connected formula consisting of a set $\mathcal{C}$ of clauses each containing 3 distinct variables none of which are negated.

Question: Is there a feasible truth assignment to the variables of X?

Lemma 8. *CMNAE3SAT is NP-complete.*

Proof We show a reduction from MNAE3SAT. Recall that MNAE3SAT is NP-complete by Theorem 13. Let F be a MNAE3SAT formula. Let $G_1, \dots, G_t$ be the connected components of G_F. For $i = 1, \dots, t$, consider the MNAE3SAT formula F_i that consists of the variables and clauses corresponding to vertices in G_i. Observe that $G_{F_i} = G_i$ and so every F_i is an instance of CMNAE3SAT. We will show that F is a positive instance of MNAE3SAT if and only if all of the F_i are positive instances of CMNAE3SAT. First assume that there is a feasible truth assignment for F. The restriction of this assignment to the variables of F_i yields a feasible truth assignment for F_i for all $i = 1, \dots, t$. Now assume that there is a feasible truth assignment for F_i for $i = 1, \dots, t$. As every vertex corresponding to a variable is contained in exactly one component, every variable is contained in exactly one of the F_i and so we obtain a unique assignment of boolean values to all variables. As every clause of $\mathcal{C}$ is contained in some F_i, this assignment is feasible for F. This finishes the proof. ■

6.2. The construction

Let $F = (X, \mathcal{C})$ be a CMNAE3SAT formula with $X = \{x_1, \dots, x_m\}$. If there is a variable $x \in X$ that is contained in only one clause $C \in \mathcal{C}$, then F is satisfiable if and only if $(X - \{x\}, \mathcal{C} - \{C\})$ is satisfiable. We may therefore assume that every $x_i \in X$ is contained in at least 2 clauses. For $i = 1, \dots, m$, we define $\boldsymbol{p_i}$ to be the number of clauses x_i is contained in.

We now construct an instance $(G = (V, E), S)$ of DELETABILITY. For $i = 1, \dots, m$, G contains a cycle $\boldsymbol{K_i}$ of length $2p_i$. We abbreviate $V(K_i)$ to $\boldsymbol{V_i}$ and $E(K_i)$ to $\boldsymbol{E_i}$. Observe that V_i can be partitioned into two stable sets in a unique way. We call one of these sets $\boldsymbol{A_i}$ and the other one $\boldsymbol{B_i}$. Note that $|A_i| = |B_i| = p_i$. For every clause C, G contains a vertex $\boldsymbol{v_C}$. We denote $\{v_C : C \in \mathcal{C}\}$ by $\boldsymbol{V_{\mathcal{C}}}$. Further, G contains a cycle $\boldsymbol{K}$ of length $3|\mathcal{C}|$. We abbreviate $V(K)$ to

$\boldsymbol{V_K}$ and $E(K)$ to $\boldsymbol{E_K}$. We add a perfect matching between $\{v_C : x_i \in C\}$ and A_i for every $i = 1, \dots, m$ and between $\bigcup_{i=1}^m B_i$ and V_K. Observe that this is possible because $|A_i| = p_i$ and $|\bigcup_{i=1}^m B_i| = \sum_{i=1}^m p_i = 3|\mathcal{C}| = |V_K|$. Finally, we define $\boldsymbol{S} = \bigcup_{i=1}^m E_i$. Note that $|V| = 10|\mathcal{C}|$ and $|E| = 15|\mathcal{C}|$, so the construction is polynomial indeed.

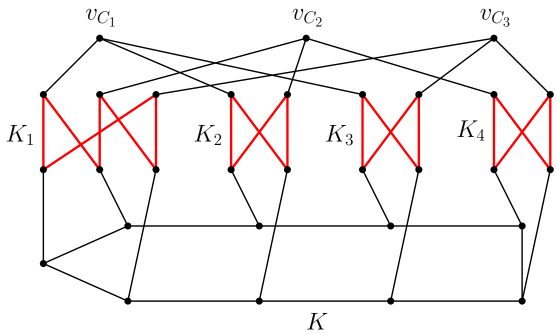

Figure 3

Figure 3 shows the constructed graph for the formula consisting of the variables $x_1, \dots, x_4$ and the clauses $C_1 = \{x_1, x_2, x_3\}, C_2 = \{x_1, x_2, x_4\}$ and $C_3 = \{x_1, x_3, x_4\}$. The edges of S are marked in red.

Observe that G is cubic as every clause contains exactly 3 variables and by construction. We show that it also satisfies the other desired structural property.

Lemma 9. *G is 3-edge-connected.*

Proof Assume for the sake of a contradiction that G contains some cut $\delta(Z)$ which consists of at most 2 edges. Without loss of generality, we may assume that $V_K \cap Z$ is nonempty.

Claim 14. *$V_K \subseteq Z$.*

Proof Assume that there is a vertex $w \in V_K - Z$. As $G - V_K$ arises from G_F by replacing vertices by cycles and G_F is connected by assumption, $G - V_K$ is connected. Then, since a perfect matching exists between $\bigcup_{i=1}^m B_i$ and V_K, we obtain that $G - E_K$ is also connected. As K is 2-edge-connected, it follows that $2 \geq d_G(Z) = d_K(Z) + d_{G-E_K}(Z) \geq 2 + 1 = 3$, a contradiction. ■

Claim 15. *$V_{\mathcal{C}} \subseteq Z$.*

Proof Consider a vertex v_C where C contains the variables x_i, x_j, x_ℓ. By construction, both v_C and K have a neighbor in each of V_i, V_j and V_ℓ and K_i, K_j and K_ℓ are connected. As $V_K \subseteq Z$ by Claim 14, there are 3 edge-disjoint paths from v_C to Z. It follows, by $d_G(Z) \leq 2$, that $v_C \in Z$. ■

By Claims 14 and 15, there exists a vertex $v \in V_i - Z$ for some $i = 1, \dots, m$ and v is connected to $V_K \cup V_{\mathcal{C}} \subseteq Z$ by a path of length 1 and two paths of length 2 and all of these are edge-disjoint. This is a contradiction to Z being separated from v by a cut of at most 2 edges. This finishes the proof of Lemma 9. ■

The remaining part of this section is dedicated to showing that our construction is indeed correct, i.e. F is a positive instance of CMNAE3SAT if and only if (G, S) is a positive instance of DELETABILITY.

6.3. *From orientation to truth assignment*

Suppose that (G, S) is a positive instance of DELETABILITY, so there is an orientation D of G such that $D - \vec{s}$ is strongly connected for all $s \in S$. Before finding a feasible truth assignment of the formula, we need the following result about the orientation.

Claim 16. *Let $i \in \{1, \ldots, m\}$. Then all the arcs between A_i and B_i are directed in the same way.*

Proof Let v be any vertex of K_i and e, f the two edges of K_i incident to v. Since $e, f \in S$, $D - e$ and $D - f$ are strongly connected. Then, as G is cubic, both of e and f are either entering or leaving v. Since K_i is connected, the claim follows. ■

Using Claim 16, we now define a truth assignment of X in the following way: a variable x_i is assigned the value true if the arcs between A_i and B_i are directed from B_i to A_i and false if the arcs between A_i and B_i are directed from A_i to B_i.

Consider a clause $C = \{x_i, x_j, x_\ell\}$. The vertex v_C has one neighbor in each of A_i, A_j and A_ℓ in G. As D is strongly connected and G is cubic, v_C has one in-neighbor w, say in A_ℓ and w has an in-neighbor in $D[V_\ell]$. It follows by construction that x_ℓ is set to true in the truth assignment. Similarly, one of x_i, x_j, x_ℓ is set to false. It follows that the assignment is feasible.

6.4. *From truth assignment to orientation*

Assume that there is a feasible truth assignment for an instance F of CM-NAE3SAT consisting of a variable set $X = \{x_1, \ldots, x_m\}$ and a clause set $\mathcal{C}$. Relabeling variables, we may assume that there is some $\boldsymbol{t} \in \{0, \ldots, m\}$ such that x_i is set to true for $i = 1, \ldots, t$ and x_i is set to false for $i = t + 1, \ldots, m$. Let $\boldsymbol{\mathcal{A}_1} = \bigcup_{i=1}^{t} A_i$, $\boldsymbol{\mathcal{A}_2} = \bigcup_{i=t+1}^{m} A_i$, $\boldsymbol{\mathcal{B}_1} = \bigcup_{i=1}^{t} B_i$ and $\boldsymbol{\mathcal{B}_2} = \bigcup_{i=t+1}^{m} B_i$.

We define an orientation $\boldsymbol{D}$ of G as follows. We orient all edges from P to R where P and R are two consecutive sets in $\mathcal{A}_1, V_\mathcal{C}, \mathcal{A}_2, \mathcal{B}_2, V_K, \mathcal{B}_1, \mathcal{A}_1$. Finally, we orient the edges of K as a circuit.

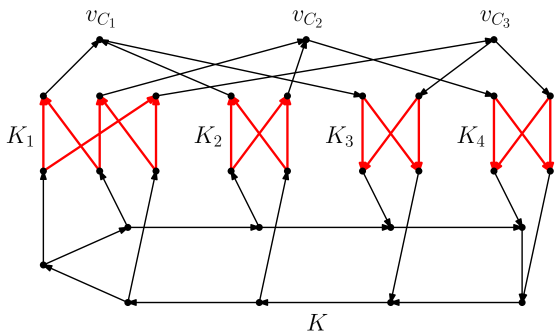

Figure 4

Figure 4 shows the obtained orientation for the formula consisting of the variables $x_1, \ldots, x_4$ and the clauses $C_1 = \{x_1, x_2, x_3\}, C_2 = \{x_1, x_2, x_4\}$ and

$C_3 = \{x_1, x_3, x_4\}$ when x_1 and x_2 are set to true and x_3 and x_4 are set to false.

The following is the orientation's decisive property:

Claim 17. *In D, every vertex $v_C \in V_{\mathcal{C}}$ has an in-neighbor in $\mathcal{A}_1$ and an out-neighbor in $\mathcal{A}_2$.*

Proof Let C contain the 3 variables x_i, x_j and x_ℓ. As the truth assignment is feasible, one of x_i, x_j, x_ℓ, say x_i, is set to true and a different one, say x_j, is set to false. Then, by construction, D contains an arc from $A_i \subseteq \mathcal{A}_1$ to v_C and an arc from v_C to $A_j \subseteq \mathcal{A}_2$. ■

The following result will finish the proof:

Claim 18. *Let $s \in S$. Then $D - \vec{s}$ is strongly connected.*

Proof Since K is oriented as a circuit, all vertices of K are in the same strongly connected component $\boldsymbol{Q}$. By construction, all vertices in $\mathcal{B}_1$ have an in-neighbor in $V_K \subseteq Q$ and all vertices in $\mathcal{A}_1$ have 2 in-neighbors in $\mathcal{B}_1$ in D, so at least one in $D - \vec{s}$. It follows, by Claim 17, that all vertices in $\mathcal{A}_1 \cup \mathcal{B}_1 \cup V_{\mathcal{C}}$ are reachable from Q. By similar arguments, Q is reachable from all vertices in $\mathcal{A}_2 \cup \mathcal{B}_2 \cup V_{\mathcal{C}}$. This yields that $V_{\mathcal{C}} \subseteq Q$. Finally, from every vertex in $\mathcal{A}_1 \cup \mathcal{B}_1$ there exists a directed path of length 1 or 2 to a vertex $v_C \in V_{\mathcal{C}}$. Similarly, to every vertex in $\mathcal{A}_2 \cup \mathcal{B}_2$ there exists a directed path of length 1 or 2 from a vertex $v_C \in V_{\mathcal{C}}$. It follows that $D - \vec{s}$ is strongly connected. ■

This reduction proves Theorem 7.

7. Conclusion

Our work shows that $f(G) \leq 7$ for every 3-edge-connected graph G and that $f(G) = 3$ if G is the Petersen graph. Also, we show a better bound for the more restricted classes of essentially 4-edge-connected graphs and 3-edge-colorable, 3-edge-connected graphs. Further, we show that a graph of Frank number bigger than 5 would imply the failure of Conjecture 1. Moreover, the decision problem whether all edges of a given subset can become deletable in one orientation is proven to be NP-complete.

The most obvious remaining problem is to improve these bounds on the Frank number in the general case. Considering the indications found during our work, we propose the following conjecture:

Conjecture 2. *Every 3-edge-connected graph G satisfies $f(G) \leq 3$.*

A possible way to make progress towards Conjecture 2 would be the following generalization of Lemmas 4 and 5. Using the fact that cubic graphs are 4-edge-colorable [14] and similar arguments as before, Conjecture 3 would imply that $f(G) \leq 4$ for any 3-edge-connected graph.

Conjecture 3. *Let M be a matching of a 3-edge-connected graph G intersecting each 3-edge-cut of G in at most one edge. Then M is deletable.*

It would also be interesting to generalize Frank numbers to arbitrary odd connectivity:

Open Problem 1. *Given a $(2k+1)$-edge-connected graph G, what is the minimum number of k-arc-connected orientations such that each edge becomes an arc whose deletion does not destroy k-arc-connectivity in at least one of these orientations?*

It follows from a theorem in [1] that this number is bounded by a constant depending only upon k. We are particularly interested in whether or not this number can be bounded by a constant not depending upon k.

8. Acknowledgements

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