Coloring $(2K_2, W_4)$-free graphs

Aurélie Lagoutte

LIMOS, University Clermont Auvergne, France

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Joint work with N. Bousquet

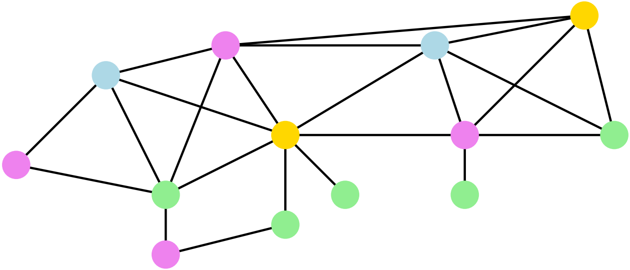
Coloring

Goal

Properly color the vertices of G with the fewest number of colors.

$\omega(G)$: size of the largest clique

$\chi(G)$: smallest number of colors needed to properly color $V(G)$.



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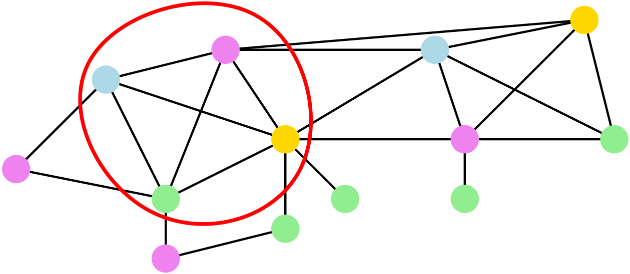


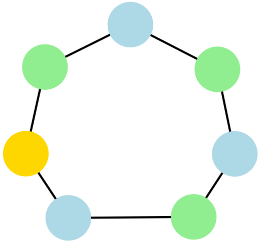
$$\omega(G) \leq \chi(G)$$

Definition: perfect graphs

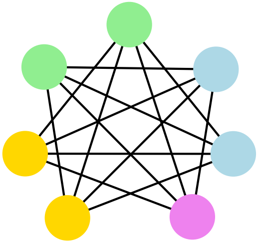
A graph G is *perfect* iff for every induced subgraph H of G, we have $\omega(H) = \chi(H)$.

Strong perfect graph theorem [CRST]

A graph is perfect iff it does not contain any odd hole and any odd antihole.



Odd hole



Odd antihole

$$\chi(G) = \omega(G) + 1$$

χ-boundedness

Gyárfás generalized the notion of perfect graphs to "reasonably colorable":

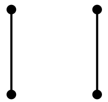
Definition: χ-bounded class

A hereditary class $\mathcal{C}$ of graphs is said to be *χ-bounded by function f* if for every $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$.

Examples:

- Perfect graphs are χ-bounded by the identity function.
- Triangle-free graphs (and even graphs with fixed girth g) are not χ-bounded by any function.
- Even-hole-free graphs are χ-bounded by $f : x \mapsto 2x - 1$.
- Graphs with no odd hole of length $\geq k$ are χ-bounded by an exponential function [Chudnovsky, Scott, Seymour, Spirkl].

Coloring $2K_2$-free graphs



$$2K_2$$

Question [Gyárfás 87]

What is the order of magnitude of the smallest χ-binding function for $2K_2$-free graphs?

Best upper bound: $\mathcal{O}(\omega^2)$. [Wagon 80]
Best lower bound: $\frac{R(C_4, K_{\omega+1})}{3}$ which is $\Omega(\omega^{1+\varepsilon})$ for some $\varepsilon > 0$ [Chung 80]

Question: Close the gap?

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- a linear χ-binding function?
  - No for $(2K_2, 3K_1)$-free graphs [1]
  - $\chi(G) \leq \lfloor \frac{3\omega(G)}{2} \rfloor$ for $(2K_2, \overline{P_5})$-free graphs and this bound is tight [4]

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  - Yes for $(2K_2, K_5 - e)$-free graphs [3]

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- a $\omega + 1$ χ-binding function?
  - Yes for $(2K_2, K_4 - e)$-free graphs [2]
  - Yes for $(2K_2, C_4)$-free graphs

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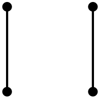
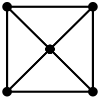


$2K_2$



W_4

Theorem [Bousquet, L., 2018]

$(2K_2, W_4)$-free graphs are $\omega(G) + 1$ colorable, and this bound is tight (on a C_5 for example).

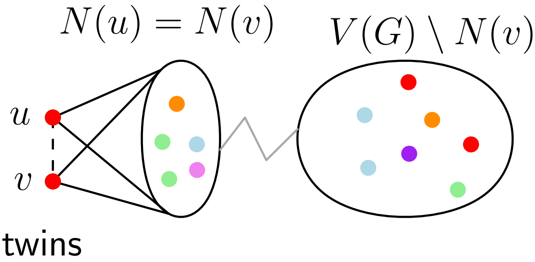
Best previous bound [Brause, Randerath, Shiermeyer, Vumar 2016]: $5\omega + 5$

Main idea of the proof

Study carefully the structure of $(2K_2, W_4)$-free graphs to know how vertices can be linked with one another.
$\Rightarrow$ Once structure is known, it is easy to identify the largest clique and to give a proper coloring.

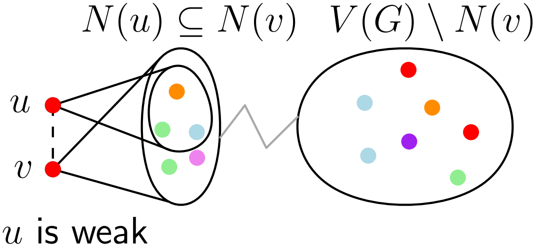
Classical techniques

We may assume that G is twin-free :



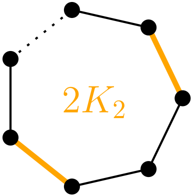
Classical techniques

We may assume that G is twin-free or even weak-free: $\nexists u \neq v$ s.t. $N(u) \subseteq N(v)$.

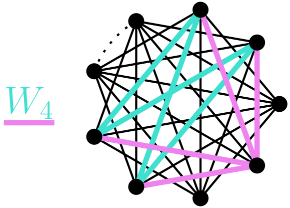


Odd holes and antiholes

- Case A: G is perfect
- Case B: G contains a $\overline{C_7}$
- Case C: G contains a C_5



Hole of length ≥ 7



Antihole of length ≥ 9

Odd holes and antiholes

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- Case B: G contains a $\overline{C_7}$
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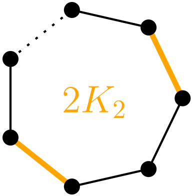

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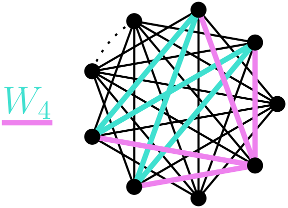

Antihole of length ≥ 9

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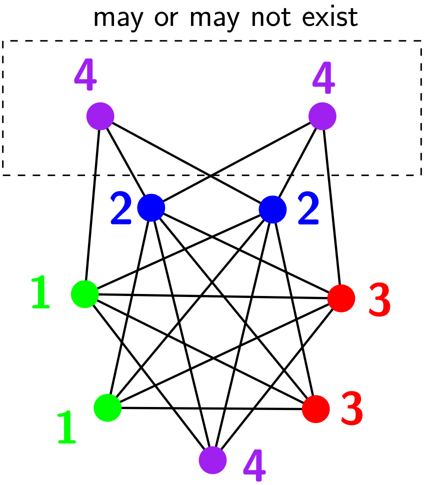
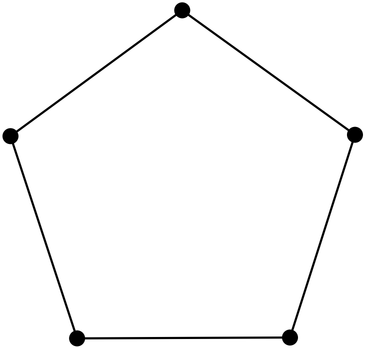


$$\omega = 3, \chi = 4$$

Case C: G contains a C_5



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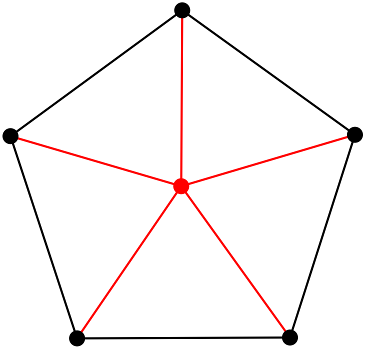
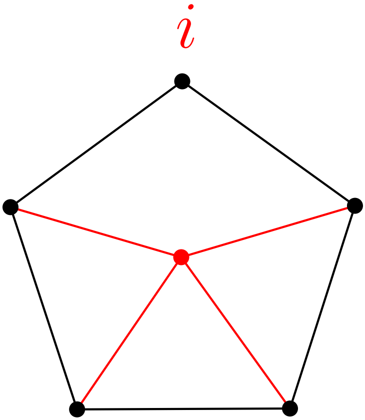


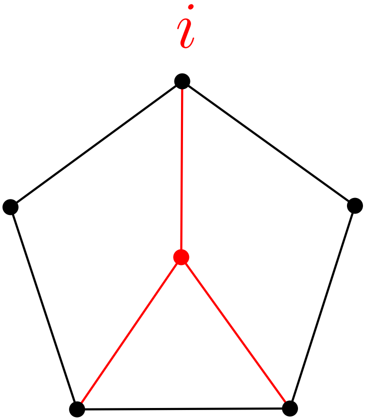
Type 5 : T_5

Case C: G contains a C_5



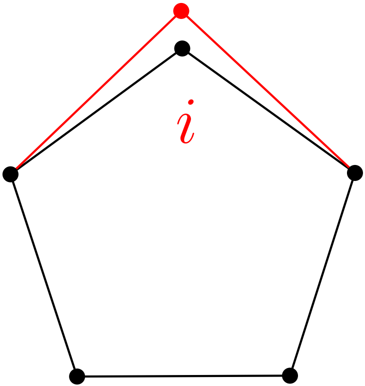
Type 4 : $\cup_i \; T_4^i$

Case C: G contains a C_5



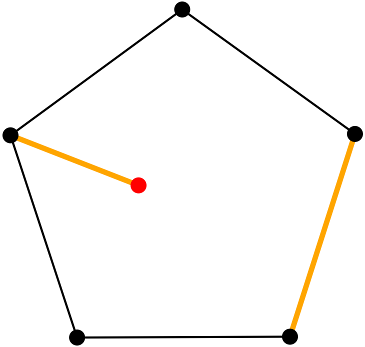
Type 3 : $\cup_i \ T_3^i$

Case C: G contains a C_5



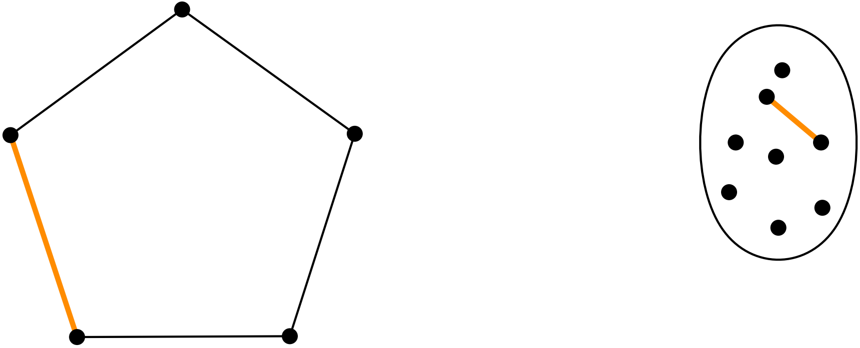
Type 2 : $\cup_i \ T_2^i$

Case C: G contains a C_5



Type 1 : $\emptyset$

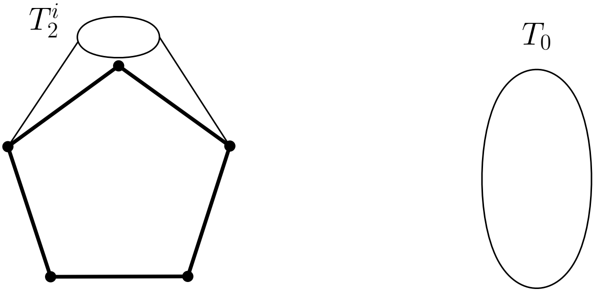
Case C: G contains a C_5



Type 0 : stable set

A few facts on types of vertices

About T_2:



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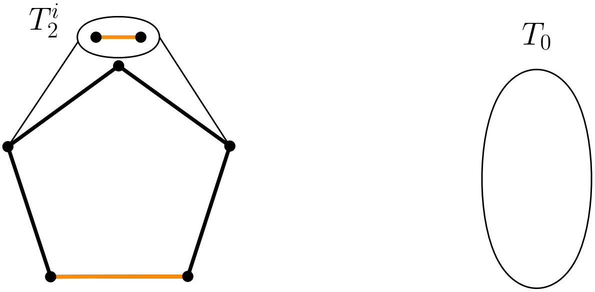

T_2 must be an stable set otherwise $2K_2$

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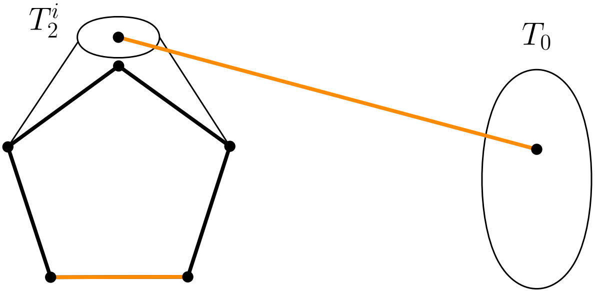

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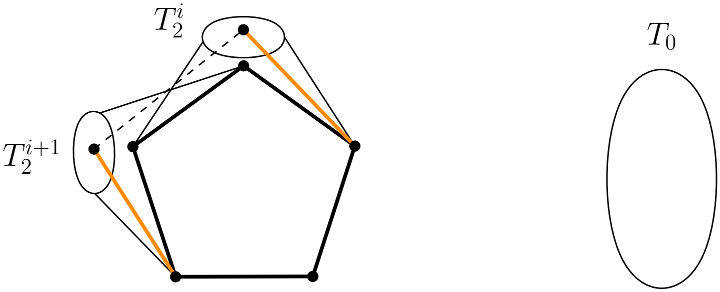

T_2 must be an stable set otherwise $2K_2$

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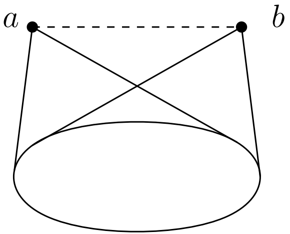
T_2 must be complete to T_2^{i+1} otherwise $2K_2$

A few facts on types of vertices

Observation

If a and b are non-adjacent vertices of G, then $N(a) \cap N(b)$ is the disjoint union of a clique and a stable set (either of which can be empty).



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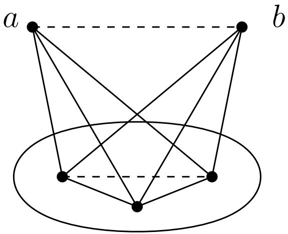

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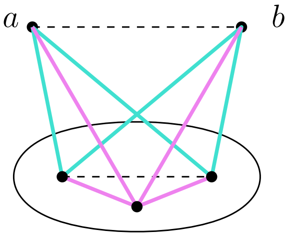

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No P_3 otherwise W_4

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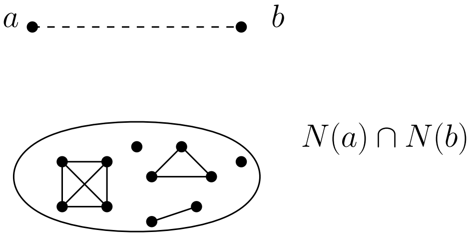

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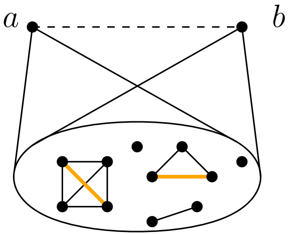

$N(a) \cap N(b)$

Only one clique can be ≥ 2 otherwise $2K_2$

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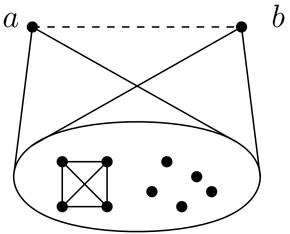

$N(a) \cap N(b)$

Lemma: Contrapositive

If $N(a) \cap N(b)$ contains an induced P_3, then $ab \in E$.

From this we deduce that:

- for each i, $T_5 \cup T_4^i$ is a clique
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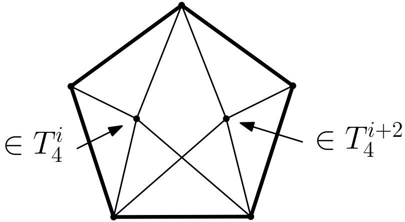
Moreover: T_4^i and T_4^{i+2} cannot be both non empty.



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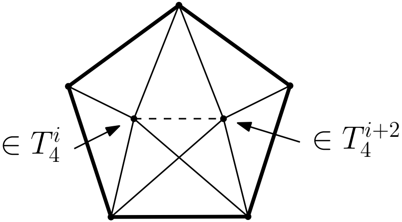

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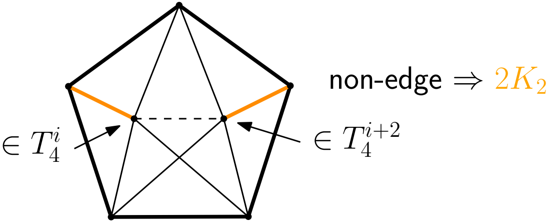

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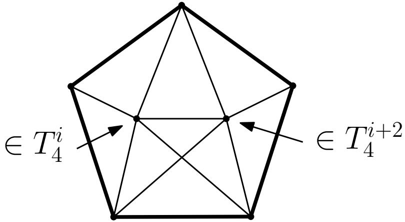

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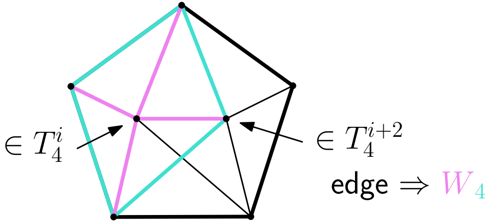

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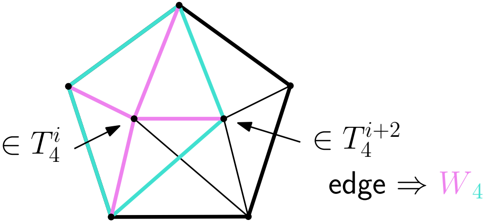


$\Rightarrow$ T_4 is reduced to $T_4^i \cup T_4^{i+1}$ for some i, and must be a clique.

Three cases when G contains a C_5

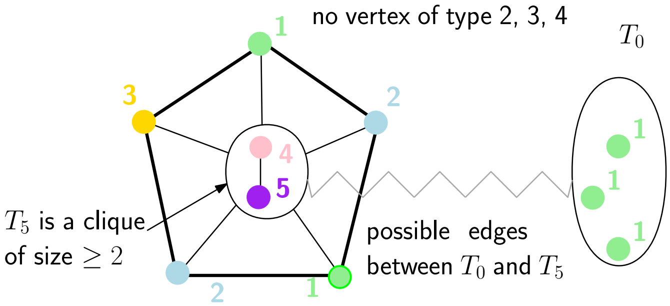
Case C.1: G contains a vertex of type 5

Case C.2: G contains a vertex of type 4 and no vertex of type 5

Case C.3: G contains no vertex of type 4 or 5

Case C.1: G contains a vertex of type 5

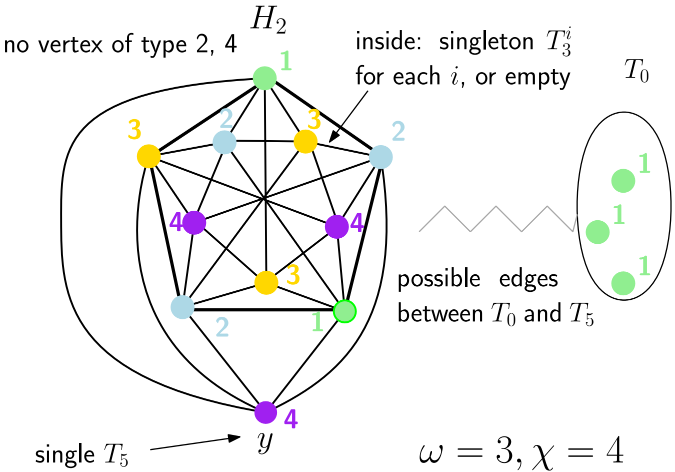
Case C.1.1: If $|T_5| \geq 2$



$$\omega = |T_5| + 2 = \chi - 1$$

Case C.1: G contains a vertex of type 5

Case C.1.2: If $|T_5| = 1$

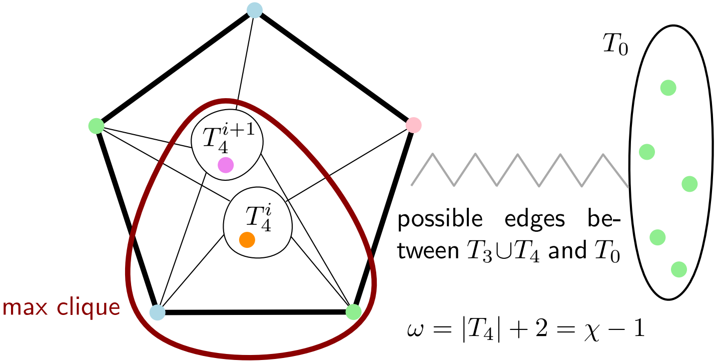


$\omega = 3, \chi = 4$

Case C.2: G contains a vertex of type 4 and no vertex of type 5

We know that $T_4 = T_4^i \cup T_4^{i+1}$ for some i, and it is a clique.
Case C.2.1: Both T_4^i and T_4^{i+1} are non-empty

no vertex of type 2, 3



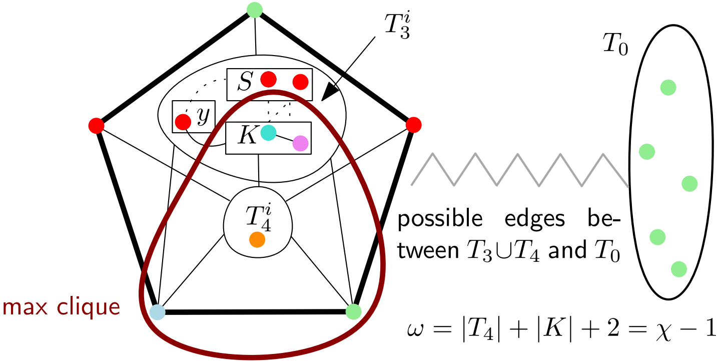
$\omega = |T_4| + 2 = \chi - 1$

Case C.2: G contains a vertex of type 4 and no vertex of type 5

We know that $T_4 = T_4^i \cup T_4^{i+1}$ for some i, and it is a clique.

Case C.2.2: $T_4 = T_4^i$

no vertex of type 2, 3

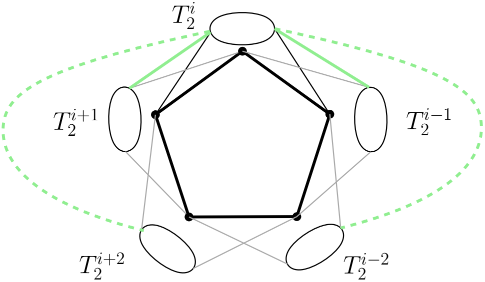


$\omega = |T_4| + |K| + 2 = \chi - 1$

Case C.3: G contains no vertex of type 4 or 5

Trouble is coming!

T_2 might now be non-empty, and might misbehave.



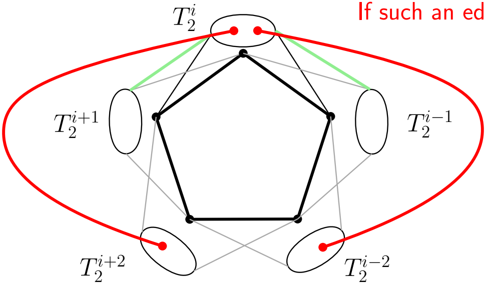
The way we'd like T_2^i to behave: just like vertex i of the cycle

- **TRUE** T_2 must be an stable set otherwise $2K_2$
- **TRUE** T_2 must be anticomplete to T_0 otherwise $2K_2$
- **TRUE** T_2 must be complete to $T_2^{i+1} \cup T_2^{i-1}$ otherwise $2K_2$
- **?** T_2 must be anticomplete to $T_2^{i+2} \cup T_2^{i-2}$

Case C.3: G contains no vertex of type 4 or 5

Trouble is coming!

T_2 might now be non-empty, and might misbehave.



If such an edge exists, we say that T_2^i is bad

The way we'd like T_2^i to behave: just like vertex i of the cycle

- **TRUE** T_2 must be an stable set otherwise $2K_2$
- **TRUE** T_2 must be anticomplete to T_0 otherwise $2K_2$
- **TRUE** T_2 must be complete to $T_2^{i+1} \cup T_2^{i-1}$ otherwise $2K_2$
- **FALSE** T_2 must be anticomplete to $T_2^{i+2} \cup T_2^{i-2}$

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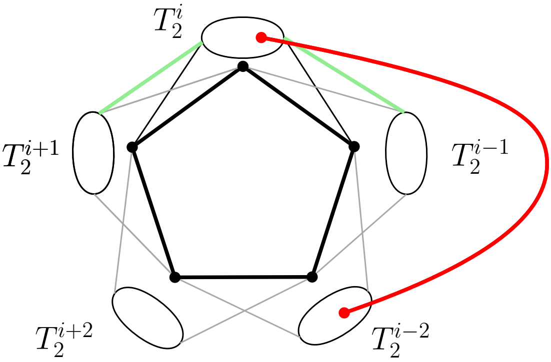

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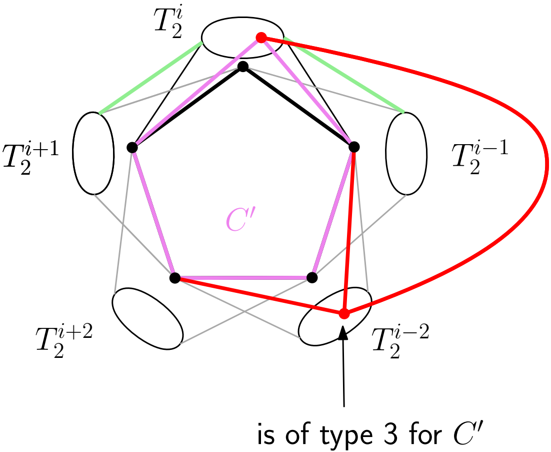

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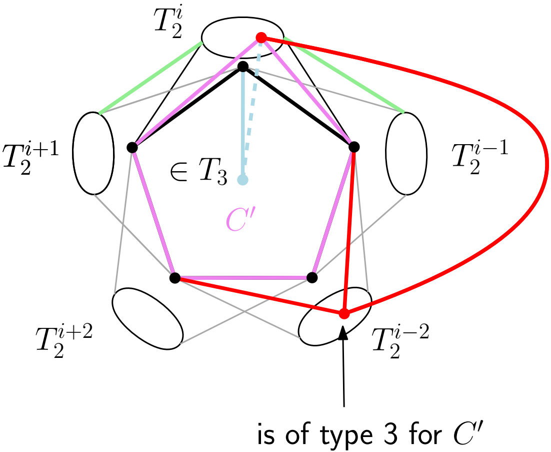

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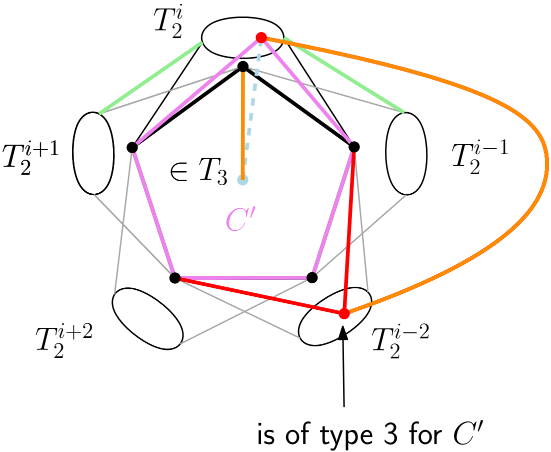

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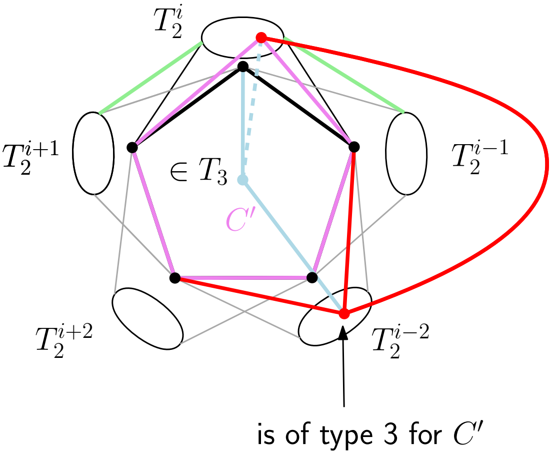

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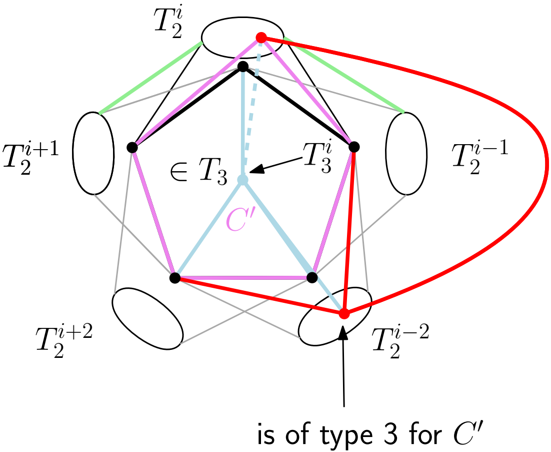
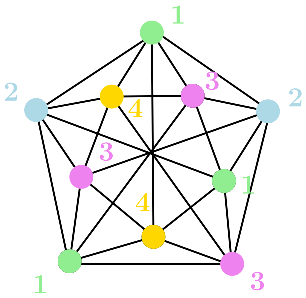
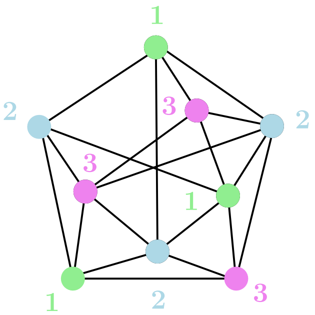


Case C.3.1: T_2 is empty, and non-empty T_3^j are consecutives

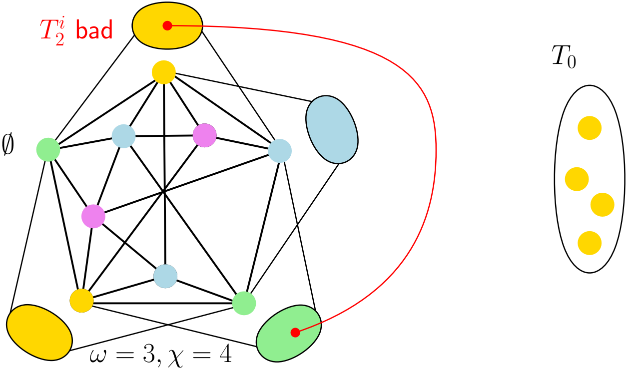


$\omega = 3, \chi = 4$

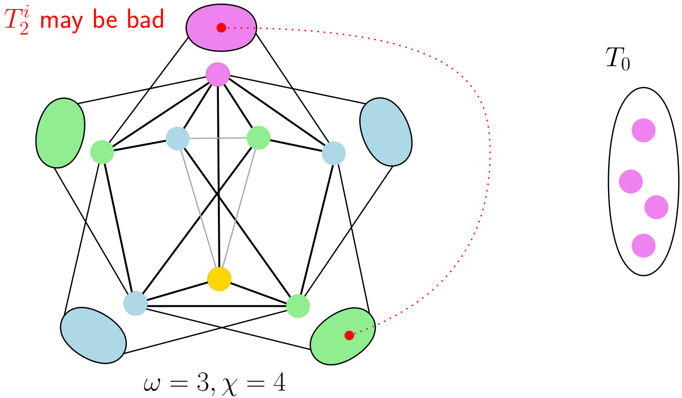


$\omega = 3 = \chi$

Case C.3.2: T_2 is non-empty, and at most one T_3^j is non-empty



Case C.3.3: Two non-consecutive T_3^j are empty



Algorithms

Following the outline of the proof, we also obtain algorithms:

Theorem [Bousquet, L. 2018]

In $(2K_2, W_4)$-free graphs, we can compute in polynomial time:

- a clique of maximum size, and
- an optimal coloring with $\omega(G)$ or $\omega(G) + 1$ colors.

Note: it is known by previous results that

- The coloring problem is NP-complete on $2K_2$-free graphs (even $(2K_2, \text{net})$-free graphs)
- The max clique problem is NP-complete on $2K_2$-free graphs.

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Thank you for your attention!