**Master 2 Internship report**

# Quasi-P versus P


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**Abstract**

This report studies problems coming from three different domains of theoretical computer science: Clique versus Independent set in communication complexity, the Alon-Saks-Seymour conjecture in graph coloring and the stubborn problem in constraint satisfaction problem. Each problem admits a quasi-polynomial size solution, meaning of size $n^{\log n}$ where $n$ is the parameter, and the problem is to find a polynomial size solution. We prove an equivalence theorem between the existence of a polynomial solution for every problem. We study further the Clique versus Independent set problem on two particular classes of graphs: random graphs and split-free graphs, for which we find a polynomial size solution.

# 1 Introduction

This report links problems coming from three different domains of theoretical computer science. Let us make a brief overview of each domain in the light of the problems under study. The goal is to give some context and intuition, while formal definitions will be given later.

**Communication complexity and the Clique-Stable Set separation.** Yannakakis introduced in [23] the following communication complexity problem, called $CL-IS$: given a publicly known graph $\Gamma$ on $n$ vertices, Alice and Bob agree on a protocol, then Alice is given a clique and Bob is given a stable set. They do not know which clique or which stable set was given to the other one, and their goal is to decide whether the clique and the stable set intersect or not, by minimizing the worst-case number of exchanged bits. Note that the intersection of a clique and a stable set is at most one vertex. In the deterministic version, Alice and Bob send alternatively messages one to each other, and the minimization is on the number of bits exchanged between them. It is a long standing open problem to prove a $\mathcal{O}(\log^2 n)$ lower bound for the deterministic communication complexity. In the non-deterministic version, for $b \in \{0, 1\}$, an all powerful prover sends a certificate in order to convince both Alice and Bob that the result is $b$. Then, Alice and Bob exchange one final bit, saying whether they agree or disagree with the certificate. The aim is to minimize the size of the certificate.

In this particular setting, a certificate proving that the clique and the stable set intersect is just the name of a vertex in the intersection. Such a certificate has clearly a logarithmic size. Convincing Alice and Bob that the clique and the stable set do not intersect is much more complicated. A certificate can be a partition of the vertices into two parts such that the whole clique is included in the first part, and the whole stable set is included in the second part. Such a partition is called a cut that separates the clique and the stable set. A family of $m$ cuts such that for every clique and for every stable set, there is a cut in the family that separates the clique and the stable set is called a CS-separator. Observe that Alice and Bob can agree on a CS-separator at the beginning, and then the prover just sends the name of a cut that separates the clique and the stable set: the certificate has size $\log_2 m$. Hence if there is a CS-separator of size polynomial in $n$, one can ensure a non-deterministic certificate of size $\mathcal{O}(\log_2 n)$.

Yannakakis proved that there is a $\mathcal{O}(\log_2 n)$ certificate for the $CL-IS$ problem if and only if there is a CS-separator of polynomial size. The existence of such a CS-separator is called in the following the clique-stable set separation problem. It appears from a geometric problem which was studied both by Yannakakis [23] and by Lovász [16]. The question is to determine if the stable set polytope of a graph is the projection of a polytope in higher dimension, with a polynomial number or facets (called extended formulation). The existence of such a polytope in higher dimension implies the existence of a polynomial CS-separator for the graph. Moreover, they proved that the answer is positive for several subclasses of perfect graphs, such as comparability graphs and their complements, chordal graphs and their complements, and $t$-perfect graphs which are a generalization of series-parallel graphs. The existence of an extended formulation for general graphs has recently been answered negatively by Fiorini et al. [8].

**Constraint satisfaction problem and the stubborn problem.** The complexity of the so-called *list-$M$ partition problems* has been widely studied in the last decades (see [20] for an overview). $M$ stands for a fixed $k \times k$ symmetric matrix filled with $0, 1$ and $*$ as illustrated on Fig. 1. The input is a graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, \dots, A_k\})$ and the question is to determine whether the vertices of $G$ can be partitioned into $k$ sets $A_1, \dots, A_k$ respecting two types of requirements. The first one is given by the list assignments, that is to say $v$ can be put in $A_i$ only if $A_i \in \mathcal{L}(v)$. The second one is described in $M$, namely: if $M_{i,i} = 0$

$$\begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 1 \end{pmatrix}$$

Figure 1: Matrix $M$ for the stubborn problem.

(resp. $M_{i,i} = 1$), then $A_i$ is a stable set (resp. a clique), and if $M_{i,j} = 0$ (resp. $M_{i,j} = 1$), then $A_i$ and $A_j$ are completely non-adjacent (resp. completely adjacent). If $M_{i,i} = *$ (resp. $M_{i,j} = *$), then $A_i$ can be any set (resp. $A_i$ and $A_j$ can have any kind of adjacency).

Feder et al. ([7], [6]) proved a *quasi-dichotomy theorem*. The list-$M$ partition problems are classified between NP-complete and quasi-polynomial time solvable (i.e. time $\mathcal{O}(n^{c \log n})$ where $c$ is a constant). Moreover, many investigations have been made about small matrices $M$ ($k \leq 4$) to get a *dichotomy theorem*, meaning a classification of the list-$M$ partition problems between polynomial time solvable and NP-complete. Cameron et al. [3] reached such a dichotomy for $k \leq 4$, except for one special case (and its complement) then called the *stubborn problem* (see Fig. 1 for the corresponding matrix), which remained only quasi-polynomial time solvable. Cygan et al. [4] closed the question by finding a polynomial time algorithm solving the stubborn problem. More precisely, they found a polynomial time algorithm for 3-Compatible Coloring, which was introduced in [5] and said to be no easier than the stubborn problem. 3-Compatible Coloring has also been introduced and studied in [14] under the name Adapted List Coloring, and was proved to be a model for some strong scheduling problems.

These two problems under study are defined in the following way:

3-Compatible Coloring Problem (3-CCP)
**Input:** An edge coloring $f_E$ of the complete graph on $n$ vertices with 3 colors $\{A, B, C\}$.
**Question:** Is there a coloring of the vertices with $\{A, B, C\}$, such that no edge has the same color as both its endpoints?

Stubborn Problem
**Input:** A graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$.
**Question:** Can $V$ be partitioned into four sets $A_1, \ldots, A_4$ such that $A_4$ is a clique, both $A_1$ and $A_2$ are stable sets, there is no edge between $A_1$ and $A_3$, and each vertex $v$ belongs to $A_i$ only if $A_i \in \mathcal{L}(v)$?

**Graph coloring and the Alon-Saks-Seymour conjecture.** Given a graph $G$, the bipartite packing, denoted by **bp**, is the minimum number of edge-disjoint complete bipartite graphs needed to partition the edges of $G$. The Alon-Saks-Seymour conjecture [13] states that if a graph has bipartite packing $k$, then its chromatic number $\chi$ is at most $k+1$. It is inspired from the Graham Pollak theorem [10] which states that $\mathbf{bp}(K_n) = n - 1$, and the conjecture has interested several authors ([19],[9]). Huang and Sudakov found in [12] a counterexample to the Alon-Saks-Seyour conjecture, twenty-five years after its statement. Actually they proved that there is an infinite family of graphs for which $\chi \geq \mathbf{bp}^{6/5}$. The Alon-Saks-Seymour conjecture can now be restated as the *polynomial* Alon-Saks-Seymour conjecture: is the chromatic number polynomially upper bounded in terms of **bp**? Moreover, Alon and Haviv [1] observed that a gap between $\chi$ and **bp** such as Huang and Sudakov proved, implies for the Clique-Stable Set separation problem a lower bound of $n^{6/5}$. This in turns implies a $6/5 \log_2(n) - \mathcal{O}(1)$ lower bound on the non-deterministic communication complexity of $CL - IS$ when the clique and the stable set do not intersect.

A generalization of the bipartite packing of a graph is the $t$-biclique number, denoted by $\mathbf{bp}_t$. It is the minimum number of complete bipartite graphs needed to cover the edges of the graph such that each edge is covered at most $t$ times. It was introduced by Alon [17] to model neighborly families of boxes, and the most studied question so far is finding tight bounds for $\mathbf{bp}_t(K_n)$.

**Contribution** The Clique-Stable set separation problem will be considered as our reference problem, since it seems the easiest to handle and to work with. Our main result states equivalence between the previously mentioned problems.

More precisely, we start in section 3 by proving that there is a polynomial CS-separator for two classes of graphs: random graphs and split-free graphs. The proof for random graphs is based on random cuts. For split-free graphs, it is based on Vapnik-Chervonenkis-dimension. The interest is that random graphs seem totally unstructured, while on the contrary split-free graph have strong structure properties.

In section 4, we highlight links between the clique-stable set separation problem and both the stubborn problem and 3-CCP. The quasi-dichotomy theorem for list-$M$ partitions proceeds by covering all the solutions by $\mathcal{O}(n^{\log n})$ particular instances of 2-SAT, called 2-list assignments. A natural extension would be a covering of all the solutions with a polynomial number of 2-list assignments. We prove that the existence of a polynomial covering of all the maximal solutions (to be defined later) for the stubborn problem is equivalent to the existence of such a covering for all the solutions of 3-CCP, which in turn is equivalent to the clique-stable set separation problem.

In section 5, we extend Alon and Haviv's observation and prove the equivalence between then polynomial Alon-Saks-Seymour conjecture and the Clique-Stable separation. It follows from an intermediate result which is also interesting in itself: for every integer $t$, the chromatic number $\chi$ can be bounded polynomially in terms of $\mathbf{bp}$ if and only if it can be bounded polynomially on $\mathbf{bp}_t$.

## 2 Definitions

Let $G = (V, E)$ be a graph and $k$ be an integer. An edge $uv \in E$ links its two *endpoints* $u$ and $v$. The *neighborhood* $N(x)$ *of* $x$ is the set of vertices $y$ such that $xy \in E$. The *non-neighborhood* $N^C(x)$ of $x$ is $V \setminus (N(x) \cup \{x\})$. For oriented graphs, $N^+(x)$ (resp. $N^-(x)$) denote the outcoming (resp. incoming) neighborhood of $x$, i.e. the set of vertices $y$ such that $xy \in E$ (resp. $yx \in E$). The subgraph *induced by* $X \subseteq V$ denoted by $G[X]$ is the graph with vertex set $X$ and edge set $E \cap (X \times X)$. A *clique of size* $n$, denoted by $K_n$, is a complete induced subgraph. A *stable set* is an induced subgraph with no edge. Note that a clique and a stable set intersect on at most one vertex. Two subsets of vertices $X, Y \subseteq V$ are *completely adjacent* if for all $x \in X$, $y \in Y$, $xy \in E$. They are *completely non-adjacent* if there are no edge between them. A graph $G = (V, E)$ is *split* if $V = V_1 \cup V_2$ and the subgraph induced by $V_1$ is a clique and the subgraph induced by $V_2$ is a stable set. A *vertex-coloring* (resp. *edge-coloring*) of $G$ with a set $\textsc{Col}$ of $k$ colors is a function $f_V$ (resp. $f_E$) from $V$ (resp. $E$) to $\textsc{Col}$.

A graph $G$ is *bipartite* if $V$ can be partitioned into $(U, W)$ such that both $U$ and $W$ are stable sets. Moreover, $G$ is *complete* if $U$ and $W$ are completely adjacent. An *oriented bipartite graph* is a bipartite graph together with an edge orientation such that all the edges go from $U$ to $W$. A *hypergraph* $H = (V, E)$ is a generalization of a graph and is composed of a set of vertices $V$ and a set of *hyperedges* $E \subseteq \mathcal{P}(V)$.

## 3 Clique-Stable Set separation conjecture

The communication complexity problem $CL - IS$ can be formalized by a function $f : X \times Y \to \{0, 1\}$, where $X$ is the set of cliques and $Y$ the set of stable sets. Alice is given a clique $x$, Bob is given a stable set $y$, and then $f(x, y) = 1$ if and only if $x$ and $y$ intersect. It can also be represented by a $|X| \times |Y|$ matrix $M$ with $M_{x,y} = f(x, y)$. As previously mentioned, we study the non-deterministic version, where for $b \in \{0, 1\}$, an all powerful prover sends a certificate of size $N^b(f)$ in order to convince both Alice and Bob that the value of $f$ is $b$. Then, Alice and Bob exchange one final bit, saying whether they agree or disagree with the certificate. The aim is to minimize $N^b(f)$ in the worst case.

The best upper bound so far on $N^0(f)$ is $\mathcal{O}(\log^2(n))$ [23], which is actually a bound on the deterministic communication complexity.

It is known in theory of communication complexity [15] that, for general $f$, $N^b(f) = \lceil \log_2 C^b(f) \rceil$ where $C^b(f)$ is the minimum number of monochromatic combinatorial rectangles needed to cover the $b$-inputs of the communication matrix $M$ (a monochromatic combinatorial rectangle is a set of rows and columns such that the matrix restricted to this set of rows and this set of columns is entirely filled by $b$). One can observe that a monochromatic combinatorial rectangle that cover some 0-inputs of $M$ corresponds to a cut that separates the cliques and the stable sets involved in the rectangle. Thus finding the minimum $N^0(f)$ is equivalent to finding the minimum number of cuts needed to separate all the cliques and the stable sets. In particular, there is a $\mathcal{O}(\log n)$ certificate for the $CL - IS$ problem if and only if there is a CS-separator of polynomial size.

A *cut* is a pair $(A, B)$ such that $A \cup B = V$ and $A \cap B = \varnothing$. It *separates* a clique $C$ and a stable set $S$ if $C \subseteq A$ and $S \subseteq B$. Note that a clique and a stable set can be separated if and only if they do not intersect. Let $\mathcal{C}_G$ be the set of cliques of $G$ and $\mathcal{S}_G$ be the set of stable sets of $G$. We say that a family $\mathcal{F}$ of cuts *is a CS-separator* if for all $(C, S) \in \mathcal{C}_G \times \mathcal{S}_G$ which do not intersect, there exists a cut in $\mathcal{F}$ that separates $C$ and $S$.

**Conjecture 1.** *(Clique-Stable Set separation Conjecture) There is a polynomial $Q$, such that for every graph $G$ on $n$ vertices, there is a CS-separator of size at most $Q(n)$.*

**Proposition 2.** *Conjecture 1 holds if and only if a polynomial family $\mathcal{F}$ of cuts separates all the maximal (in sense of inclusion) cliques from the maximal stable sets that do not intersect.*

*Proof.* First note that one direction is direct. Let us prove the other one. Assume $\mathcal{F}$ is a polynomial family that separates all the maximal cliques from the maximal stable sets that do not intersect. Let $Cut_{1,x}$ be the cut $(N(x) \cup x, V \setminus \{N(x), x\})$ and $Cut_{2,x}$ be the cut $(N(x), V \setminus N(x))$. Let us prove that $\mathcal{F}'$ which is the family $\mathcal{F}$ together with the families $Cut_{1,x}$ and $Cut_{2,x}$ for all $x$ is a CS-separator.

Let $(C, S)$ be a pair of clique and stable set. Extend $C$ and $S$ by adding vertices to get a maximal clique $C'$ and a maximal stable set $S'$. Either $C'$ and $S'$ do not intersect, and there is a cut in $\mathcal{F}$ that separate $C'$ from $S'$ (thus $C$ from $S$). Or $C'$ and $S'$ intersect in $x$ (recall that a clique and a stable set intersect on at most one vertex): if $x \in C$, then $Cut_{1,x}$ separates $C$ from $S$, otherwise $Cut_{2,x}$ does. □

Consequently, in the following, we will only focus on separating the maximal cliques from the maximal stable sets. In this section, we study the Clique-Stable set separation conjecture on random graphs and split-free graphs.

### 3.1 Random graphs

Random graphs are a typical example of unstructured graphs, hence they appear as a natural candidate for a counterexample to the Clique-Stable set conjecture. However, the size of their cliques and stable sets will enable some random cuts to separate them. Let $n$ be a positive integer and $p \in [0,1]$. Several models of random graphs have been studied [2], and we will work on the Erdős-Rényi model. The random graph $G(n,p)$ is a probability space over the set of graphs on the vertex set $\{1,\dots,n\}$ determined by $\Pr[ij \in G] = p$, with these events mutually independent. We say that $G(n,p)$ has clique number $\omega$ if $\omega$ satisfies $\mathbb{E}(\text{number of cliques of size } \omega) = 1$. We define similarly the independent number of $G(n,p)$. An event $\mathcal{E}$ occurs *with high probability* if the probability of this event tends to 1 when $n$ tends to infinity.

A family $\mathcal{F}$ of cuts on a graph $G$ with $n$ vertices is a *complete* $(a,b,n)$*-separator* if for every pair $(A,B)$ of subsets of vertices with $|A| \leq a$, $|B| \leq b$, there exists a cut $(U, V \setminus U) \in \mathcal{F}$ separating $A$ and $B$, namely $A \subseteq U$ and $B \subseteq V \setminus U$. We say that $G(n,p)$ *has a polynomial complete* $(a,b,n)$*-separator* if there exists a polynomial $P$ such that for all $p \in [0,1]$, there exists a complete $(a,b,n)$-separator of size $P(n)$ in $G(n,p)$ with high probability.

**Theorem 3.** *$G(n,p)$ has a polynomial complete $(\omega,\alpha,n)$-separator where $\omega$ and $\alpha$ are respectively the clique number and the independent number of $G(n,p)$.*

*Proof.* In the following, $\log_b$ denotes the logarithm to base $b$, and log denotes the logarithm to base 2. Without loss of generality, we assume $p = 1 - 2^{-2\log n/a(n)}$, where $a(n)$ is a function of $n$. Let $p' = 1-p$, $b = 1/p$ and $b' = 1/p'$. The independence number and clique number of $G(n,p)$ are given by the following formulas, depending on $p$ (see [2]):

$$\omega = 2\log_b(n) - 2\log_b(\log_b n) + 2\log_b(e/2) + 1 + o(1)$$
$$\alpha = 2\log_{b'}(n) - 2\log_{b'}(\log_{b'} n) + 2\log_{b'}(e/2) + 1 + o(1)$$

Draw a random partition $(V_1, V_2)$ where each vertex is put in $V_1$ independently from the others with probability $p$. Let $(C,S)$ be a pair of clique and stable set of the graph. There are at most $4^n$ such pairs. The probability that $C \subseteq V_1$ and $S \subseteq V_2$ is at least $p^\omega(1-p)^\alpha$. Assume for a moment that $p^\omega(1-p)^\alpha \geq 1/n^6$. Then $(C,S)$ is separated by at least $1/n^6$ of all the partitions. By double counting, there exists a partition that separates at least $1/n^6$ of all the pairs. We delete these separated pairs, and there remains at most $(1-1/n^6)\cdot 4^n$ pairs. The same probability for a pair $(C,S)$ to be cut by a random partition still holds, hence we can iterate the process $k$ times until $(1-1/n^6)^k \cdot 4^n \leq 1$. This is satisfied for $k = 2n^7$ which is a polynomial in $n$. Thus there is a complete $(\alpha,\omega,n)$-separator of size polynomial in $n$.

The proof that $p^\omega(1-p)^\alpha \geq 1/n^6$ is detailed in Appendix A and uses Taylor series computation. For simplicity, we only show here the case when $p = 1/2$. Then :

- $\omega = 2\log(n) + o(\log n)$
- $\alpha = 2\log(n) + o(\log n)$

Thus $p^\omega(1-p)^\alpha = 1/2^{4\log n + o(\log n)} = n^{4+o(1)}$. □

Note here that no optimization was made on the constant of the polynomial. Some refinements in the proof can lead to a complete $(\omega,\alpha,n)$-separator of size $\mathcal{O}(n^{5+\varepsilon})$. Moreover, an interesting question would be a lower bound on the constant of the polynomial needed to separate the cliques and the stable sets in random graphs, in particular for the special case $p = 1/2$.

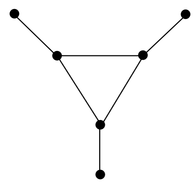


Figure 2: A net

### 3.2 Graph classes: the case of split-free graphs.

A graph $\Gamma$ is called *split* if it is the union of a clique and a stable set. A graph $G = (V, E)$ *has an induced* $\Gamma$ if there exists $X \subseteq V$ such that the induced graph $G[X]$ is isomorphic to $\Gamma$. We denote by $\mathcal{C}_\Gamma$ the class of graph with no induced $\Gamma$. For instance, if $\Gamma$ is the split graph described on Fig. 2 and called a *net*, then $\mathcal{C}_\Gamma$ is the class of net-free graphs and contains comparability graphs.

Let us first state some definitions concerning hypergraphs and VC-dimension. Let $H = (V, E)$ be a hypergraph. The *transversality* $\tau(H)$ is the minimum cardinality of a subset of vertices intersecting each hyperedge. The transversality corresponds to an optimal solution of the following integer linear program.

$$\text{Objective function:} \qquad \min \sum_{x \in V} w(x)$$

Subject to:

- for all $x \in V$, $w(x) \in \{0, 1\}$.
- for all $e \in E$, $\sum_{x \in e} w(x) \geq 1$.

The *fractional transversality* $\tau^*$ is the fractional relaxation of the above linear programming. The first condition is then replaced by: for all $x \in V$, $0 \leq w(x) \leq 1$. Note that removing the constraint $w(x) \leq 1$ for all $x \in V$ does not change the solution as we want to minimize the objective function. Indeed if $w(y) > 1$ for some $y \in V$ in a feasible solution $w$, getting $w(y)$ down to 1 will not violate any constraint, and will reduce strictly the objective function.

The *Vapnik-Chervonenkis dimension* or *VC-dimension* of a hypergraph $H = (V, E)$ is the maximum cardinality of a set of vertices $A \subseteq V$ such that for every $B \subseteq A$ there is an edge $e \in E$ so that $e \cap A = B$. The following bound due to Haussler and Welzl [11] links the transversality, the VC-dimension and the fractional transversality.

**Lemma 4.** *Every hypergraph $H$ with VC-dimension $d$ satisfies*

$$\tau(H) \leq 16d\tau^*(H)\log(d\tau^*(H)).$$

Let us introduce the dual measure of VC-dimension. A set $T$ of hyperedges forms a *complete Venn diagram* is for all $T' \subseteq T$, there exists a vertex $v$ such that $v \in e$ if and only if $e \in T'$. The *dual Vapnik-Chervonenkis dimension* (*dual VC-dimension* for short) of $H$ is the maximum size of a complete Venn diagram in $H$. In the remaining of the report, we only consider hypergraphs which are neighborhood hypergraphs, i.e. hypergraphs so that the hyperedges are the neighborhoods of vertices in a given graph $G$. In this context, the VC-dimension and the dual VC-dimension of a hypergraph coincide, since for all vertices $x$ and $y$, $x \in N(y)$ if and only if $y \in N(x)$. Consequently, for simplicity, in the following the dual VC-dimension will be called VC-dimension.

**Theorem 5.** *Let $\Gamma$ be a fixed split graph. Then the clique-stable set conjecture is verified on $\mathcal{C}_\Gamma$.*

*Proof.* The vertices of $\Gamma$ are partitioned into $(V_1, V_2)$ where $V_1$ is a clique and $V_2$ is a stable set. Let $\varphi = \max(|V_1|, |V_2|)$. Let $t$ be the constant $32\varphi(\log(\varphi) + 1)$. Let $G = (V, E) \in \mathcal{C}_\Gamma$ and let $\mathcal{F}$ be the following family of cuts. For every subset $\{x_1, \dots, x_r\}$ of at most $t$ vertices which is a clique (resp. a stable set), take $U = \cap_{1\leq i\leq r} N(x_i)$ (resp. $U = \cup_{1\leq i\leq r} N(x_i)$), and put $(U, V \setminus U)$ in $\mathcal{F}$. Since each member of $\mathcal{F}$ is defined with a set of at most $t$ vertices, the size of $\mathcal{F}$ is at most $\mathcal{O}(n^t)$. Let us now prove that $\mathcal{F}$ is a CS-separator.

Let $(C, S)$ be a pair of maximal clique and stable set. We prove that $(C, S)$ is separated by $\mathcal{F}$. Build an oriented graph $B$ with vertex set $C \cup S$. For all $x \in C$ and $y \in S$, put the arc $xy$ if $xy \in E$, and put the arc $yx$ otherwise (see Fig. 3(b)). We use the following variant of Farkas' lemma, from which we derive Lemma 7:

**Lemma 6** (Farkas' lemma [21]). *Let $A$ be a $m \times n$ matrix and $b \in \mathbb{R}^m$. Then either:*

*1. There is a $w \in \mathbb{R}^n$ such that $Aw \leq b$,*

*or 2. There is a $y \in \mathbb{R}^m$ such that $y \geq 0$, $yA = 0$ and $yb < 0$.*

**Lemma 7.** *For all oriented graph $G = (V, E)$, there exists a weight function $w : V \to [0, 1]$ such that $w(V) = 1$ and for all vertex $x$, $w(N^+(x)) \geq w(N^-(x))$.*

*Proof.* Let us define a $(2n + 1) \times n$ matrix $A$ obtained from the vertical concatenation of three matrices. First, the transpose ${}^t\mathrm{Adj}(G)$ of the adjacency matrix of $G$ which will ensure the constraints of type $w(N^+(x)) \geq w(N^-(x))$. Second, the matrix $-\mathrm{Id}_n$ which will ensure the constraints of type $w(x) \geq 0$. And finally the auxiliary line vector $(-1, \dots, -1) \in \mathbb{R}^n$. Define $b = {}^t(0, \dots, 0, -1) \in \mathbb{R}^{2n+1}$.

Then apply lemma 6. Either case one occurs and then $Aw \leq b$: as expected, we get $w(N^+(x)) \geq w(N^-(x))$ for all $x \in V$ thanks to ${}^t\mathrm{Adj}(G)$. Thanks to the other lines, we get $w(x) \geq 0$ for all $x$ and $w(V) \geq 1$. We conclude by rescaling the weight function with a factor $1/w(V)$.

Otherwise, case two occurs and there is $y \in \mathbb{R}^{2n+1}$ : let $w \in \mathbb{R}^n$ be the projection of $y$ on the first $n$ coordinates. Then for all $j \in \{1, \dots, 2n + 1\}$, $y_j \geq 0$ so $w(x) \geq 0$ for all $x$; moreover, since $yA = 0$ then for all $x_i \in V$, we have $w(N^+(x)) - w(N^-(x)) - y_{n+i} - y_{2n+1} = 0$ hence $w(N^+(x)) \geq w(N^-(x))$. Since $yb < 0$, then $y_{2n+1} > 0$ which ensures $w(V) > 0$: indeed, otherwise for all $x$, $w(x) = 0$ and the previous equality becomes $y_{2n+1} = -y_{n+i} \leq 0$, which is a contradiction. We conclude as before by rescaling with a factor $1/w(V)$. □

**Corollary 8.** *In $B$, there exists either:*

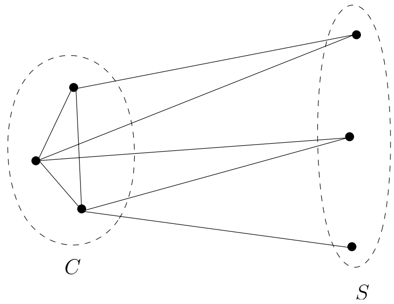
*(i) a weight function $w : C \to \mathbb{R}^+$ such that $w(C) = 2$ and for all vertex $x \in S, w(N^+(x)) \geq 1$.*

*or (ii) a weight function $w : S \to \mathbb{R}^+$ such that $w(S) = 2$ and for all vertex $x \in C, w(N^+(x)) \geq 1$.*

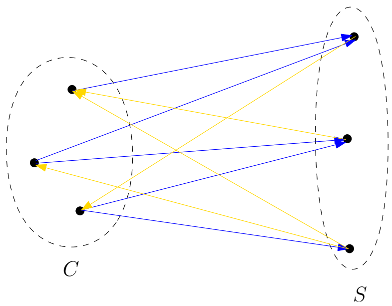
*Proof.* Let $w : V \to [0, 1]$ be a weight function satisfying conditions of Lemma 7. Since $w(V) = 1$, either $w(C) > 0$ or $w(S) > 0$. Assume $w(C) > 0$ (the other case is handled symmetrically). Take a new weight function defined by $w'(x) = 2w(x)/w(C)$ if $x \in C$, and 0 otherwise. Then for all $x \in S$, on one hand $w'(N^+(x)) \geq w'(N^-(x))$ by extension of the property of $w$, and on the other hand, $N^+(x) \cup N^-(x) = C$ by construction of $B$. Thus $w'(N^+(x)) \geq w'(C)/2 = 1$ since $w'(C) = 2$. □

In the following, let assume we are in case (i). Case (ii) is handled symmetrically by switching $C$ (resp. neighborhood) and $S$ (resp. non-neighborhood).

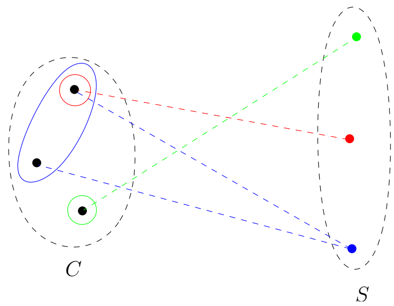
Let us now build $H$ a hypergraph with vertex set $C$. For all $x \in S$, build the hyperedge $C \setminus N_G(x)$, that is the complementary in $C$ of the neighbors of $x$ (see Fig. 3(c)).




(a) A clique $C$ and a stable $S$ in $G$.




(b) Graph $B$ build from $C$ and $S$. Edges in $G$ are replaced by forward arcs, and non-edges are replaced by backward arcs.




(c) Hypergraph $H$ where hyperedges are built from the non-neighborhood of vertices from $S$.

Figure 3: Illustration of proof of Theorem 5. For more visibility in 3(b), forward arcs are drawn in blue and backward arcs in yellow.

**Lemma 9.** *The hypergraph $H$ has fractional transversality $\tau^* \leq 2$.*

*Proof.* Let $w$ be the weight function given by Corollary 8. Let $h$ be a hyperedge built from the non-neighborhood of $x \in S$. Recall that this non-neighborhood is precisely $N^+(x)$ in $B$, then we have:

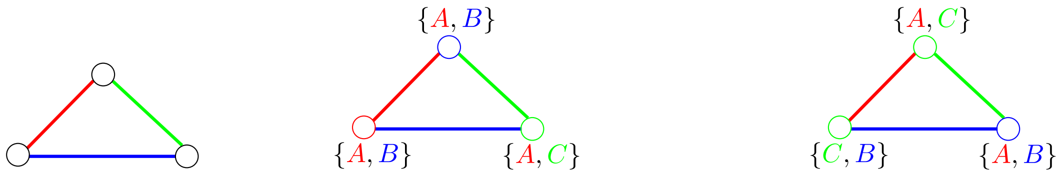
$$\sum_{y \in h} w(y) = w(N^+(x)) \geq 1.$$

Thus $w$ satisfies the constraints of the fractional transversality, and $w(C) \leq 2$, i.e. $\tau^* \leq 2$. □

**Lemma 10.** *$H$ has VC-dimension bounded by $2\varphi - 1$.*

*Proof.* Assume that there is a complete Venn-diagram $\mathcal{D}$ of size $2\varphi$ in $H$. The aim is to exploit the shattering to find an induced $\Gamma$, which builds a contradiction. Let $s_1, \dots s_\varphi, t_1, \dots, t_\varphi$ be the hyperedges composing $\mathcal{D}$. In the following, we will abuse notation by calling $s_i$ (resp. $t_i$) both the hyperedge and the vertex of $S$ whose non-neighborhood is precisely the hyperedge $s_i$ (resp. $t_i$). Recall that the forbidden split graph $\Gamma$ is the union of a clique $V_1 = \{x_1, \dots, x_r\}$ and a stable set $V_2 = \{y_1, \dots, y_{r'}\}$ (with $r, r' \leq \varphi$). Let $x_i \in V_1$ and let $\{y_{i_1}, \dots, y_{i_k}\} = N_\Gamma(x_i) \cap V_2$ be the set of its neighbors in $V_2$.

Consider $\mathcal{S} = \{s_{i_1}, \dots, s_{i_k}\} \cup \{t_i\}$ (possible because $|V_1|, |V_2| \leq \varphi$). As $\mathcal{D}$ is a complete Venn diagram, there exists $x'_i \in C$ such that $x'_i \in \cap_{s \in \mathcal{S}} s$ and $x'_i \notin \cup_{s' \notin \mathcal{S}} s'$, meaning that the set of hyperedges containing $x'_i$ is precisely $\mathcal{S}$. Now, forget about the existence of $t_1, \dots, t_\varphi$, and look at the subgraph of $G$ induced by $x'_1, \dots, x'_r$ and $s_1, \dots, s_{r'}$: $x'_i$ has exactly the same shape of neighborhood in $\{s_1, \dots, s_{r'}\}$ as the neighborhood of $x_i$ in $V_2$. Thus we have found an induced $\Gamma$, which is impossible. □




(a) An instance of 3-CCP

(b) A solution to the instance (vertex coloring) together with a compatible 2-list assignment: each vertex has a 2-constraint.

(c) Another solution to the instance with a compatible 2-list assignment.

Figure 4: Illustration of definitions. Color correspondence: A=red ; B=blue ; C=green. Both 2-list assignments together form a 2-list covering because any solution is compatible with at least one of them.

Note that the presence of $t_1, \dots, t_\varphi$ is useful in case where two vertices of $V_1$ are twins with respect to $V_2$, meaning that their neighborhoods intersected with $V_2$ are the same. Then, the complete Venn diagram does not ensure that at least two vertices are contained in exactly the set of hyperedges $\mathcal{S} = \{s_{i_1}, \dots, s_{i_k}\}$, and no more. In fact, this remark leads us to assert that the VC-dimension of $H$ is bounded by $\varphi + \log \varphi$. Indeed, we need only $t_1, \dots, t_{\log \varphi}$ in addition to $s_1, \dots, s_\varphi$: for $x_i \in V_1$, code $i$ in binary over $\log \varphi$ bits and define $\mathcal{S}$ to be the union of $\{s_{i_1}, \dots, s_{i_k}\}$ with the set of $t_j$ such that the $j$-th bit is one. This ensures that no two $x_i, x_{i'}$ have the same $\mathcal{S}$.
Applying Lemmas 4, 9 and 10 to $H$, we obtain

$$\tau(H) \leq 16d\tau^*(H)\log(d\tau^*(H)) \leq 32\varphi(\log(\varphi) + 1) = t.$$

Hence $\tau$ is bounded by $t$ which only depends on $H$. There must be $x_1, \dots, x_\tau \in C$ such that each hyperedge of $H$ contains at least one $x_i$. Thus for all $y \in S$, there is an $i$ such that $x_i \in N^C(y)$. Consequently, $S \subseteq \cup_{1 \leq i \leq t} N^C_G(x_i)$. Moreover, $C \subseteq U = \cap_{1 \leq i \leq t} N_G(x_i) \cup \{x_i\}$ since $x_1, \dots, x_\tau$ are in the same clique $C$. This means that the cut $(U, V \setminus U) \in \mathcal{F}$ built from the subclique $x_1, \dots x_\tau$ separates $C$ and $S$.

When case (ii) of Corollary 8 occurs, there are $\tau$ vertices $x_1, \dots, x_\tau \in S$ such that for all $y \in C$, there exists $x_i \in N(y)$. Thus $C \subseteq U = \cup_{1 \leq i \leq t} N_G(x_i)$ and $S \subseteq \cup_{1 \leq i \leq t} N^C_G(x_i)$. The cut $(U, V \setminus U) \in \mathcal{F}$ built from the stable set $x_1, \dots x_\tau$ separates $C$ and $S$. $\square$

## 4 3-CCP and the stubborn problem

The following definitions are illustrated on Fig. 4 and deal with list colorings. Formally, let $G$ be a graph and Col a set of $k$ colors. A set of possible colors, called *constraint*, is associated to each vertex. A vertex has an *$l$-constraint* if its set of possible colors has size $l$. An *$l$-list assignment* is a function $\mathcal{L} : V \to \mathcal{P}(\text{Col})$ that give each vertex an $l$-constraint. A solution $\mathcal{S}$ is a coloring of the vertices $S : V \to \text{Col}$ that respects some requirements depending on the problem. We can equivalently consider $S$ as a partition $(A_1, \dots, A_k)$ with $x \in A_i$ if and only if $S(x) = A_i$ (note then that $A_i$ denotes both the color and the set of vertices having this color). An $l$-list assignment $\mathcal{L}$ is *compatible* with a solution $S$ if for each vertex $x$, $S(x) \in \mathcal{L}(x)$. A set of $l$-list assignment *covers* a solution $\mathcal{S}$ if at least one of the $l$-list assignment is compatible with $\mathcal{S}$.

We recall the definitions of 3-CCP and the stubborn problem:

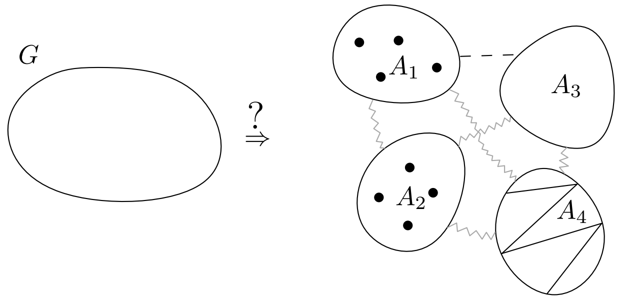



Figure 5: Diagram representing the stubborn problem. Cliques are represented by hatched sets, stable sets by dotted sets. Completely non-adjacent sets are linked by a dashed edge. Grey lines represent edges that may or may not appear in the graph.

3-Compatible Coloring Problem (3-CCP)
**Input:** An edge coloring $f_E$ of $K_n$ with 3 colors $\{A, B, C\}$.
**Question:** Is there a coloring of the vertices with $\{A, B, C\}$, such that no edge has the same color as both its endpoints?

Stubborn Problem (see Fig. 5)
**Input:** A graph $G = (V, E)$ together with a list assignments $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$.
**Question:** Can $V$ be partitioned into four sets $A_1, \ldots, A_4$ such that $A_4$ is a clique, both $A_1$ and $A_2$ are stable sets, $A_1$ and $A_3$ are completely non-adjacent, and the partition is compatible with $\mathcal{L}$?

Given an edge-coloring $f_E$ on $K_n$, we say that a set of 2-list assignment is a *2-list covering for 3-CCP on* $(K_n, f_E)$ if it covers all the solutions of 3-CCP on this instance. Moreover, 3-CCP is said to have a *polynomial* 2-list covering if for all $n$ and for all edge-coloring $f_E$, there is a 2-list covering on $(K_n, f_E)$ whose cardinality is bounded by a polynomial in $n$. Symmetrically, we want to define a *2-list covering for the stubborn problem*. However, we will not be able to cover polynomially all the solutions: for example on a stable set with the trivial 4-constraint on each vertex, any partition of the vertices into three sets gives a solution $(A_1, A_2, A_3, \emptyset)$. Thus we need a notion of maximal solutions. This notion is extracted from the notion of domination (here $A_3$ dominates $A_1$) in the language of general list-$M$ partition problem (see [7]). Intuitively, if $\mathcal{L}(v)$ contains both $A_1$ and $A_3$ and $v$ belongs to $A_1$ in some solution $S$, we can build a simpler solution by putting $v \in A_3$ and leaving everything else unchanged. A solution $(A_1, A_2, A_3, A_4)$ of the stubborn problem on $(G, \mathcal{L})$ is a *maximal solution* if no member of $A_1$ satisfies $A_3 \in \mathcal{L}(v)$. We may note that if $A_3$ is contained in every $\mathcal{L}(v)$ for $v \in V$, then every maximal solution of the stubborn problem on $(G, \mathcal{L})$ let $A_1$ empty. Now, a set of 2-list assignments is a *2-list covering for the stubborn problem on* $(G, \mathcal{L})$ if it covers all the maximal solutions on this instance. Moreover, it is called a *polynomial* 2-list covering if its size is bounded by a polynomial in the number of vertices in $G$.

For edge-colored graphs, an $\alpha_1, ..., \alpha_k$*-clique* is a clique for which every edge has a color in $\{\alpha_1, ..., \alpha_k\}$. A *split* graph is the union of an $\alpha$-clique and a $\beta$-clique. The $\alpha$*-edge-neighborhood* of $x$ is the set of vertices $y$ such that $xy$ is an $\alpha$*-edge*, i.e an edge colored with $\alpha$. The *majority color of* $x \in V$ is the color which appears the most often for the edges with endpoint $x$ (in case of ties, we arbitrarily cut them).

In this section, we prove that the existence of a polynomial 2-list covering for the stubborn problem is equivalent to the existence of a polynomial one for 3-CCP, which in turn is equivalent to the existence of a polynomial CS-separator. We may observe that the existence of a polynomial

2-list covering does not imply the polynomial solvability of the problem: indeed, such a family may not be computable in polynomial time.

We start by justifying the interest of 2-list covering and observing that we can always find a quasi-polynomial 2-list covering for 3-CCP.

**Observation 11.** *Given a polynomial number of* 2*-list assignments for 3-CCP, it is possible to decide in polynomial time if there exists a solution covered by them.*

*Proof.* Each 2-list assignment can be translated into an instance of 2-SAT. Each vertex has a 2-constraint $\{\alpha, \beta\}$ from which we construct two variables $x_\alpha$ and $x_\beta$ and a clause $x_\alpha \vee x_\beta$. Turn $x_\alpha$ to true will mean that $x$ is given the color $\alpha$. Then we need also the clause $\neg x_\alpha \vee \neg x_\beta$ saying that only one color can be given to $x$. Finally for all edge $xy$ colored with $\alpha$, we add the clause $\neg x_\alpha \vee \neg y_\alpha$ if both variables exists. □

**Theorem 12.** *[5] There exists an algorithm giving a 2-list covering of size $O(n^{\log n})$ for 3-CCP. By Observation 11, this gives an algorithm in time $O(n^{\log n})$ which solves* 3*-CCP.*

*Proof.* Let us build a tree of maximum degree $n+1$ and height $\mathcal{O}(\log n)$ whose leaves will exactly be the 2-list assignments needed to cover all the solutions. By a counting argument, such a tree will have at most $O(n^{\log n})$ leaves. Let $x$ be a vertex, without loss of generality we can assume that $x$ has majority color $A$. The solutions are easily partitioned between those where $x$ is given its majority color $A$, and those where $x$ is given color $B$ or $C$. From this simple remark, we can build a tree with an unlabelled root, $n$ children each labelled by a different vertex, and an extra leave corresponding to the solutions where no vertex is colored by its majority color. The latter forms a 2-list assignment since we forbid one color for each vertex. Each labelled child of the root, say its label is $x$, will consider only solutions where $x$ is given its majority color $A$, thus $x$ has constraint $\{A\}$. Then in every such solution, each vertex linked to $x$ by an $A$-edge will be given the color $B$ or $C$. Thus we associate the 2-constraint $\{B, C\}$ to the whole $A$-edge-neighborhood of $x$. Since the graph is complete and $A$ is the majority color, this $A$-edge-neighborhood represents at least $1/3$ of all the vertices. We iterate the process on the graph restricted to unconstrained vertices, and build a subtree rooted at node $x$. We do so for the other labelled children of the root. The tree is ensured to have height $\mathcal{O}(\log n)$ because we erase at least $1/3$ of the vertices at each level. □

The main result of this section is the following theorem:

**Theorem 13.** *The following are equivalent:*

1. *For every graph $G$ and every list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on $(G, \mathcal{L})$.*

2. *For every $n$ and every edge-coloring $f : E(K_n) \to \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on $(K_n, f)$.*

3. *For every graph $G$, there is a polynomial CS-separator.*

We decompose the proof into three lemmas, each of which describing one implication.

**Lemma 14.** $(1 \Rightarrow 2)$*: Suppose for every graph $G$ and every list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, \ldots, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on $(G, \mathcal{L})$. Then for every graph $n$ and every edge-coloring $f : E(K_n) \to \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on $(K_n, f)$.*

*Proof.* Let $n \in \mathbb{N}$, $(K_n, f)$ be an instance of 3-CCP, and $x$ a vertex of $K_n$. Let us build a polynomial number of 2-list assignments that cover all the solutions where $x$ is given color $A$. Since the colors are symmetric, we just have to multiply the number of 2-list assignments by 3 to cover all the solutions. Let $(A, B, C)$ be a solution of 3-CCP where $x \in A$.

**Claim 15.** *Let $x$ be a vertex and $\alpha, \beta, \gamma$ be the three different colors. Let $U$ be the $\alpha$-edge-neighborhood of $x$. If there is a $\beta\gamma$-clique $Z$ of $U$ which is not split, then there is no solution where $x$ is colored with $\alpha$.*

*Proof.* Consider a solution in which $x$ is colored with $\alpha$. All the vertices of $Z$ are of color $\beta$ or $\gamma$ because they are in the $\alpha$-edge-neighborhood of $x$. The vertices colored with $\beta$ form a $\gamma$-clique, those colored by $\gamma$ form a $\beta$-clique. Hence $Z$ is split. □

A vertex $x$ is *really* 3*-colorable* if for each color $\alpha$, every $\beta\gamma$-clique of the $\alpha$-edge-neighborhood of $x$ is a split graph. If a vertex is not really 3-colorable then, in a solution, it can be colored by at most 2 different colors. Hence if $K_n(V \backslash x)$ has a polynomial 2-list covering, the same holds for $K_n$ by assigning the only two possible colors to $x$ in each 2-list assignment.

Thus we can assume that $x$ is really 3-colorable, otherwise there is a natural 2-constraint on it. Since we assume that the color of $x$ is $A$, we can consider that in all the following 2-list assignments, the constraint $\{B, C\}$ is given to the $A$-edge-neighborhood of $x$. Let us abuse notation and still denote by $(A, B, C)$ the partition of the $C$-edge-neighborhood of $x$, induced by the solution $(A, B, C)$. As $x$ is really 3-colorable, Claim 15 ensures that $C$ is a split graph $C' \uplus C''$ with $C'$ a $B$-clique and $C''$ a $A$-clique. The situation is described in Fig. 7(a). Let $H$ be the non-colored graph with vertex set the $C$-edge-neighborhood of $x$ and there is an edge $e$ if and only if $f(e) = B$ or $f(e) = C$ (see Fig. 7(b)). Moreover, let $H'$ be the non-colored graph with vertex set the $C$-edge-neighborhood of $x$ and there is an edge $e$ if and only if $f(e) = B$ (see Fig. 7(c)). We consider $(H, \mathcal{L}_0)$ and $(H', \mathcal{L}_0)$ as two instances of the stubborn problem, where $\mathcal{L}_0$ is the trivial list assignment that gives each vertex the constraint $\{A_1, A_2, A_3, A_4\}$.

By assumption, there exists $\mathcal{F}$ (resp. $\mathcal{F}'$) a polynomial 2-list covering for the stubborn problem on $(H, \mathcal{L}_0)$ (resp. $(H', \mathcal{L}_0)$). We construct $\mathcal{F}''$ the set of 2-list assignment $f''$ built from all the pairs $(f, f') \in \mathcal{F} \times \mathcal{F}'$ according to the rules described in Fig. 6 (intuition for such rules is given in the next paragraph). $\mathcal{F}''$ aims at being a polynomial 2-list covering for 3-CCP on the $C$-edge-neighborhood of $x$.

The following is illustrated on Fig. 7(b) and 7(c). Let $\mathcal{S}$ be the partition defined by $A_1 = \emptyset$, $A_2 = C''$, $A_3 = B \cup C'$ and $A_4 = A$. We can check that $A_2$ is a stable set and $A_4$ is a clique (the others restrictions are trivially satisfied by $A_1$ being empty and $\mathcal{L}_0$ being trivial). In parallel, let $\mathcal{S}'$ be the partition defined by $A'_1 = \emptyset$, $A'_2 = B$, $A'_3 = A \cup C''$ and $A_4 = C'$. We can also check that $A'_2$ is a stable set and $A'_4$ is a clique. Thus $\mathcal{S}$ (resp. $\mathcal{S}'$) is a maximal solution for the stubborn problem on $(H, \mathcal{L}_0)$ (resp. $(H', \mathcal{L}_0)$) inherited from the solution $(A, B, C = C' \uplus C'')$ for 3-CCP.

Let $f \in \mathcal{F}$ (resp. $f' \in \mathcal{F}'$) be a 2-list assignment compatible with $\mathcal{S}$ (resp. $\mathcal{S}'$). Then $f'' \in \mathcal{F}''$ built from $(f, f')$ is a 2-list assignment compatible with $(A, B, C)$.

Doing so for the $B$-edge-neighborhood of $x$ and pulling everything back together gives a polynomial 2-list covering for 3-CCP on $(K_n, f)$.

□

**Lemma 16.** $(2 \Rightarrow 3)$*: Suppose for every $n$ and every edge-coloring $f : E(K_n) \to \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on $(K_n, f)$. Then for every graph $G$, there is a polynomial CS-separator.*

| $f(v)$ | $f'(v)$ | $f''(v)$ |
|---|---|---|
| $A_2$ or $A_1, A_2$ | $*$ | $C$ |
| $A_3$ or $A_1, A_3$ | $*$ | $B, C$ |
| $A_4$ or $A_1, A_4$ | $*$ | $A$ |
| $A_2, A_4$ | $*$ | $A, C$ |
| $A_2, A_3$ | $*$ | $B, C$ |
| $A_3, A_4$ | $A'_2$ or $A'_1, A'_2$ | $B$ |
| $A_3, A_4$ | $A'_3$ or $A'_1, A'_3$ | $A, C$ |
| $A_3, A_4$ | $A'_4$ or $A'_1, A'_4$ | $C$ |
| $A_3, A_4$ | $A'_2, A'_4$ | $B, C$ |
| $A_3, A_4$ | $A'_2, A'_3$ | $A, B$ |
| $A_3, A_4$ | $A'_3, A'_4$ | $A, C$ |

Figure 6: This table describes the rules used in proof of lemma 14 to built a 2-list assignment $f''$ for 3-CCP from a pair $(f, f')$ of 2-list assignment for two instances of the stubborn problem. Symbol $*$ stands for any constraint. For simplicity, we write $X, Y$ (resp. $X$) instead of $\{X, Y\}$ (resp. $\{X\}$).

*Proof.* Let $G = (V, E)$ be a graph on $n$ vertices. Let $f$ be the coloring on $K_n$ defined by $f(e) = A$ if $e \in E$ and $f(e) = B$ otherwise. In the following $(K_n, f)$ is considered as an instance of 3-CCP. By hypothesis, there is a polynomial 2-list covering $\mathcal{F}$ for 3-CCP on $(K_n, f)$. Let us prove that we can derive from $\mathcal{F}$ a polynomial CS-separator $\mathcal{C}$.

Let $\mathcal{L} \in \mathcal{F}$ be a 2-list assignment. Define $X$ (resp. $Y$, $Z$) the set of vertices which have the constraint $\{A, B\}$ (resp. $\{B, C\}$, $\{A, C\}$). Since no edge has color $C$, $X$ is split. Indeed, the vertices of color $A$ form a $B$-clique and conversely. Given a graph, there is a linear number of decompositions into a split graph [7]. Thus there are a linear number of decomposition $(U_k, V_k)_{k \leq cn}$ of $X$ into a split graph where $U_k$ is a $B$-clique. For all $k$, the cut $(U_k \cup Y, V_k \cup Z)$ is added in $\mathcal{C}$. For each 2-list assignment we create a linear number of separators.

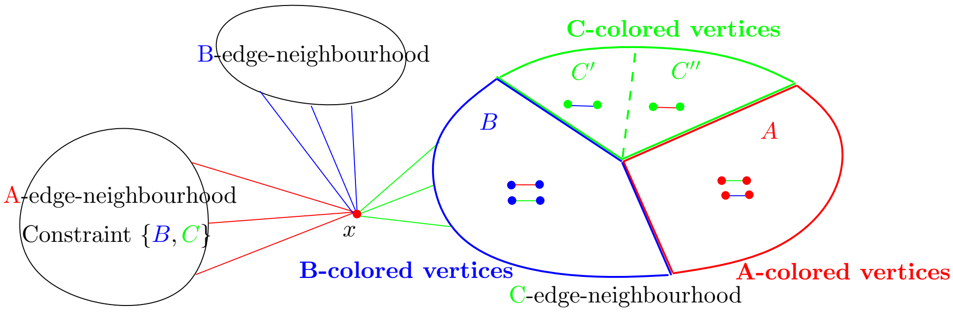
Let $K$ be a clique and $S$ a stable set of $G$ which do not intersect. The edges of $K$ are colored by $A$, and those of $S$ are colored by $B$. Then the coloration $\mathcal{S}(x) = B$ if $x \in K$, $\mathcal{S}(x) = A$ if $x \in S$ and $\mathcal{S}(x) = C$ otherwise is a solution of $(K_n, f)$. Left-hand side of Fig. 8 illustrates the situation. There is a 2-list assignment $\mathcal{L}$ in $\mathcal{F}$ which is compatible with this solution. As before, let $X$ (resp. $Y$, $Z$) be the set of vertices which have the constraint $\{A, B\}$ (resp. $\{B, C\}$, $\{A, C\}$). Since the vertices of $K$ are colored $B$, we have $K \subseteq X \cup Y$ (see right hand-side of Fig. 8). Likewise, $S \subseteq X \cup Z$. Then $(K \cap X, S \cap X)$ forms a split partition of $X$. So, by construction, there is a cut $((K \cap X) \cup Y, (S \cap X) \cup Z) \in \mathcal{C}$ which ensures that $(K, S)$ is separated by $\mathcal{C}$. □

**Lemma 17.** $(3 \Rightarrow 1)$: *Suppose for every graph $G$, there is a polynomial CS-separator. Then for every graph $G$ and every list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on $(G, \mathcal{L})$.*

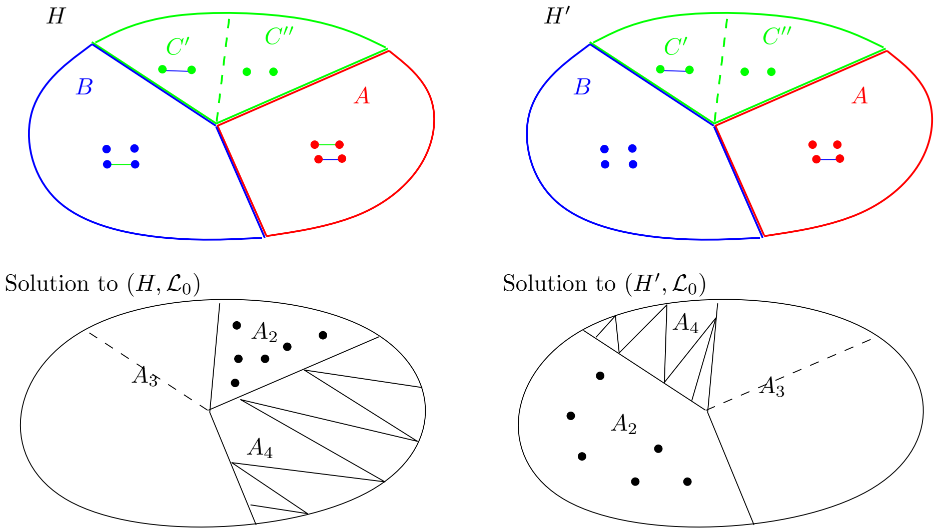
*Proof.* Let $(G, \mathcal{L})$ be an instance of the stubborn problem. By assumption, there is a polynomial CS-separator for $G$.

**Claim 18.** *If there are $p$ cuts that separate all the cliques from the stable sets, then there are $p^2$ cuts that separate all the cliques from the unions $S \cup S'$ (where $S$ and $S'$ are stable sets).*

*Proof.* Indeed, if $(V_1, V_2)$ separates $C$ from $S$ and $(V'_1, V'_2)$ separates $C$ from $S'$, then the new cut $(V_1 \cap V'_1, V_2 \cup V'_2)$ satisfies $C \subseteq V_1 \cap V'_1$ and $S \cup S' \subseteq V_2 \cup V'_2$. □




(a) Vertex $x$, its A-edge-neighborhood subject to the constraint $\{B, C\}$, and its C-edge-neighborhood separated in different parts.




(b) Above, the graph $H$ obtained from the C-edge-neighborhood by keeping only B-edges and C-edges. Below, the solution of the stubborn problem.

(c) Above, the graph $H'$ obtained from the C-edge-neighborhood by keeping only B-edges. Below, the solution of the stubborn problem.

Figure 7: Illustration of proof of lemma 14. Color correspondence: A=red ; B=blue ; C=green. As before, cliques are represented by hatched sets, stable sets by dotted sets.

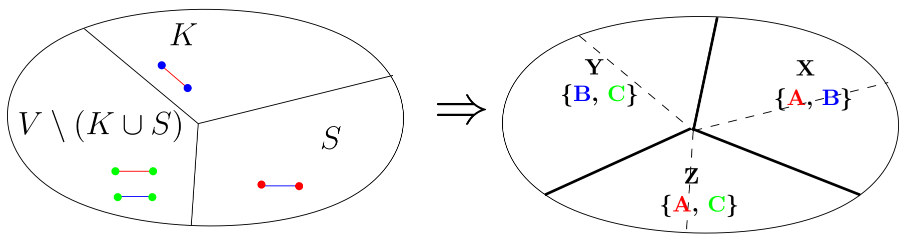



Figure 8: Illustration of proof of lemma 16. On the left hand-side, $G$ is separated in 3 parts: $K$, $S$, and the remaining vertices. Each possible configuration of edge- and vertex-coloring are represented. On the right-hand-side, $(X, Y, Z)$ is a 2-list assignment compatible with the solution. $X$ (resp. $Y$, $Z$) has constraint $\{A, B\}$ (resp. $\{B, C\}$, $\{A, C\}$). Color correspondence: A=red ; B=blue ; C=green.

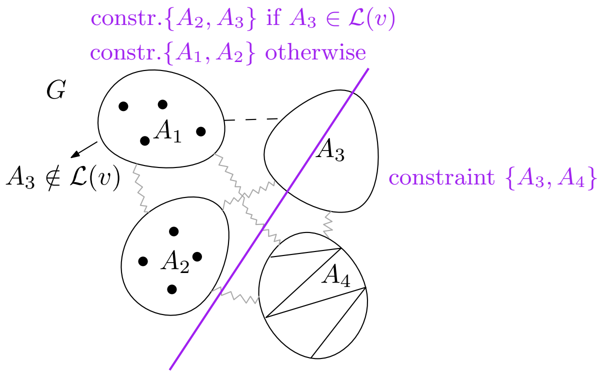



Figure 9: Illustration of proof of lemma 17. A solution to the stubborn problem together with the cut that separates $A_4$ from $A_1 \cup A_2$. The 2-list assignment built from this cut is indicated in purple.

Let $\mathcal{F}_2$ be a polynomial family of cuts that separate all the cliques from union of two stable sets, which exists by Claim 18. Then for all $(U, W) \in \mathcal{F}_2$, we build the following 2-list assignment $\mathcal{L}'$:

1. If $v \in U$, let $\mathcal{L}'(v) = \{A_3, A_4\}$.
2. If $v \in W$ and $A_3 \in \mathcal{L}(v)$, then let $\mathcal{L}'(v) = \{A_2, A_3\}$.
3. Otherwise, $v \in W$ and $A_3 \notin \mathcal{L}(v)$, let $\mathcal{L}'(v) = \{A_1, A_2\}$.

Now the set $\mathcal{F}'$ of such 2-list assignment $\mathcal{L}'$ is a 2-list covering for the stubborn problem on $(G, \mathcal{L})$: let $\mathcal{S} = (A_1, A_2, A_3, A_4)$ be a maximal solution of the stubborn problem on this instance. Then $A_4$ is a clique and $A_1, A_2$ are stable sets, so there is a separator $(U, W) \in \mathcal{F}_2$ such that $A_4 \subseteq U$ and $A_1 \cup A_2 \subseteq W$ (see Fig. 9), and there is a corresponding 2-list assignment $\mathcal{L}' \in \mathcal{F}'$. Consequently, the 2-constraint $\mathcal{L}'(v)$ built from rules 1 and 3 are compatible with $\mathcal{S}$. Finally, as $\mathcal{S}$ is maximal, there is no $v \in A_1$ such that $A_3 \in \mathcal{L}(v)$: the 2-constraints built from rule 2 are also compatible with $\mathcal{S}$. □

*Proof of theorem 13.* Lemmas 14, 16 and 17 conclude the proof of Theorem 13. □

# 5 Bipartite packing and graph coloring

The aim of this section is to prove that the polynomial Alon-Saks-Seymour conjecture is equivalent to the Clique-Stable set separation conjecture. We first need an intermediate step using a new version of the Alon-Saks-Seymour conjecture, called the Oriented Alon-Saks-Seymour conjecture. Then we prove that bounding the chromatic number $\chi$ polynomially in terms of the $t$-biclique number $\mathbf{bp}_t$ is equivalent to bounding $\chi$ polynomially in terms of the bipartite packing $\mathbf{bp}$. Combining these two properties prove the statement.

## 5.1 Oriented Alon-Saks-Seymour conjecture

Given a graph $G$, the *chromatic number* $\chi(G)$ of $G$ is the minimum number of colors needed to color the vertices such that two vertices connected by an edge do not have the same color. The *bipartite packing* $\boldsymbol{bp}(G)$ of a graph $G$ is the minimum number of edge-disjoint complete bipartite graphs needed to partition the edges of $G$. The Alon-Saks-Seymour conjecture states the following.

**Conjecture 19.** *(Alon, Saks, Seymour) If $\boldsymbol{bp}(G) \le k$, then $\chi(G) \le k+1$ .*

For complete graphs it is a well-known result, due to Graham and Pollak [10]. Indeed, $n-1$ edge-disjoint complete bipartite graphs are needed to partition the edges of $K_n$. A beautiful algebraic proof of this theorem is due to Tverberg [22]. Conjecture 19 was disproved by Huang and Sudakov in [12] who proved that $\chi \geq k^{6/5}$ for some graphs using a construction based on Razborov's graphs [18]. Nevertheless the existence of a polynomial bound is still open. In the following we will consider an oriented version of the Alon-Saks-Seymour conjecture. The *oriented bipartite packing* $\boldsymbol{bp}_{\mathrm{or}}(G)$ of a non-oriented graph $G$ is the minimum number of oriented complete bipartite graphs such that each edge is covered by an arc in at least one direction (it can be in both directions), but it cannot be covered twice in the same direction.

**Conjecture 20.** *(Oriented Alon-Saks-Seymour Conjecture) There exists a polynomial $P$ such that for every $G$, $\chi(G) \le P(\boldsymbol{bp}_{\mathrm{or}}(G))$.*

**Lemma 21.** *If the oriented Alon-Saks-Seymour conjecture is verified, then the Clique-Stable set separation conjecture is verified.*

*Proof.* Let $G$ be a graph on $n$ vertices. We want to separate all the pairs of cliques and stable sets which do not intersect. Consider all the pairs $(C,S)$ such that the clique $C$ does not intersect the stable set $S$. Construct an auxiliary graph $H$ as follows. The vertices of $H$ are the pairs $(C,S)$ and there is an edge between a pair $(C,S)$ and a pair $(C',S')$ if and only if there is a vertex $x \in S \cap C'$. Observe that the number of vertices of $H$ is at most $4^n$. The bipartite packing of this graph is at most the number $n$ of vertices of $G$. Indeed, let $H_x$ be the graph $H$ restricted to the edges for which $x \in S \cap C'$. Put an orientation $(C,S)(C',S')$ on these edges. Observe that the union of the oriented graphs $H_x$ for all $x$ covers the graph $H$ because if $(C,S)(C',S')$ is an edge, then $S \cap C' \neq \emptyset$. In addition, the graph $H_x$ is a complete bipartite graph: if there is an edge which starts in $(C,S)$ and if there is an edge which ends in $(C',S')$ then $x \in S$ and $x \in C'$ and finally there is an arc $(C,S)(C',S')$. The graphs $H_x$ and $H_y$ cannot share an arc because otherwise the intersection between a clique and a stable set would be at least 2 which is impossible. Hence the oriented bipartite packing of this graph is at most $n$.
If the oriented Alon-Saks-Seymour conjecture is verified, $\chi(H) < P(n)$. Consider a color of this polynomial coloring. Let $A$ be the set of vertices of this color. There is no edge between two vertices of $A$, then the union of all the second components (stable sets) of the vertices of $A$ do not intersect the union of all the first components (cliques) of $A$. Indeed, if they intersect, there is a clique $C$ which intersects a stable set $S$, hence there is an edge which is impossible.

The union of the cliques and the union of the stable sets do not intersect, hence it defines a cut which separates all the pairs of $A$. The same can be done for every color. Then we can separate all the pairs $(C,S)$ by $\chi(H)$ cuts, which is a polynomial in $n$ if the Alon-Saks-Seymour conjecture is verified. This achieves the proof. □

**Lemma 22.** *If the Clique-Stable set separation conjecture is verified, then the oriented Alon-Saks-Seymour conjecture is verified.*

*Proof.* Let $G=(V,E)$ be a graph with bipartite packing $k$. Construct an auxiliary graph $H$ as follows. It has $k$ vertices which are the oriented complete bipartite graphs that cover the edges of $G$. There is an edge between two pairs $(A_1,B_1)$ and $(A_2,B_2)$ if and only if there is a vertex $x \in A_1 \cap A_2$. Hence the complete bipartite graphs in which $x$ appears at the left form a clique of $H$ (say the clique $C_x$ *associated to* $x$) and the complete bipartite graphs for which $y$ appears at the right form a stable set in $H$ (say the stable set $S_y$ *associated to* $y$) . Indeed, it is quite clear for the clique, and it is also true for the stable set because if $y \in B_1 \cap B_2$ and there is an edge resulting from $x \in A_1 \cap A_2$, then the arc $xy$ is covered twice which is impossible. Note that a clique or a stable set associated to a vertex can be empty, but this does not trigger any problem.

By assumption there are $P(k)$ (with $P$ a polynomial) cuts which separate all the pairs $(C, S)$, in particular which separate all the pairs $(C_x, S_x)$ for $x \in V$. We color each vertex $x$ by the color of the cut separating $(C_x, S_x)$. This coloring is proper: assume there is an edge from $x$ to $y$, and that $x$ and $y$ are given the same color. Then there exists a bipartite graph $(A, B)$ that cover the edge $xy$, hence $(A, B)$ is in the clique associated to $x$ and in the stable set associated to $y$, which means they intersect: no cut can separate at the same time both $C_x$ from $S_x$ and $C_y$ from $S_y$, because it would then separate $C_x$ from $S_y$. This is impossible. Then we have a coloring with at most $P(k)$ colors, which is a polynomial in $k$. □

**Theorem 23.** *The oriented Alon-Saks-Seymour conjecture is verified if and only if the Clique-Stable set separation conjecture is verified.*

*Proof.* This is straightforward using Lemmas 21 and 22. □

## 5.2 Generalization: $t$-biclique covering numbers

We study here a natural generalization of the Alon-Saks-Seymour conjecture, studied by Huang and Sudakov in [12]. While the Alon-Saks-Seymour conjecture deals with partitioning the edges, we relax here to a covering of the edges by complete bipartite graphs, meaning that an edge can be covered several times. Formally, a *$t$-biclique covering* of an undirected graph $G$ is a collection of bipartite graphs that cover every edge of $G$ at least once and at most $t$ times. The minimum size of such a covering is called the *$t$-biclique covering number*, and is denoted by $\mathbf{bp}_t(G)$. In particular, $\mathbf{bp}_1(G)$ is the usual biclique partition number $\mathbf{bp}(G)$.

In addition to being an interesting parameter to study in its own right, the $t$-biclique covering number of complete graphs is also closely related to a question in combinatorial geometry about neighborly families of boxes. It was studied by Zaks [24] and then by Alon [17], who proved that $\mathbb{R}^d$ has a $t$-neighborly family of $k$ standard boxes if and only if the complete graph $K_k$ has a $t$-biclique covering of size $d$ (see [12] for definitions and further details). Alon also gives asymptotic bounds for $\mathbf{bp}_t(K_k)$:

$$(1 + o(1))(t!/2^t)^{1/t} k^{1/t} \le \mathbf{bp}_t(K_k) \le (1 + o(1)) t k^{1/t} \ .$$

Our results are concerned not only with $K_k$ but for every graph $G$. It is natural to ask the same question for $\mathbf{bp}_t(G)$ as for $\mathbf{bp}(G)$, namely:

**Conjecture 24** (Generalized Alon-Saks-Seymour conjecture of order $t$)**.** *There exists a polynomial $P$ such that for all graphs $G$, $\chi(G) \le P(\boldsymbol{bp}_t(G))$.*

**Observation 25.** *A $t$-biclique covering is a fortiori a $t'$-biclique covering for all $t' \ge t$. Moreover, the set of $\boldsymbol{bp}_{\mathrm{or}}(G)$ oriented bipartite graphs covering each edge at most once in each direction can be seen as a non-oriented biclique covering which covers each edge at most twice. Hence, we have the following inequalities:*

$$\ldots \le \boldsymbol{bp}_{t+1}(G) \le \boldsymbol{bp}_t(G) \le \boldsymbol{bp}_{t-1}(G) \le \ldots \boldsymbol{bp}_2(G) \le \boldsymbol{bp}_{\mathrm{or}}(G) \le \boldsymbol{bp}_1(G) \ .$$

*In particular, if the generalized Alon-Saks-Seymour conjecture of order $t$ holds, then $\chi(G)$ is bounded by a polynomial in $\boldsymbol{bp}_t(G)$ and thus by a polynomial in $\boldsymbol{bp}_1(G)$, so the generalized Alon-Saks-Seymour of order 1 holds.*

We prove that the reverse is also true.

**Theorem 26.** *Let $t \in \mathbb{N}^*$. The generalized Alon-Saks-Seymour conjecture of order $t$ holds if and only if it holds for order 1.*

Before going to the proof, we need a few definitions: let $G$ be a graph and let $\mathcal{B} = \{B_1, ..., B_k\}$ be a family of bipartite complete graphs which covers the edges of $G$. Given an edge $e$, the *multiplicity* $m(e)$ *of* $e$ is the number of bipartite graphs which contain the edge $e$. We associate a *positive side* and a *negative side* to each bipartite graph, meaning that $B_i = (B_i^+, B_i^-)$ with $B_i^+$ (resp. $B_i^-$) being a stable set and referred to as the positive (resp. negative) side of $B_i$. Given an edge $xy$, two *half edges* are associated to $xy$: $(xy, x)$ and $(xy, y)$. If $B_i$ covers the edge $xy$, the half edge $(xy, x)$ is *a positive (resp. negative) half edge* for $B_i$ if $x$ is in the positive (resp. negative) side of the bipartite graph and $y$ in the other side. The *vertex associated to the half edge* $(e, x)$ is the vertex $x$. A *tag* is a pair consisting into a bipartite graph $B_i \in \mathcal{B}$ and a sign ($+$ or $-$). The *opposite tag* of a tag is the same bipartite graph and the opposite sign. A *bag* is a set of tags and two bags $X$ and $Y$ are *opposite* if for each tag in $X$, the opposite tag is in $Y$.

*Proof of Theorem 26.* As Observation 25 proves one direction, we focus on the other, and assume that the generalized Alon-Saks-Seymour conjecture of order 1 holds. Let us prove the result by induction on $t$, initialization for $t = 1$ being obvious. Let $G$ be a graph and let $\mathcal{B} = (B_1, ..., B_k)$ be a $t$-biclique covering. Since each edge is covered by at most $t$ bipartite graphs, each half edge $(e, x)$ of the graph can be represented by a bag with at most $t$ tags, containing all the $(B_i, +)$ such that $(e, x)$ is a positive half edge in $B_i$, and all the $(B_j, -)$ such that $(e, x)$ is a negative half edge in $B_j$. Note that there are $2k$ possible tags, thus the total number of different bags is at most $(2k)^t$.

Partition the set of half edges into $\{P_1, \ldots, P_m\}$ with $m = (2k)^t$, in such a way that two half edges are in the same $P_j$ if and only if they have exactly the same bag. The vertices *associated to* $P_j$ are the vertices associated to the half edges which appear in $P_j$. In a first time we will only consider the edges which are covered exactly $t$ times. Let $P_j$ and $P_{j'}$ be two parts such that their induced bags $X$ and $\overline{X}$ are opposite and of size $t$, and let $U$ and $U'$ be their set of associated vertices, respectively. Observe that the bipartite graphs appearing in $X$ and in $\overline{X}$ are the same, and that the vertices of $U$ and $U'$ appear in all these bipartite graphs. Note first that $U$ and $U'$ are disjoint: indeed, if $x \in U \cap U'$, then there exists $(e, x) \in P_j$ and $(e', x) \in P_{j'}$. For all tag $(B_i, \pm)$ in $X$, $(B_i, \mp)$ is in $\overline{X}$, thus $x$ has to be on both side of $B_i$, which is impossible. Moreover, there is a complete bipartite graph between $U$ and $U'$ in $G$. Indeed, for all $x \in U$, $x' \in U'$ and for all bipartite graph $B_i$ in $X$, $x$ is on one side on $B_i$ and $x'$ on the other so the edge $xx'$ is in $B_i$.

Select one bipartite graph (it can be any of those appearing in $X$) and call it $B'_j$. It covers all the edges between $U$ and $U'$. We do the same for all pair $(X, \overline{X})$ of opposite bags of size $t$ and get a family $\mathcal{B}'$ of complete bipartite graphs. All the edges of the graph $G$ of multiplicity $t$ are covered by this set of bipartite graphs. Indeed, consider an edge $xy$ of multiplicity $t$, the two half edges $(xy, x)$ and $(xy, y)$ have two opposite bags of size $t$. Then by construction, they are covered by one of the $B'_j$. Let us prove that each edge is covered at most once. Indeed, if $xy$ is covered by $B'_j$, then the two half-edges are appearing in two opposite bags $(X, \overline{X})$ of size $t$, thus there are already $t$ bipartite graphs $B_i$ which cover $xy$. If it is also covered by another bipartite graph $B'_{j'}$ selected from a different pair of opposite bags $(Y, \overline{Y})$, it means that, there is a bipartite graph $B_{i'}$ which is not in the bag $X$ but in the bag $Y$ (since all the bags have size $t$) which covers the edge $xy$. Thus, the edge $xy$ is covered by at least $t + 1$ bipartite graphs which contradicts the hypothesis on $t$. Thus this set of bipartite graphs cover the edges of multiplicity $t$ with a multiplicity one.

As we assumed that the generalized Alon-Saks-Seymour conjecture of order 1 holds, it means that the graph restricted to this set of bipartite graphs $B'_j$ has chromatic number at most $P((2k)^t)$

(with $P$ a polynomial) since $\mathcal{B}'$ ensures $\mathbf{bp}_1(G) \leq (2k)^t$. Thus the vertex set can be partitioned into $P((2k)^t)$ stable sets $S_1, \ldots, S_{P((2k)^t)}$. Since all the edges of multiplicity $t$ in $\mathcal{B}$ are covered, it means that the multiplicity of the edges of $G$ in each part $S_i$ is at most $t-1$. Hence by induction hypothesis, it means that the chromatic number of each $S_i$ is bounded by a polynomial $Q$ in $\mathbf{bp}_{t-1}(S_i)$. As $S_i$ contains no edge of multiplicity $t$, $\mathcal{B}$ restricted to the vertices of $S_i$ ensures that $\mathbf{bp}_{t-1}(S_i) \leq (2k)^t$. Thus the chromatic number of $G$ seen as a product graph is at most $(P \cdot Q)(2k^t)$, which is a polynomial in $k$. Thus the generalized Alon-Saks-Seymour conjecture of order $t$ holds if and only if it holds for order one. □

## 6 Conclusion

**Corollary 27.** *The following are equivalent:*

- ***Oriented Alon-Saks-Seymour Conjecture.***
  *There exists a polynomial $P$ such that for every graph $G$, $\chi(G) \leq P(\boldsymbol{bp}_{\mathrm{or}}(G))$.*
- ***Generalized Alon-Saks-Seymour conjecture of order $t$, $t \in \mathbb{N}^*$.***
  *There exists a polynomial $P$ such that for every graph $G$, $\chi(G) \leq P(\boldsymbol{bp}_t(G))$*
- ***Clique-Stable set Separation Conjecture.***
  *For every graph $G$, there is a polynomial CS-separator.*
- ***Polynomial 2-list covering for the stubborn problem.***
  *For every graph $G$ and every list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a polynomial 2-list covering for the stubborn problem on $(G, \mathcal{L})$.*
- ***Polynomial 2-list covering for 3-CCP.***
  *For every $n$ and every edge-coloring $f : E(K_n) \to \{A, B, C\}$, there is a polynomial 2-list covering for 3-CCP on $(K_n, f)$.*

*Proof.* Combining Observation 25 and Theorems 13, 23, 26. □

These results are interesting due to the link they make between some distant areas of theoretical computer science such as communication complexity, graph theory, constraint satisfaction problem, and even polytope geometrics, via an equivalence between long-standing open problems in each area. It has been somehow fascinating to explore such a wide range of domains and to see links appearing between them. The main question is now of course to prove or disprove one of these equivalent problems. Our results are a step forward, enabling anyone to choose his favourite domain between the three involved ones.

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# Appendix

## A Detailed proof on random graphs

We give here detailed computations using Taylor series for a result used in the proof of Th. 3.

**Proposition 28.** *If $\omega$ and $\alpha$ are respectively the clique number and the independent number of $G(n,p)$, then $p^\omega(1-p)^\alpha \geq 1/n^6$.*

*Proof.* In the following, $\log_b$ denotes the logarithm to base $b$, log denotes the logarithm to base 2, and ln denotes the logarithm to base $e$. Without loss of generality, we assume $p = 1-2^{-2\log n/a(n)}$, where $a(n)$ is a function of $n$. Let $p' = 1-p$, $b = 1/p$ and $b' = 1/p'$. The independence number and clique number of $G(n,p)$ are given by the following formulas, depending on $p$ (see [2]):

$$\omega = 2\log_b(n) - 2\log_b(\log_b n) + 2\log_b(e/2) + 1 + o(1)$$
$$\alpha = 2\log_{b'}(n) - 2\log_{b'}(\log_{b'} n) + 2\log_{b'}(e/2) + 1 + o(1)$$

We need to distinguish two cases.

**Case 1** $a(n) = o(\log n)$ and $a(n) \geq 2$.

In the following, $a(n)$ will be denoted by $a$.

Using the previous formula and $\dfrac{1}{\log b'} = \dfrac{a}{2\log n}$, we get:

$$\begin{aligned}\alpha &= 2\log_{b'}(n) - 2\log_{b'}\log_{b'} n + 2\log_{b'}(e/2) + 1 + o(1)\\ &= a - \frac{a}{\log n}\log\left(\frac{a}{2}\right) + 1 + o(1)\\ &= a - \frac{a\log a}{\log n} + 1 + o(1)\end{aligned}$$

Moreover, thanks to Taylor series we get:

$$\begin{aligned}\frac{1}{\log b} &= \frac{-1}{\log(1-2^{-2\log n/a})} && \text{by definition of } b\\ &= \frac{-\ln 2}{-2^{-2\log n/a} + \mathcal{O}(2^{-4\log n/a})} && \text{using } \ln(1+x) = x + \mathcal{O}(x^2)\\ &= \frac{\ln 2 \cdot 2^{2\log n/a}}{1+\mathcal{O}(2^{-2\log n/a})} && \text{by factorization}\\ &= \ln 2 \cdot 2^{2\log n/a} \cdot (1+\mathcal{O}(2^{-2\log n/a}))) && \text{using } \frac{1}{1-x} = 1 + \mathcal{O}(x)\end{aligned}$$

Thus, let us look at the different terms in the approximation of $\omega$:

- $$\begin{aligned} 2\log_b n &= 2\ln 2\cdot 2^{2\log n/a}\cdot(1+\mathcal{O}(2^{-2\log n/a}))\cdot\log n \\ &= 2\ln 2\cdot 2^{2\log n/a}\log n+\mathcal{O}(\log n)\end{aligned}$$

- $$\begin{aligned} -2\log_b\log_b n &= -2\ln 2\cdot 2^{2\log n/a}\cdot(1+\mathcal{O}(2^{-2\log n/a}))\cdot(\log\log n-\log\log b) \\ &\text{by substitution of } \log b \\ &= -2\ln 2\cdot 2^{2\log n/a}\cdot(1+\mathcal{O}(2^{-2\log n/a})) \\ &\quad\cdot(\log\log n+\log\ln 2-\log(2^{-2\log n/a}(1+\mathcal{O}(2^{-2\log n/a})))) \\ &\text{by previous computation} \\ &= -2\ln 2\cdot 2^{2\log n/a}\cdot(1+\mathcal{O}(2^{-2\log n/a})) \\ &\quad\cdot(\log\log n+\log\ln 2+\frac{2\log n}{a}+\mathcal{O}(2^{-2\log n/a})) \\ &\text{using } \ln(1+x)=x+\mathcal{O}(x^2) \\ &= -2\ln 2\cdot 2^{2\log n/a}\cdot(\log\log n+\log\ln 2+\frac{2\log n}{a})+\mathcal{O}(\log n) \\ &\text{by developping.}\end{aligned}$$

- $2\log_b(e/2)+1+o(1) \quad = 2\log(e/2)\ln 2\cdot 2^{2\log n/a}+\mathcal{O}(1)$

Hence:

$$\begin{aligned} w &= 2\ln 2\cdot 2^{2\log n/a}\cdot(\log n-\frac{2\log n}{a}-\log\log n-\log\ln 2+\log(e/2))+\mathcal{O}(\log n) \\ &= 2\ln 2\cdot 2^{2\log n/a}\cdot(\log n-\frac{2\log n}{a}-\log\log n)+\mathcal{O}(2^{2\log n/a})+\mathcal{O}(\log n)\end{aligned}$$

On one hand,

$$\begin{aligned} (1-p)^\alpha\geq n^{-(3+\varepsilon)} \quad &\Leftrightarrow \alpha\log(1-p)\geq-(3+\varepsilon)\log n \\ &\Leftrightarrow (a-\frac{a\log a}{\log n}+1+o(1))\cdot\frac{-2\log n}{a}\geq-(3+\varepsilon)\log n \\ &\Leftrightarrow 2\log n+\frac{2\log n}{a}+o(\log n)\leq(3+\varepsilon)\log n \\ &\text{which is true if } n \text{ is large enough.}\end{aligned}$$

On the other hand, using the previous approximations:

$$
\begin{aligned}
p^\omega \geq n^{-(2+\varepsilon)} \quad & \Leftrightarrow \omega \log p \geq -(2+\varepsilon)\log n \\
& \Leftrightarrow \left( 2\ln 2 \cdot 2^{2\log n/a} \cdot (\log n - \frac{2\log n}{a} - \log\log n) + \mathcal{O}(2^{2\log n/a}) + \mathcal{O}(\log n) \right) \\
& \cdot \left( \frac{-2^{-2\log n/a}}{\ln 2} + \mathcal{O}(2^{-4\log n/a}) \right) \geq -(2+\varepsilon)\log n \\
& \Leftrightarrow 2(\log n - \frac{2\log n}{a} - \log\log n) + \mathcal{O}(1) + \mathcal{O}(2^{-2\log n/a}\log n) \leq (2+\varepsilon)\log n
\end{aligned}
$$

which is true if $n$ is large enough.

As a conclusion, for all $\varepsilon$, $p^\alpha(1-p)^\omega \geq 1/n^{5+\varepsilon}$.

**Case 2:** $a(n) = 2d' \log n$ for some constant $d' > 0$. Define $d = -1/\log(1-2^{-1/d})$. Then $\frac{1}{\log b'} = d$ and $\frac{1}{\log b} = d$, which implies:

$$
\begin{aligned}
\alpha &= 2d' \log(n) + o(\log n) \\
\omega &= 2d \log(n) + o(\log n)
\end{aligned}
$$

Thus

$$
\begin{aligned}
(1-p)^\alpha \geq n^{-(2+\varepsilon)} \quad & \Leftrightarrow \alpha \log(1-p) \geq -(2+\varepsilon)\log n \\
& \Leftrightarrow (2d' \log(n) + o(\log n)) \cdot \frac{-1}{d'} \geq -(2+\varepsilon)\log n \\
& \Leftrightarrow 2\log(n) + o(\log n) \leq (2+\varepsilon)\log n
\end{aligned}
$$

which is true if $n$ is large enough.

Similarly

$$
\begin{aligned}
p^\omega \geq n^{-(2+\varepsilon)} \quad & \Leftrightarrow \alpha \log p \geq -(2+\varepsilon)\log n \\
& \Leftrightarrow (2d \log(n) + o(\log n)) \cdot \frac{-1}{d} \geq -(2+\varepsilon)\log n \\
& \Leftrightarrow 2\log(n) + o(\log n) \leq (2+\varepsilon)\log n
\end{aligned}
$$

which is true if $n$ is large enough.

As a conclusion, for all $\varepsilon$, $p^\omega(1-p)^\alpha \geq 1/n^{4+\varepsilon}$.

□

**Observation 29.** *In the previous proof, if $a(n) < 2$, then the independent number $\alpha$ is upper bounded by 3. Thus, the family of every cut $(U, V \setminus U)$ with $|U| \leq 3$ has size $\mathcal{O}(n^3)$ and is a complete $(\omega, \alpha, n)$-separator for $G(n,p)$.*