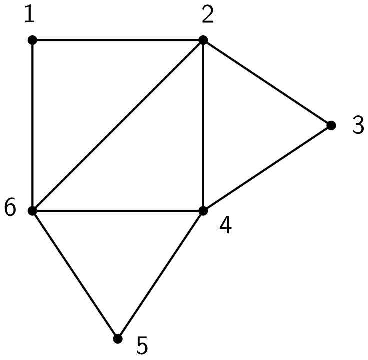
Enumeration algorithms in graphs

Aurélie Lagoutte
G-SCOP, Grenoble INP / Université Grenoble Alpes

G@g project – January 2026

Enumeration : a typical example

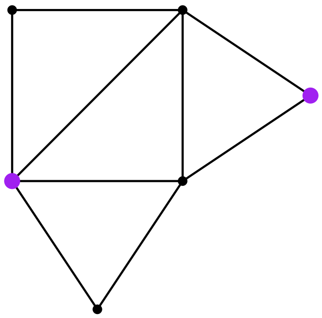
Input: Graph G
Output : The list of all **inclusion-wise maximal** stable sets of G



$\{1,3,5\},\{1,4\},\{2,5\},\{3,6\}$

Focus on easy problems

Input: Graph G
Output : one **inclusion-wise maximal** stable set. $\in$ P (greedy)



Not to be confused with :
Input: Graph G
Output : a stable set of maximum size

NP-complete

Enumerating in graphs : useful cases

- Graph databases : answer to a query
- Graph model is not exact : some solutions are *best* based on qualitative criteria, we have to examine them one by one
- Identify all problematic (or interesting !) patterns in a network

Application fields: bioinformatics (phylogenetic trees), chemistry (molecule structure), complex system modeling, databases...

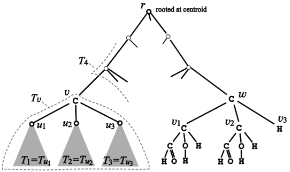


Diagram of a stereoisomer[1]

[1] Comparison and Enumeration of Chemical Graphs, T. Akutsu, H. Nagamochi, *Comp. and Struct. Biotechnology Journal*

Complexity for enumeration problems

In most cases : exponential number of solutions to output
(ex: $3^{n/3}$ max. stable sets)
$\Rightarrow$ *Good complexity measure?*

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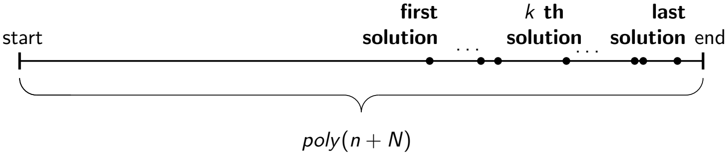
Input of size n, N solutions to output.



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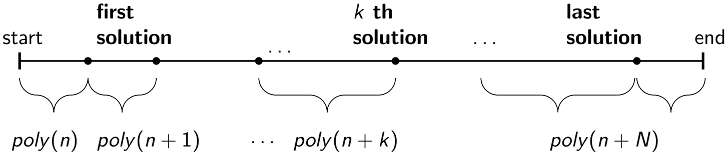

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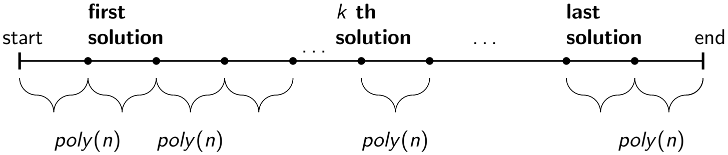

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poly space vs. exponential space

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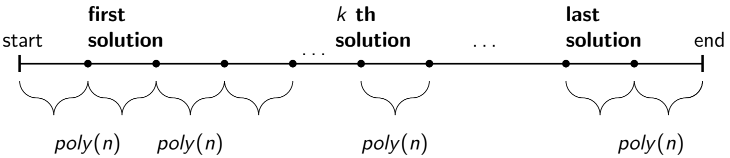


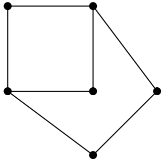
Interesting objects to enumerate

1. Inclusion-wise minimal transversal of a hypergraph
2. Inclusion-wise minimal dominating sets
3. Spanning trees
4. "Structured patterns" : inclusion-wise max. stable sets or cliques ...
5. **Inclusion-wise minimal "Π-fixings" of a graph**
   $\rightarrow$ Completions, deletion, induced subgraphs of a graph ...
   ... satisfying a given property Π

Minimal fixings

3 variants

We want to satisfy a given property Π
Example : $\Pi = C_4$-free
(contains no induced C_4)



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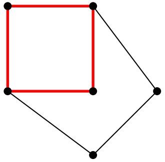

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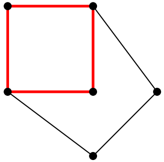
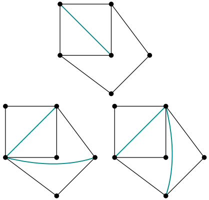


Fixing by adding edges
Min. Π-completion



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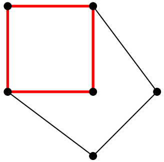

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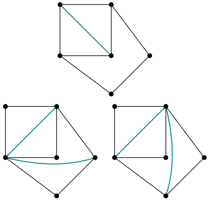

Fixing by
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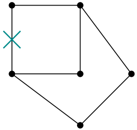
Min. Π-deletion



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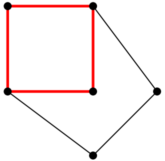
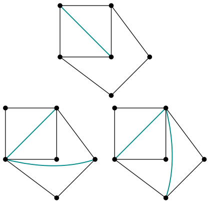
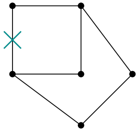
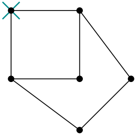
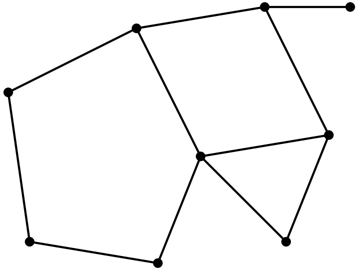


Fixing by adding edges	Fixing by deleting edges	Fixing by deleting vertices
Min. Π-completion	**Min. Π-deletion**	**Max. Π-induced subgraph**
	 + 3 others	 + 3 others

Chordal completion

Chordal completion of a graph G : a completion of G that is chordal (no chordless cycle of length ≥ 4).

A chordal completion of G is also called a *triangulation* of G or sometimes a *fill-in*.



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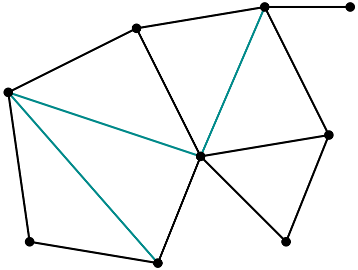
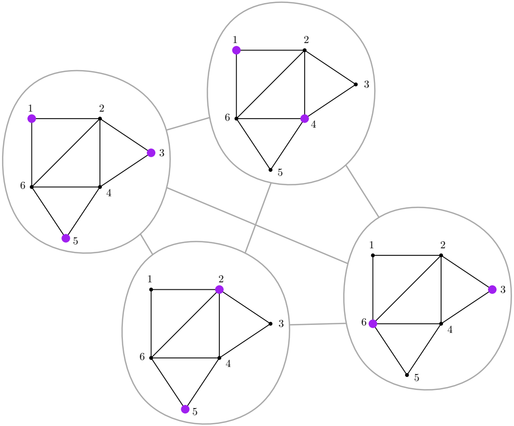


Algorithmic methods to enumerate

General principle of an enumeration algorithm

- Metagraph of solutions : traversal of this metagraph

Example with
maximal stable sets
Solution metagraph



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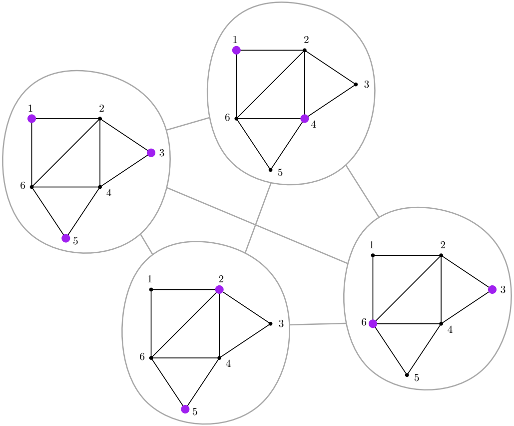

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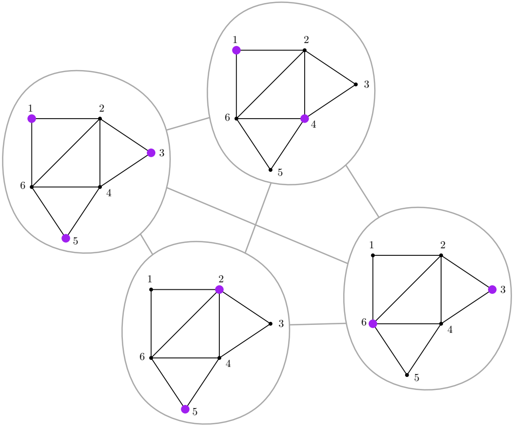

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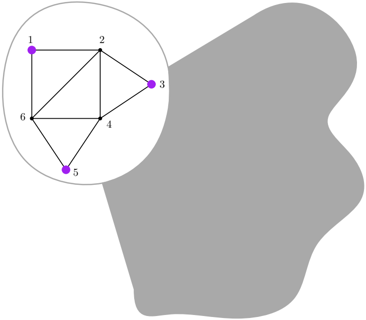

Three classical methods :

Goal : get polynomial delay
+ poly space for 1 et 2 (and sometimes 3)

1. *Flashlight search* or *Binary partition*
   [Read, Tarjan '75]
2. *Reverse search*
   [Avis, Fukuda '96]
3. *Proximity Search*
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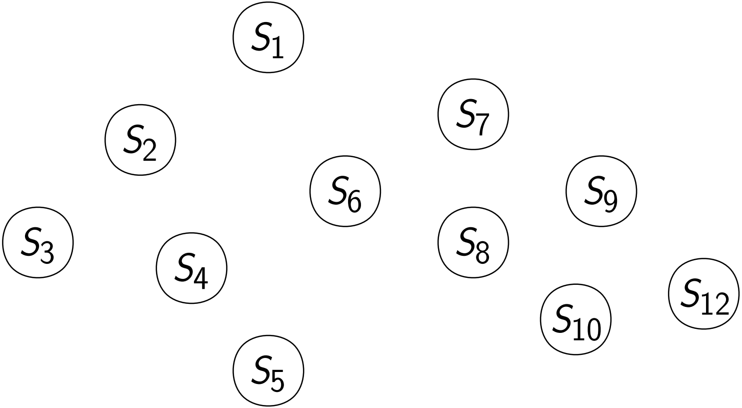
Reverse search

Solution space



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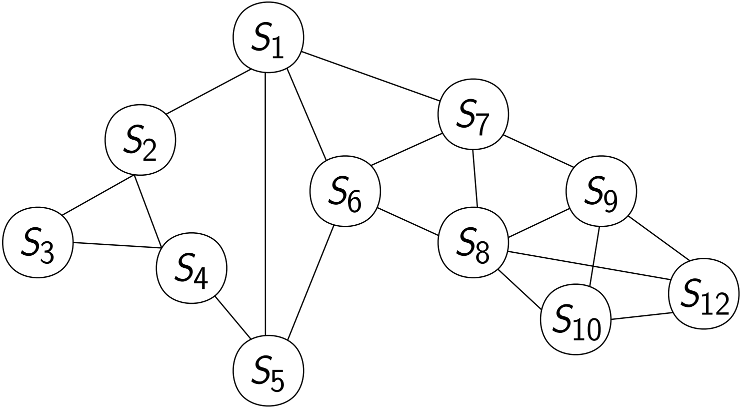
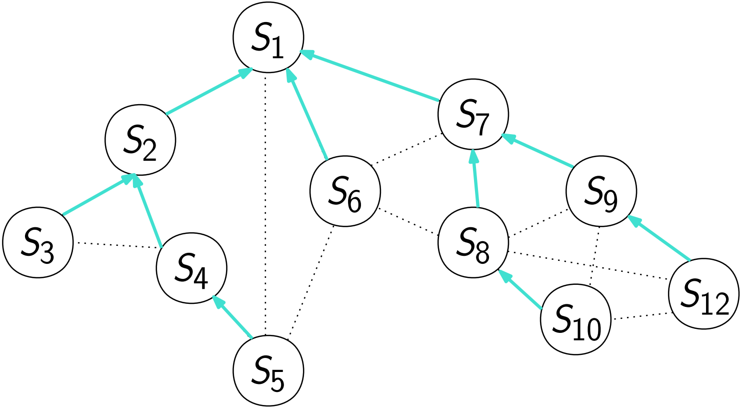


Reverse search

Solution tree



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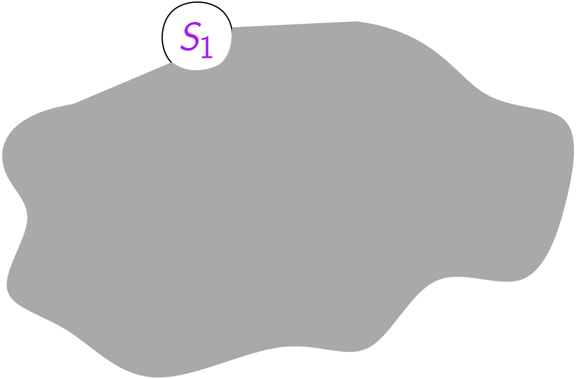

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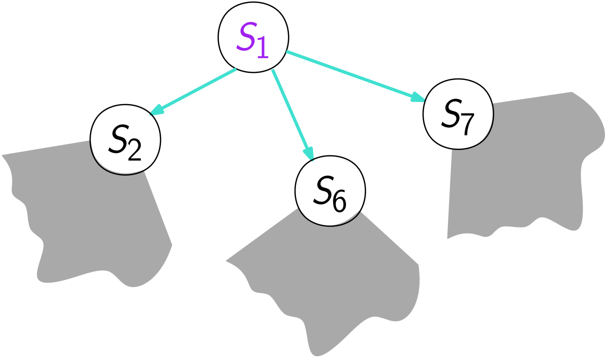

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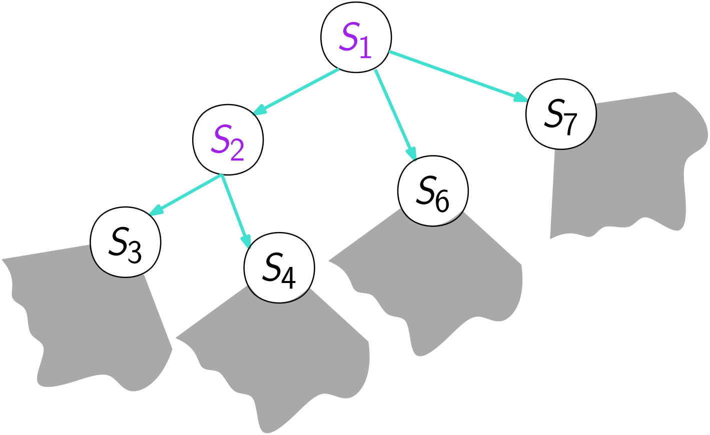

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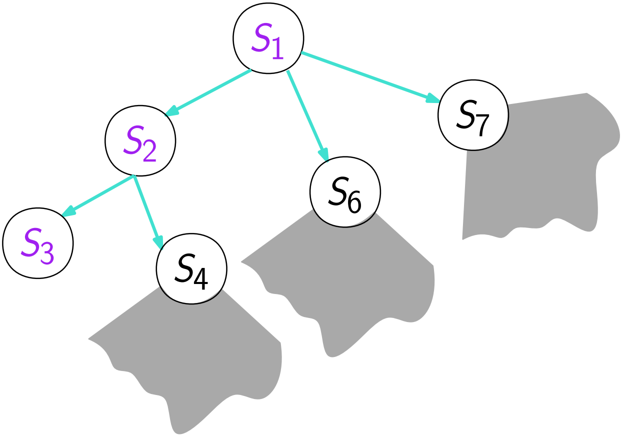

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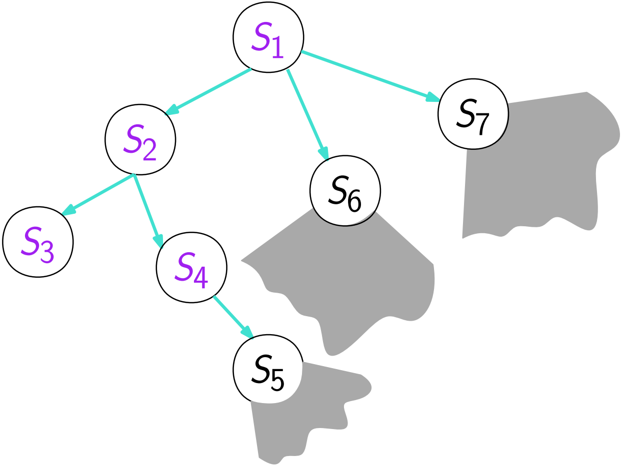

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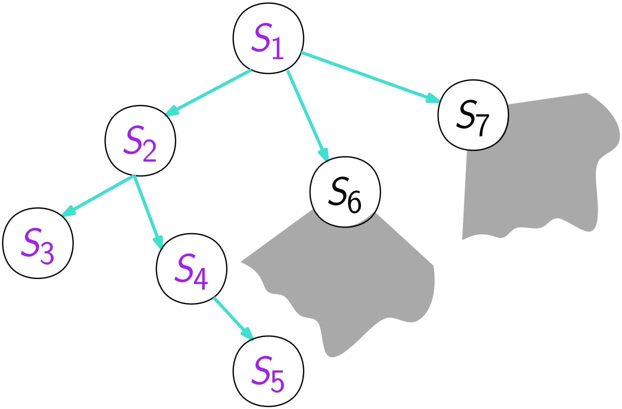

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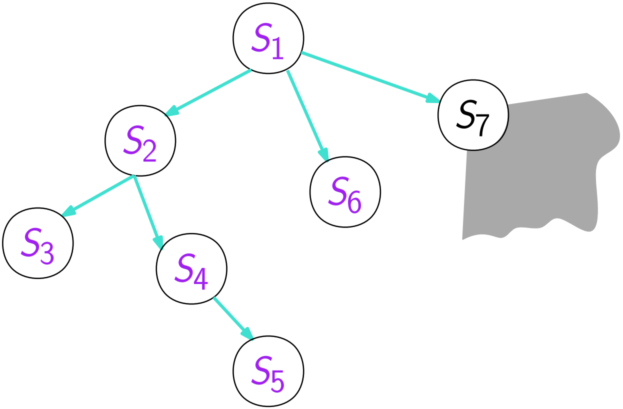
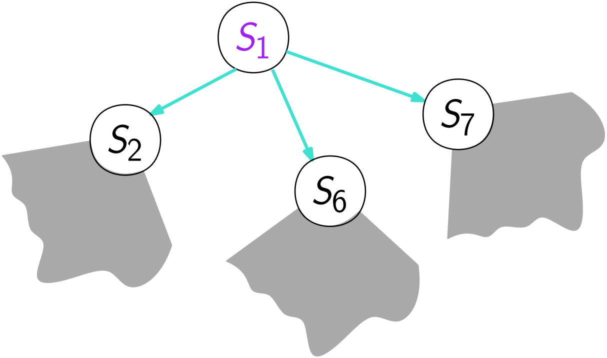


Reverse search

To have *Reverse search* run in poly delay and space :

Generate in poly time and space the *children* of a solution (each solution must have ***a single*** father)



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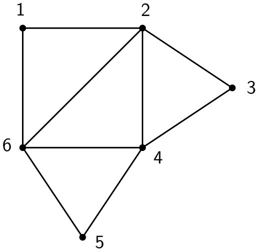
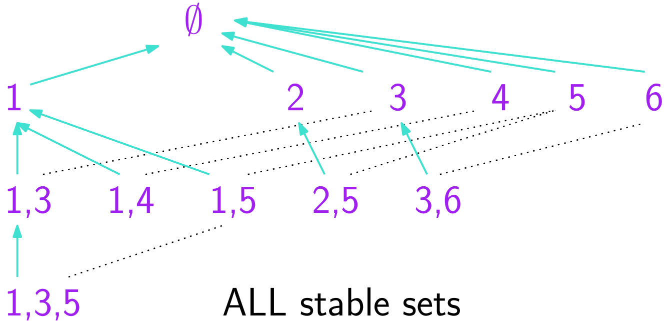
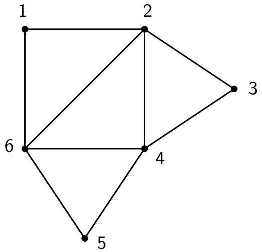
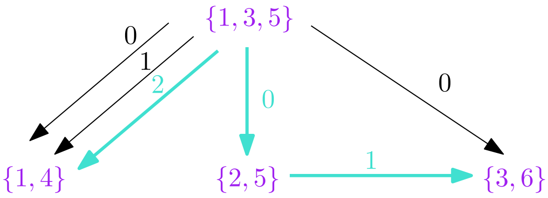




ALL stable sets

Reverse Search for Maximal Stable Sets





$S \xrightarrow{i} S'$ if $S \cap \{v_1, \ldots, v_i\} = S' \cap \{v_1, \ldots, v_i\}$ and S is the lexicographically smallest among all solutions containing $S' \cap \{v_1, \ldots, v_i\}$

P is the father of S if $P \xrightarrow{i} S$ and there is no arc to S indexed $> i$

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- generate **neighbors** of a vertex in poly time
  → *poly degree*

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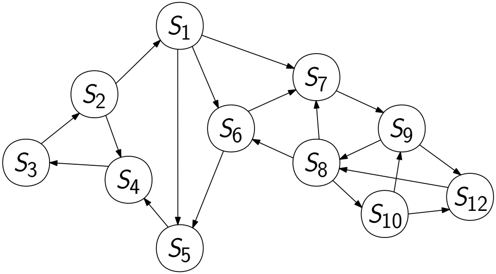

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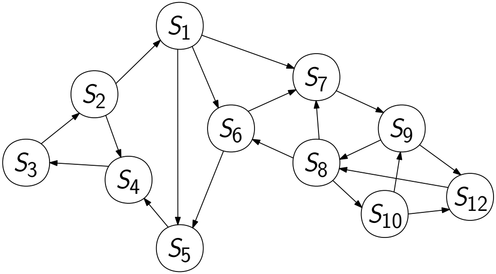

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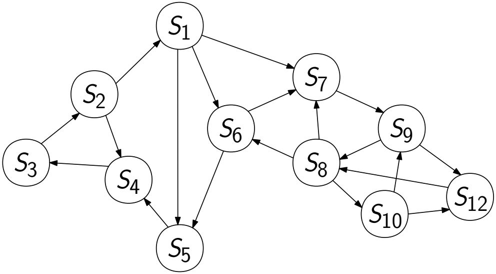

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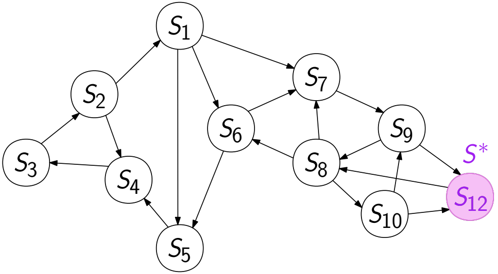

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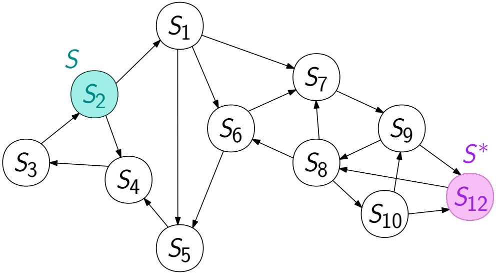

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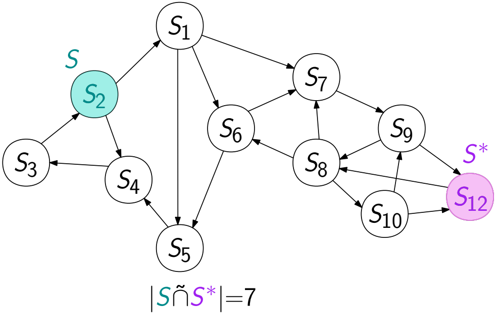

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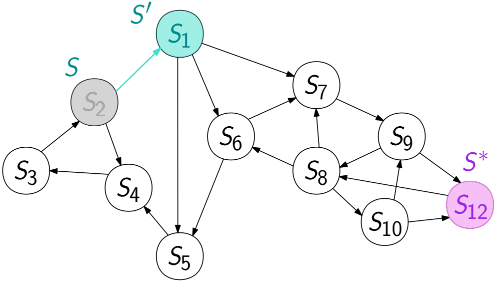


$|S' \tilde{\cap} S^*| = 10$

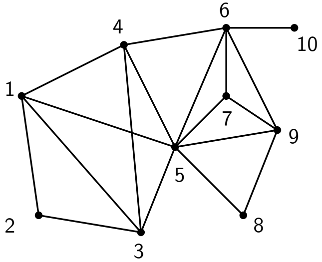
Enumerate chordal fixings

Chordal graph : no chordless cycle of length ≥ 4.

Simplicial vertex

Any chordal graph has a **simplicial vertex** : its neighborhood is a clique.

$\Rightarrow$ **Perfect elimination ordering** : remove simplicial vertices one by one.



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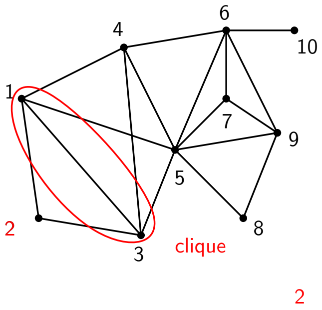

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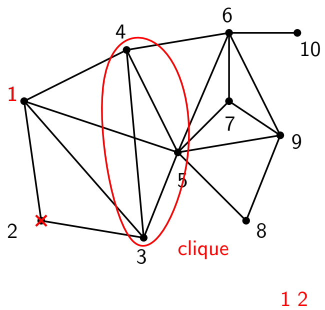

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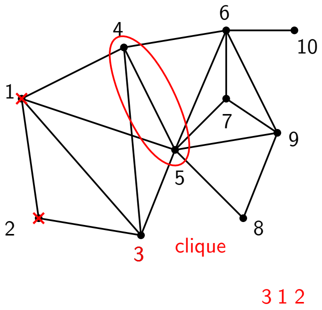

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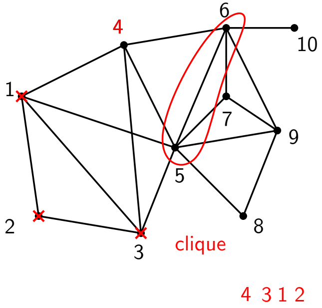

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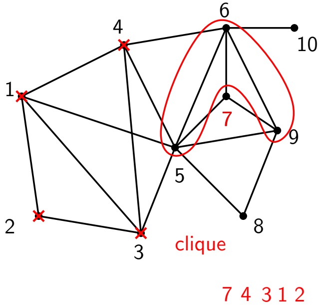

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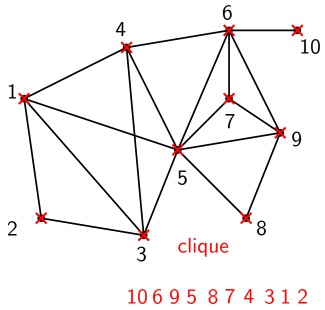
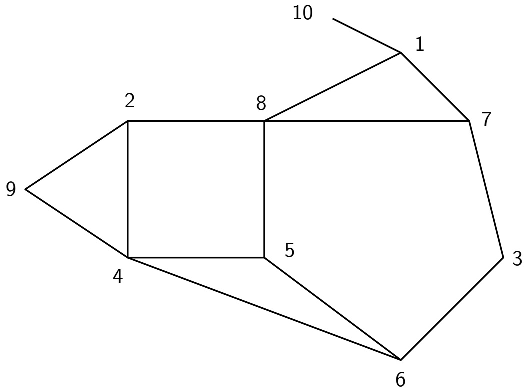


Proximity between solutions

Goal : enumerate all **induced subgraphs** of G that are **chordal** (fixing by deleting vertices)



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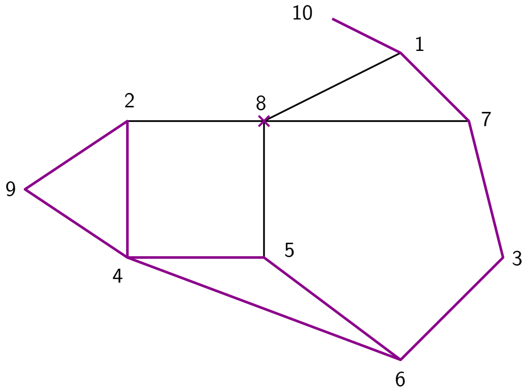

10 1 7 3 6 4 9 5 2

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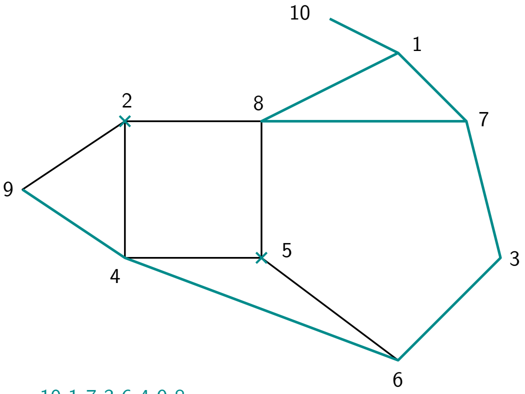

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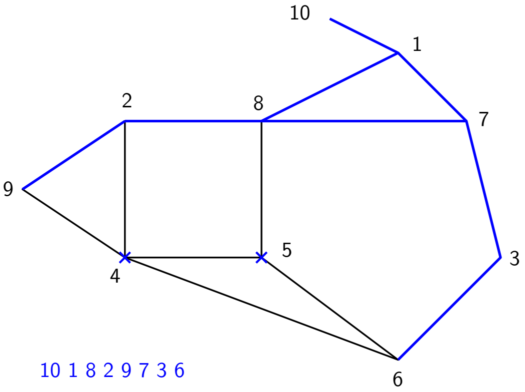

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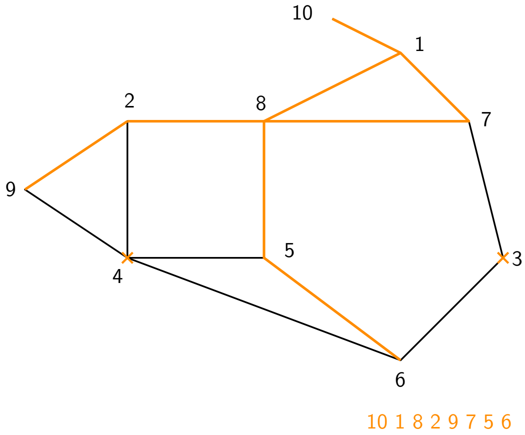

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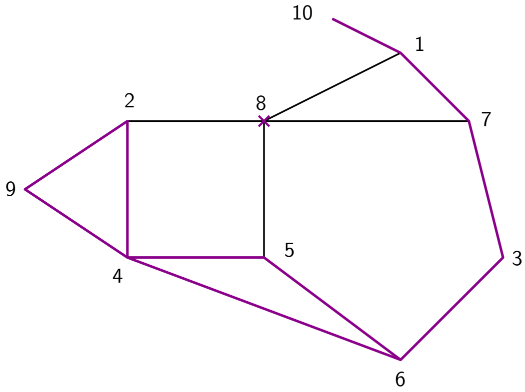

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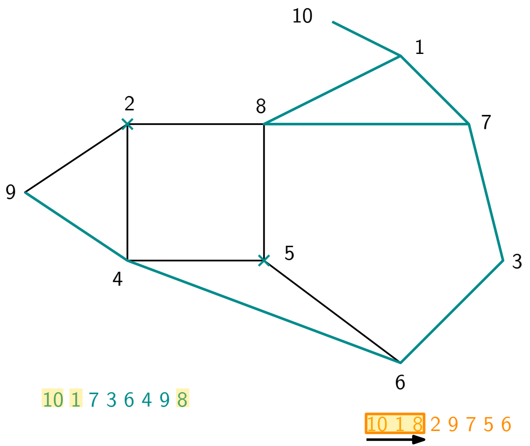

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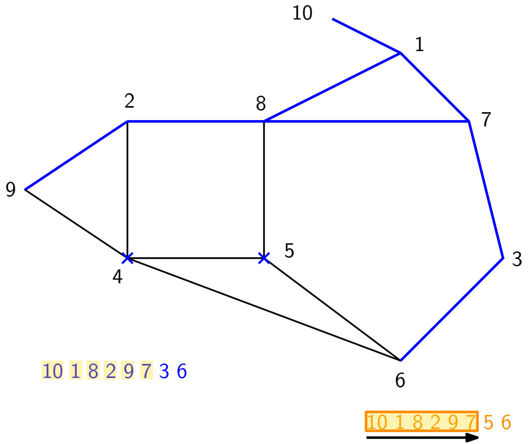
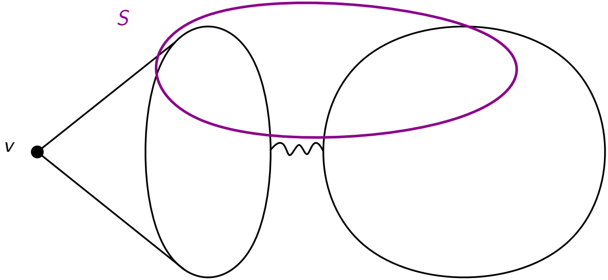


Neighbors in the solution metagraph



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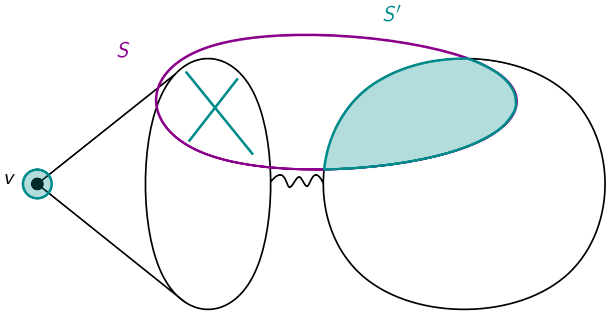

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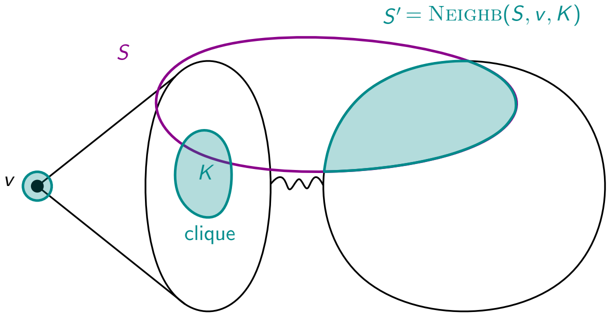

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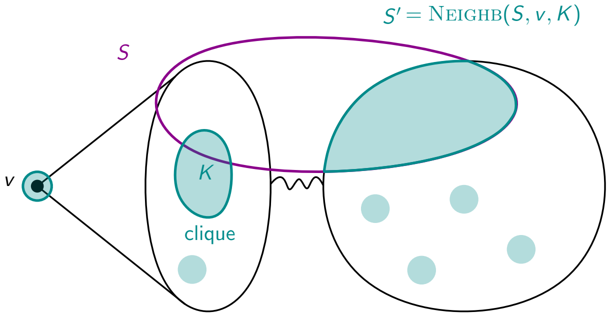
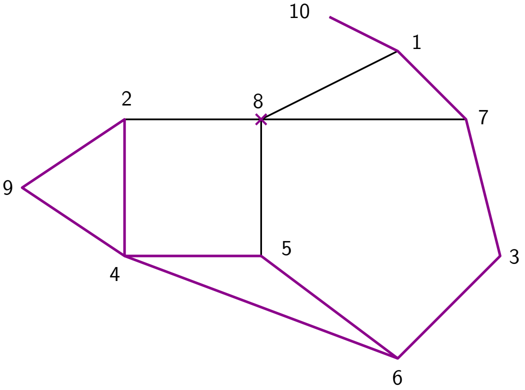


Neighbors in the solution metagraph : example



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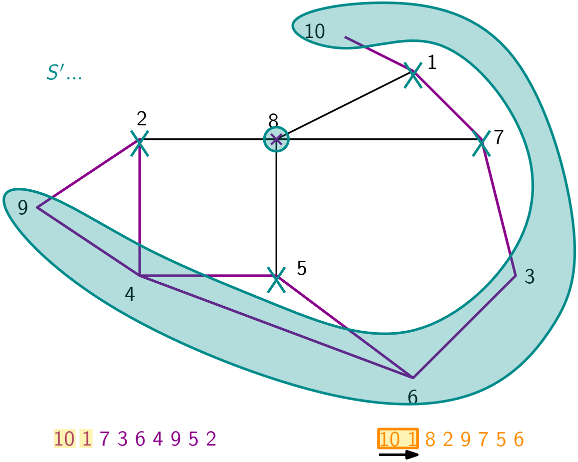

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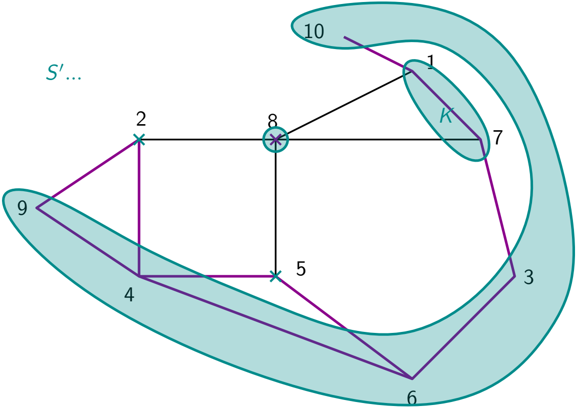

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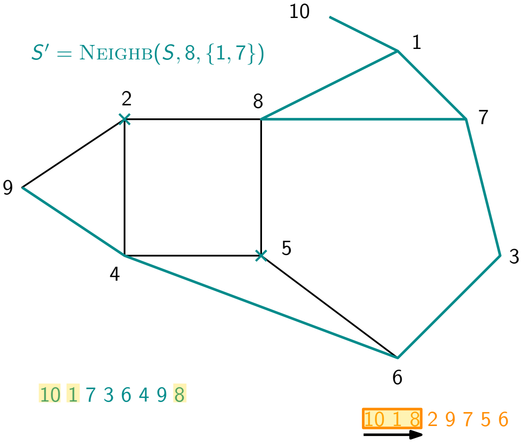
Neighbors in the solution metagraph : example

$S' = \textsc{Neighb}(S, 8, \{1, 7\})$



Chordal fixings : recap

We have all the ingredients for Proximity Search to work :

1. Proximity between two solutions S and S^* is defined
2. $\mathit{Neighbors}(S)$ is computable in polytime
3. Lemma proving that we can always find a neighbor with higher proximity with a target solution, thanks to the perfect elimination ordering (not shown here)

$\rightarrow$ Enumeration of chordal fixings by inclusion-wise min. deletion of vertices in polynomial delay

Other type of fixings : Chordal completions

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There exists a polynomial delay polynomial space algorithm to enumerate all inclusion-wise minimal chordal completion of a graph G given in input.

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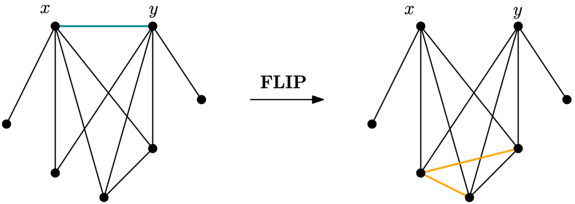

Removing edge xy → Common neighb. turned into clique

Thank you