# Local certification of geometric graph classes

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*Joint work with*
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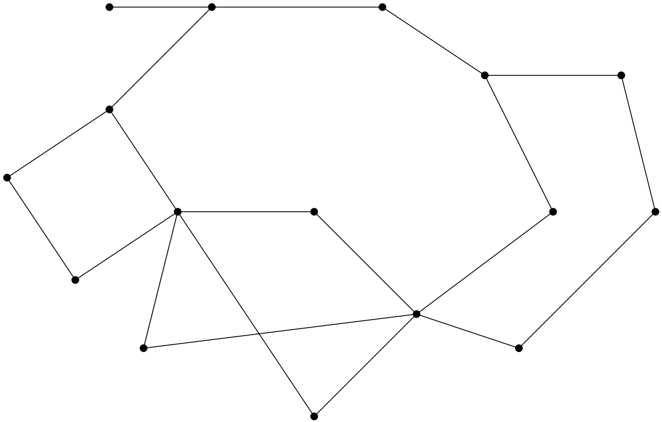
The goal is to provide small local certificates. $\rightarrow$ local complexity

# Examples

$\mathcal{F} = \{$bipartite graphs$\}$



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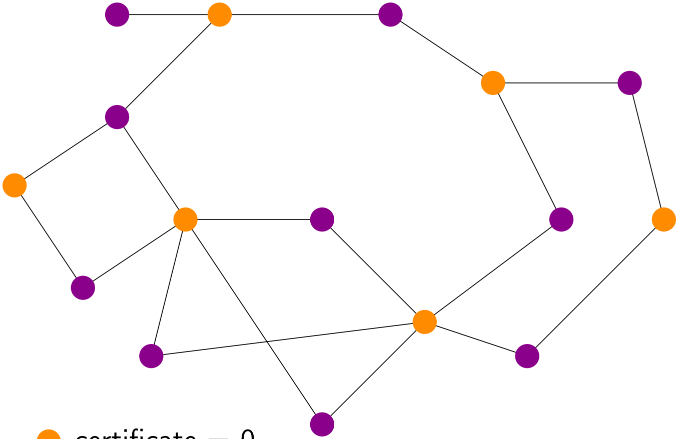

certificate $= 0$

certificate $= 1$

# From constant to quadratic

→ $k$-colorable : certificates of size $\lceil \log k \rceil$

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Any class of $n$-vertex connected graphs can be certified locally with certificates of $O(n^2)$ bits per vertex.

Certificate. Give the adjacency matrix to each vertex.

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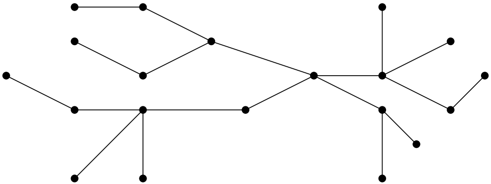
Verification. Each vertex checks that it was given the same matrix as its neighbors, and that the matrix is consistent with its local view.

# Trees

**Observation**

$n$-vertex trees can be certified locally with certificates of $O(\log n)$ bits per vertex.



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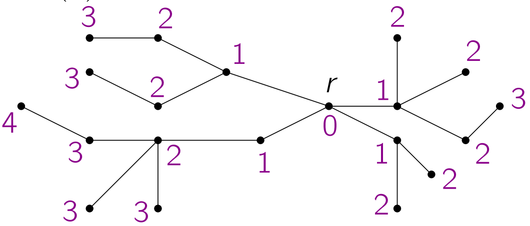

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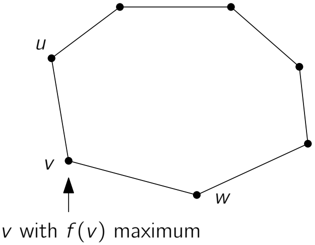

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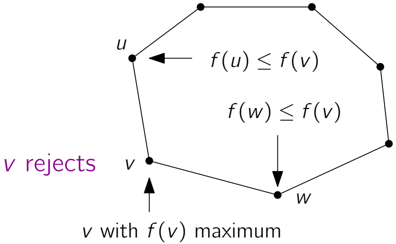

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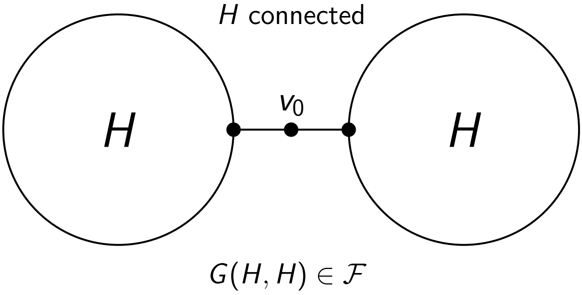
**Theorem [Göös, Suomela 2016]**

Certificates of $\Omega(n^2)$ bits per vertex are needed for some graph class $\mathcal{F}$.



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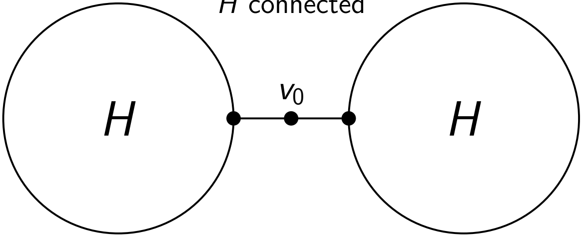


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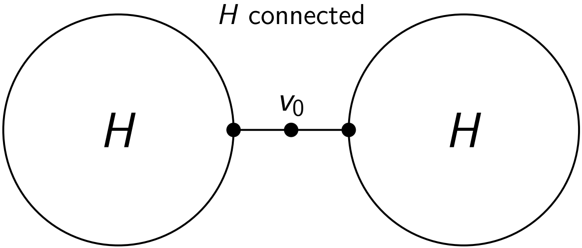


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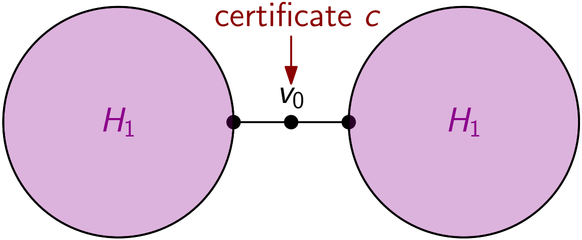


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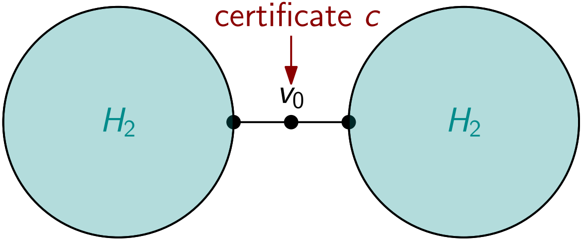


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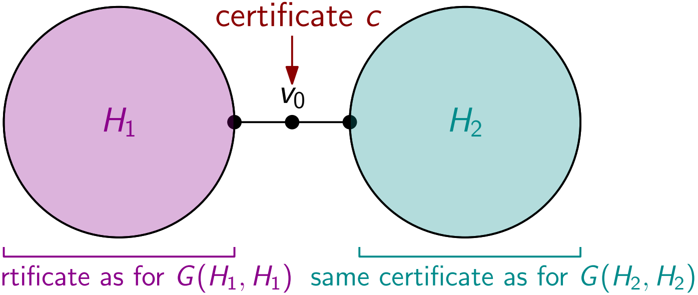


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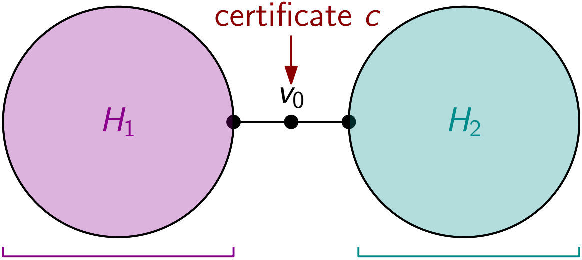


same certificate as for $G(H_1, H_1)$ same certificate as for $G(H_2, H_2)$

all vertices accept ! but $G(H_1, H_2) \notin \mathcal{F}$

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# Geometric graph classes

**Theorems [FFMRRT'21, JMRR'23[+]]**

There are local certifiers on $O(\log n)$ bits for :

- planar graphs,
- interval graphs,
- chordal graphs,
- circular-arc graphs,
- trapezoid graphs,
- permutation graphs.

→ Any class of intersection graphs has small local certifiers ?

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Otherwise, we can fool the vertices with some $A'$ intersecting $B$.

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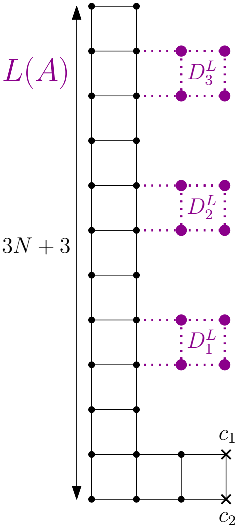
$N = 3$,
$A, B \subseteq \{1, \dots N\}$



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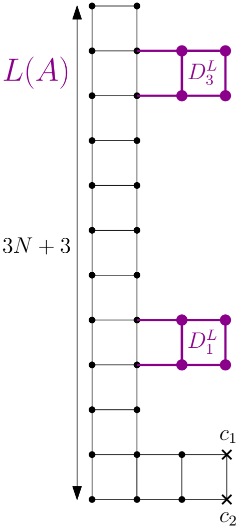


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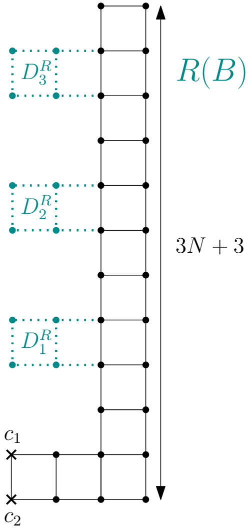

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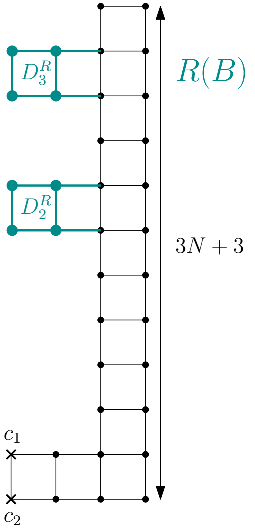


$B = \{1, 2\} \equiv 110$

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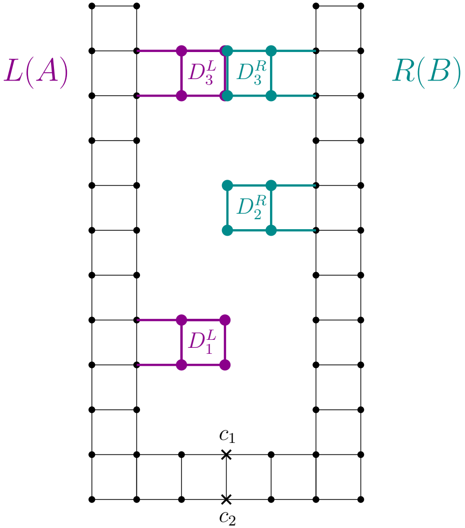


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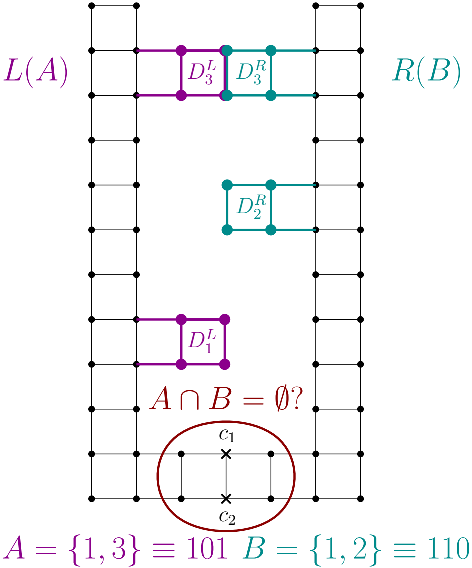


$\rightarrow \frac{n}{6} = \Omega(n)$ bits

# Upper bounds

$\mathcal{C}$ : class of graphs

$\mathcal{C}_n$ : $n$-vertex graphs in $\mathcal{C}$ (up to isomorphism)

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> $|\mathcal{C}_n| \leq 2^n \rightarrow$ certifiable with $O(n)$ bits.
> $|\mathcal{C}_n| \leq 2^{n \log n} \rightarrow$ certifiable with $O(n \log n)$ bits.

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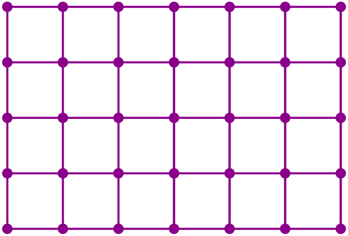
**Defrain, Esperet, L., Morin, Raymond'24**

The following classes have local complexity $\Theta(n)$:

- (induced) subgraphs of the square grid



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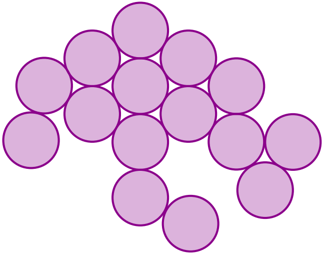

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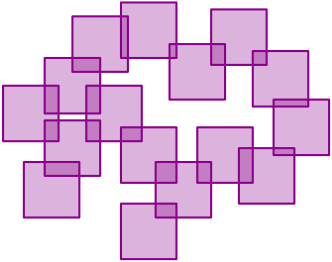

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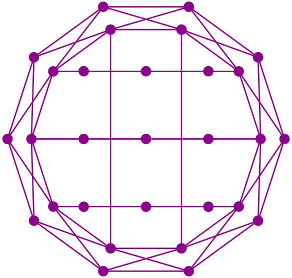

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The local complexity of :

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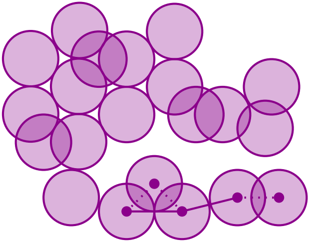

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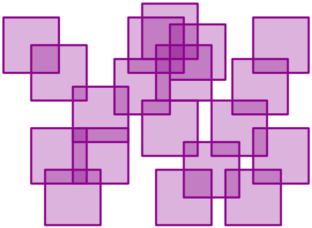

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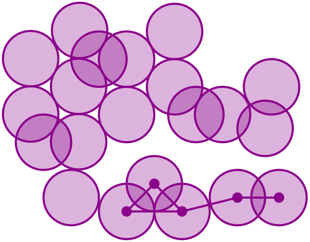
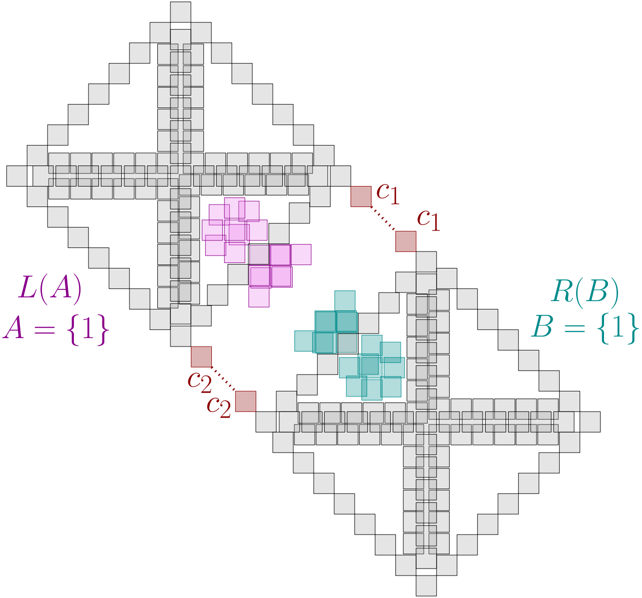
- unit-disk graphs is $\Omega(n^{1-\varepsilon})$ for any $\varepsilon > 0$ and $O(n \log n)$



# (Triangle-free) Unit square construction

$N = 1$



# Unit disk construction

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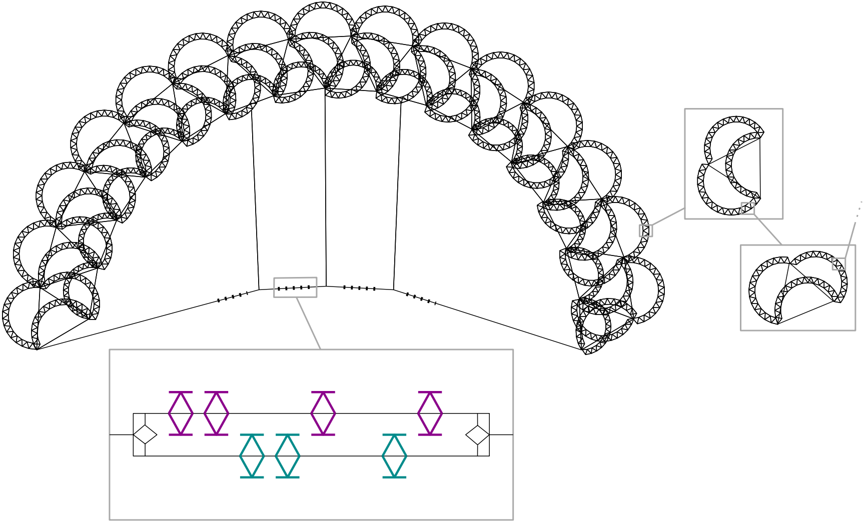

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Thank you for your attention !