Coloring graphs with no even hole of length at least 6: the triangle-free case

Aurélie Lagoutte

LIP, ENS Lyon

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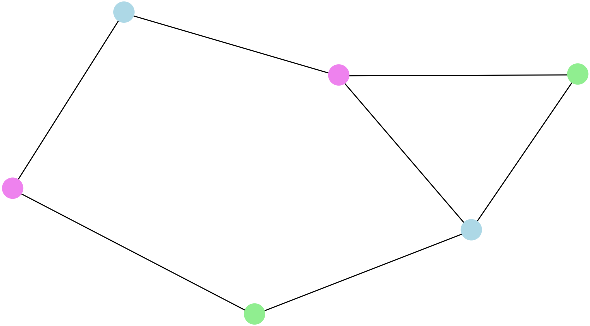
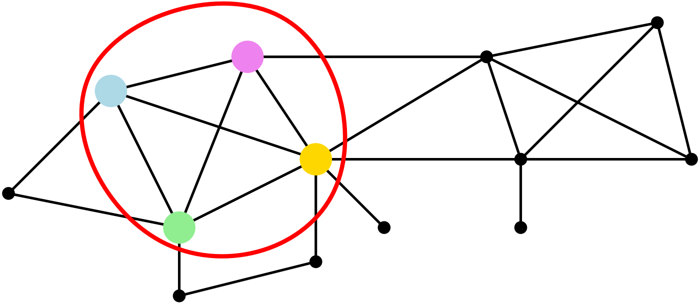
GOAL Seminar - Université Lyon 1

Proper coloring: two adjacent vertices get different colors.



- $\omega(G)$: size of the largest clique
- $\chi(G)$: min. number of colors in a proper coloring



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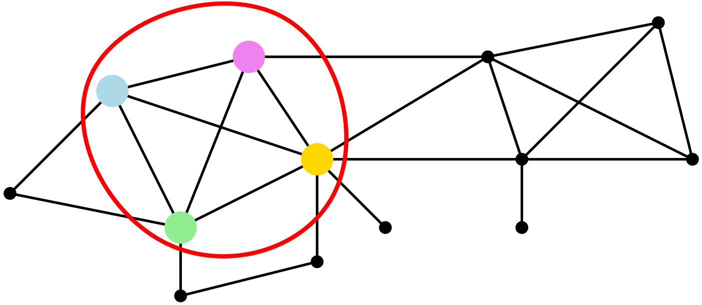

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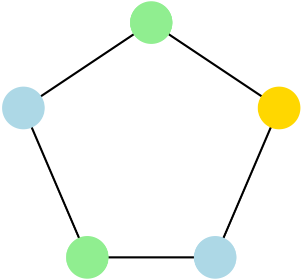

$\chi(C_5) > \omega(C_5)$!

χ-boundedness

Let $\mathcal{C}$ be a hereditary class of graphs.

Definition (Gyárfás 1987)

The class $\mathcal{C}$ is χ-*bounded* if there exists f such that for every $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$.

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- Perfect graphs are χ-bounded with $f(x) = x$.
- Triangle-free graphs is not a χ-bounded class.

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For every k, ℓ, there exist graphs with girth $\geq k$ and chromatic number $\geq \ell$.

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Proved when:

- H is a path (Gyárfás 1987)
- H is a star
- H has radius two (or three, with extra conditions)
- H is any tree but 'H-free' means *no subdivision of H* instead of *no induced subgraphs isom. to H* (Scott 1997).

Hole Parity & Length

Conjectures (Gyárfás 1987)

- The class of graphs with no **odd** hole is χ-bounded.
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  (Scott, Seymour 2014 & Chudnovsky, Scott, Seymour, 2015)
- Triangle-free case of the third conjecture has just been proved (Scott, Seymour 2015): For every k, there exists ℓ such that every triangle-free graph G with $\chi(G) \geq \ell$ has a sequence of holes of k consecutive lengths.

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x is bisimplicial if $N(x)$ is the union of two cliques.

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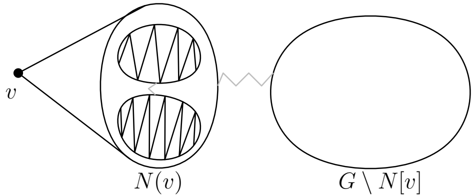

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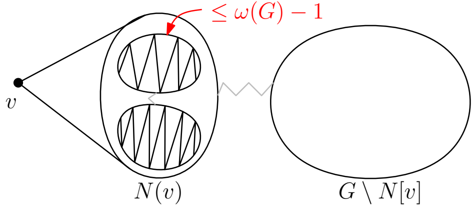

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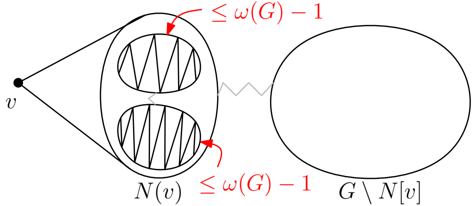

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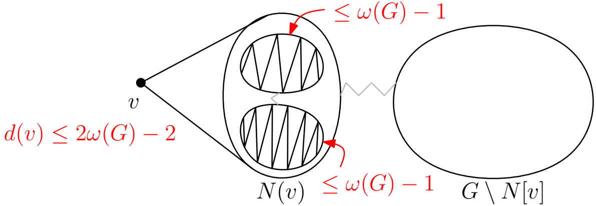

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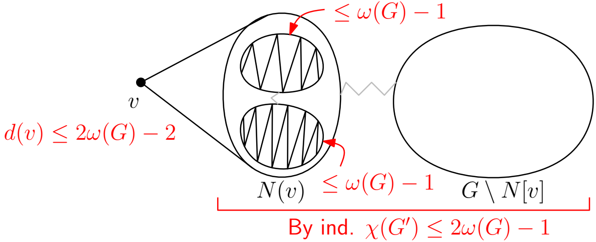

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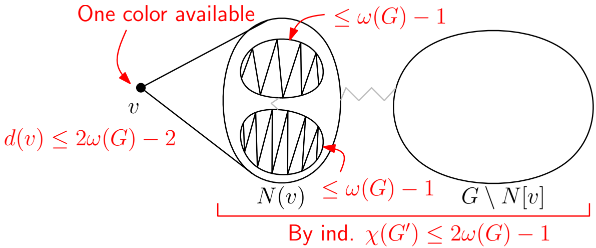

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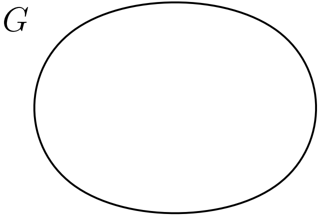
G contains an ind. subdiv of K_t.

G has no triangle, no ind. C_4, no hole of length divisible by k.

Theorem

Every graph with large average degree and no C_4 subgraph contains an ind. (≥ 1)-subdivision of K_t.



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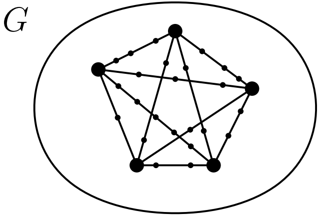

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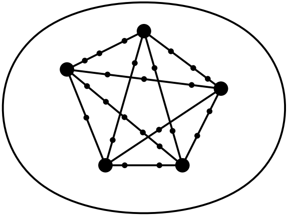

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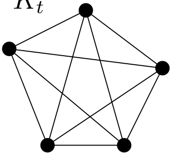

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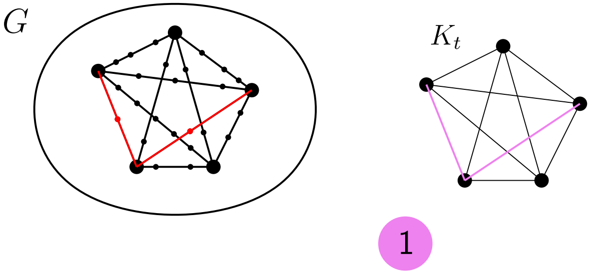

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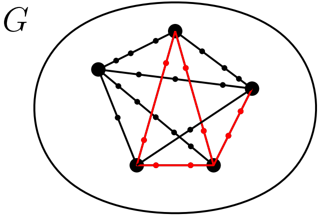

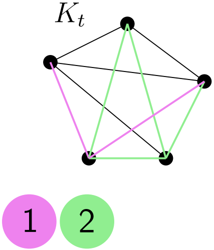

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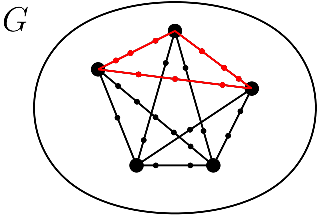

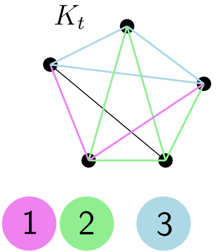

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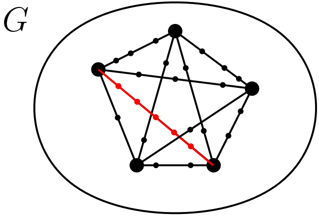

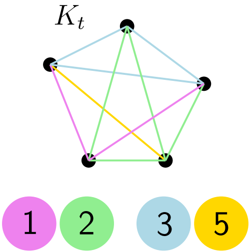

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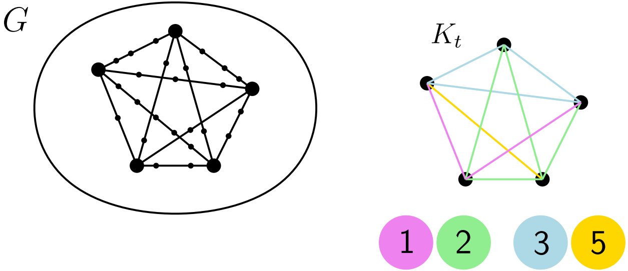

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$\Rightarrow$ Ramsey: $\exists$ monochr. clique of size k

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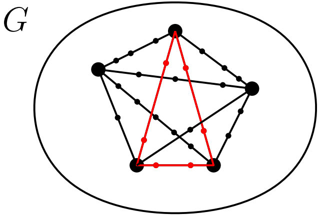

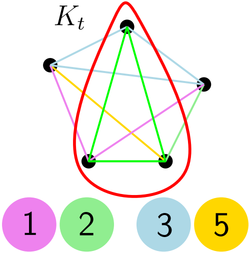

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The Result

Let $\mathcal{C}_{3,2k\geq 6}$ be the class of graphs with no triangle and no hole of even length at least 6.

Theorem (L. 2015^{+})

There exists $c > 0$ such that for every graph $G \in \mathcal{C}_{3,2k\geq 6}$, $\chi(G) \leq c$.

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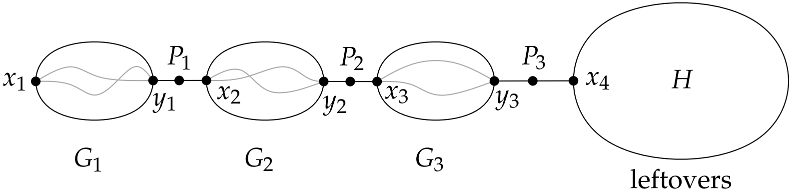
Let $\mathcal{C}_{3,5,2k\geq 6}$ be the class of graphs with no triangle, no C_5 and no hole of even length at least 6.

Lemma

There exists $c' > 0$ such that for every graph $G \in \mathcal{C}_{3,5,2k\geq 6}$, $\chi(G) \leq c'$.

Parity Changing Path



A Parity Changing Path (PCP) of order ℓ is a sequence $(G_1, P_1, \dots, G_\ell, P_\ell, H)$ such that:

- There is an odd and an even path from x_i to y_i, $\forall i$.
- P_i has length ≥ 2, $\forall i$.
- H is connected and $\chi(H)$ is the *leftovers*.
- $\chi(G_i) \leq 4$

x_1 is the *origin* of the PCP.

Sketch of proof

1. Big $\chi \Rightarrow$ Grow a PCP
2. Big $\chi \Rightarrow$ Grow a rooted PCP in N_k
3. Having a neighbor in $H \Rightarrow$ having neighbors everywhere
4. The active lift ($\sim$ parents of the PCP) has big χ.
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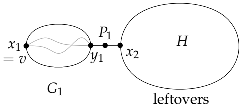
Lemma

Let $G \in \mathcal{C}_{3,5,2k\geq 6}$ be connected, $v \in V(G)$ and $\delta = \chi(G)$. Then $\exists$ a PCP of order 1 with origin v and leftovers $\geq h(\delta) = \delta/2 - 8$.



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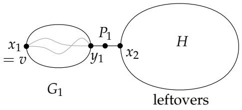

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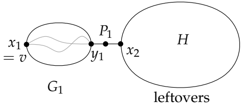

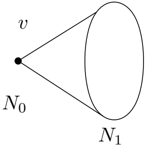

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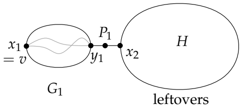

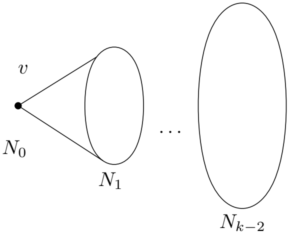

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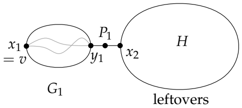

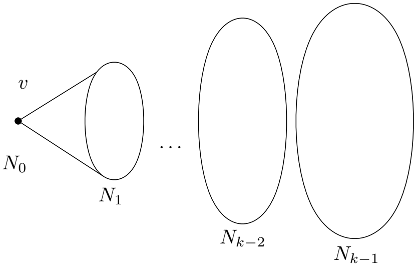

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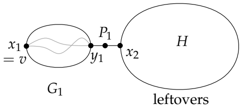

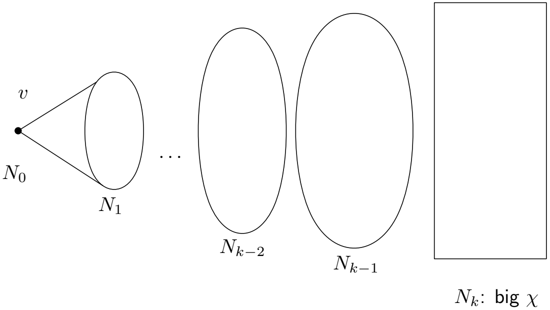

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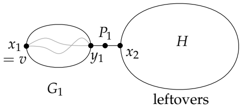

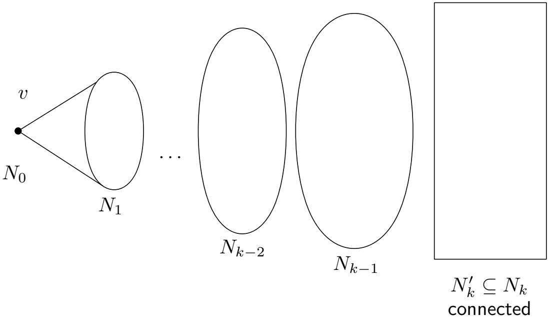

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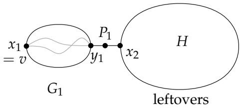

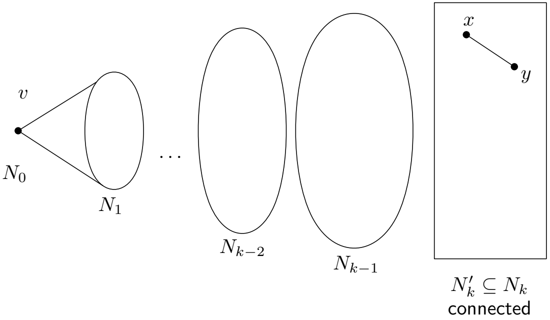

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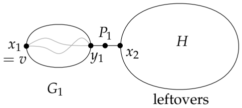

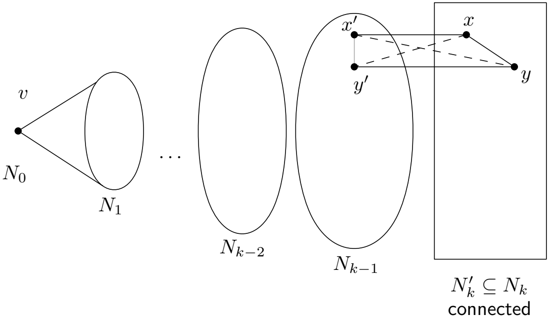

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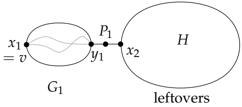

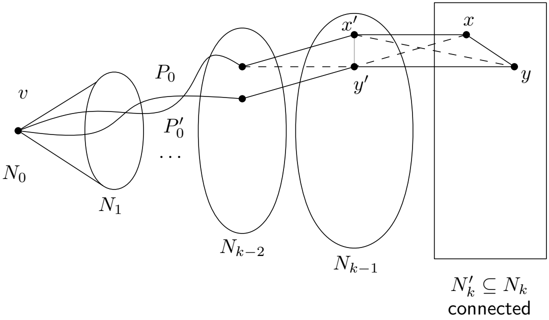

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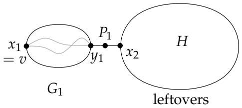

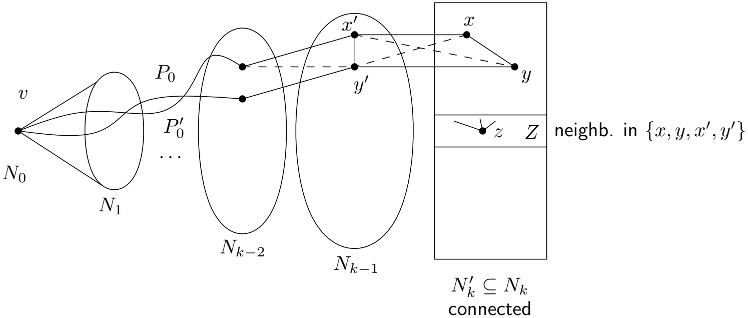

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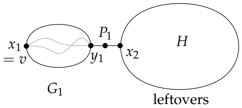

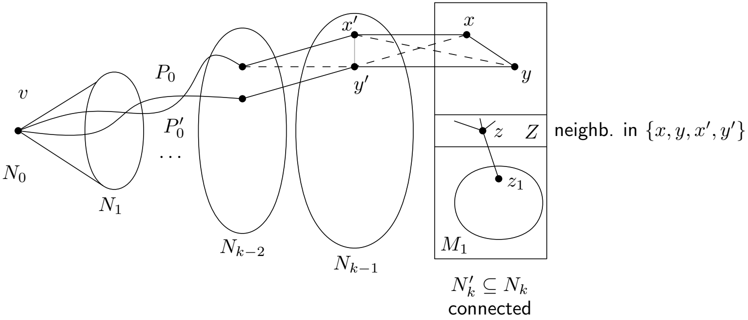

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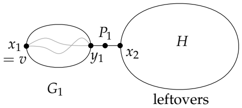
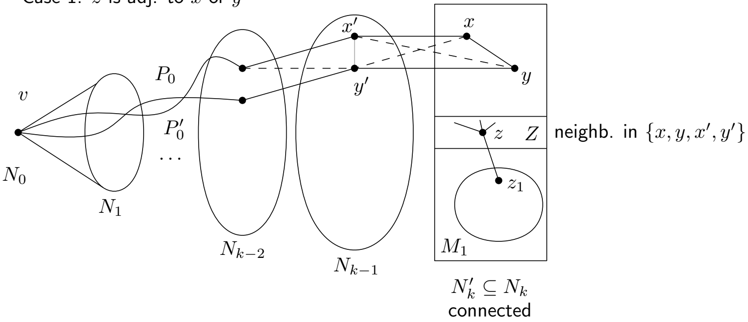


Case 1: z is adj. to x or y



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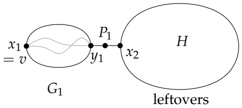

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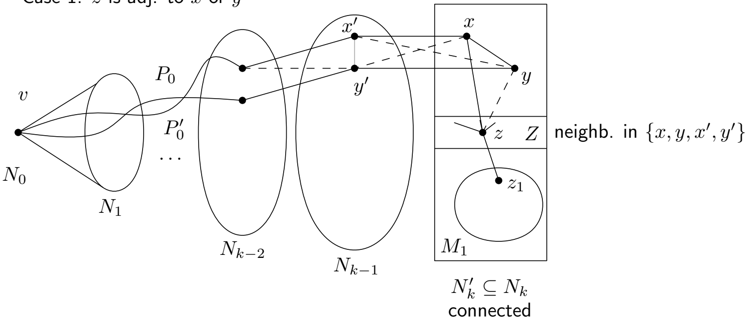

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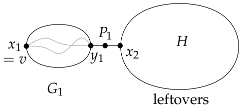

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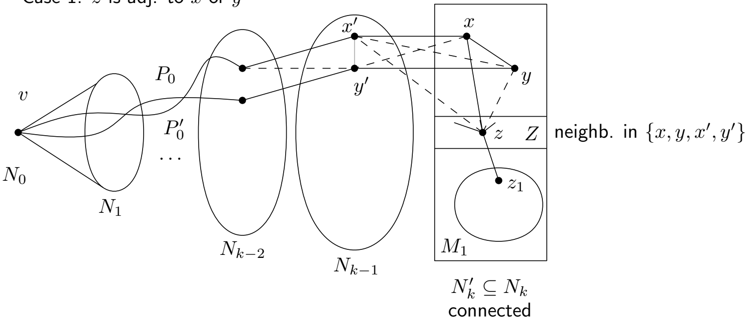

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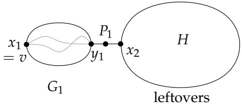

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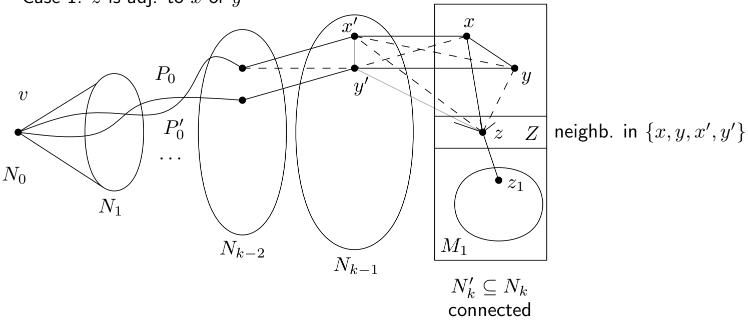

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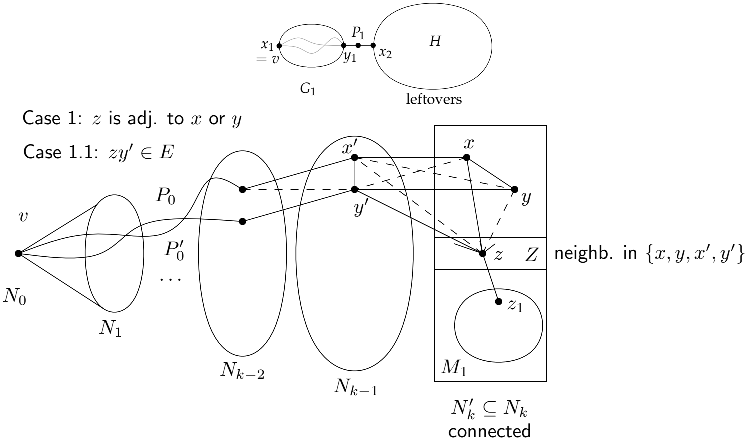

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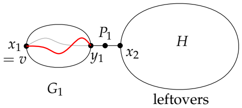

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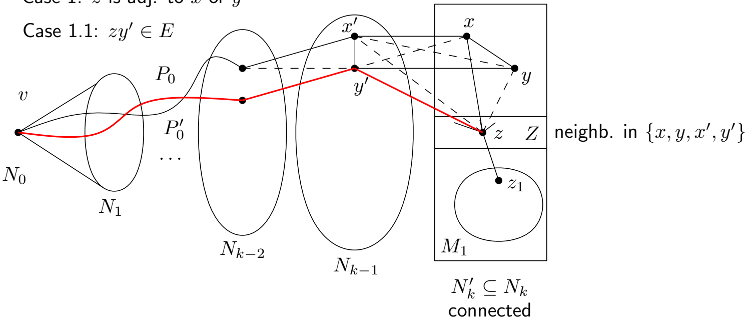
Case 1.1: $zy' \in E$



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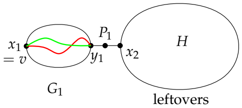

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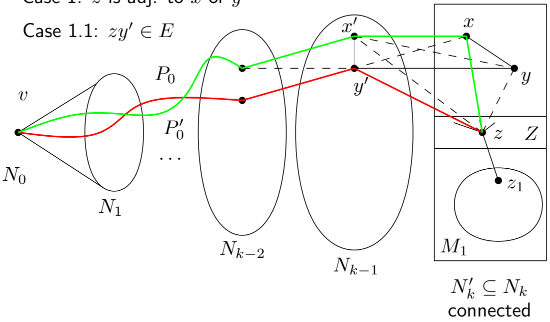

neighb. in $\{x, y, x', y'\}$

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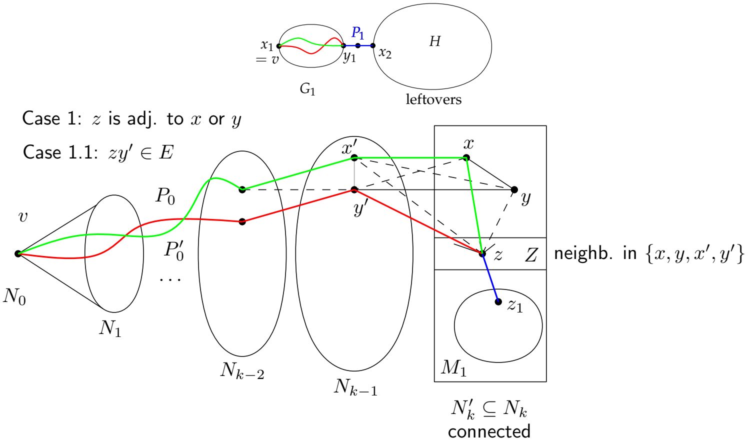

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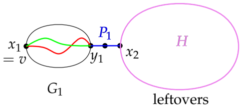

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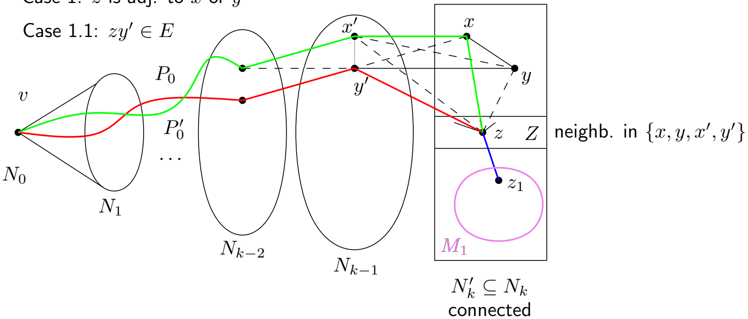

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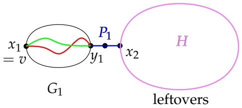

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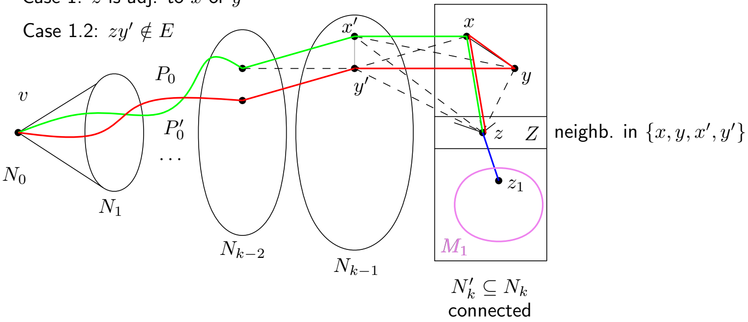

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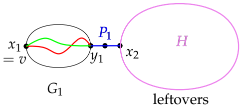

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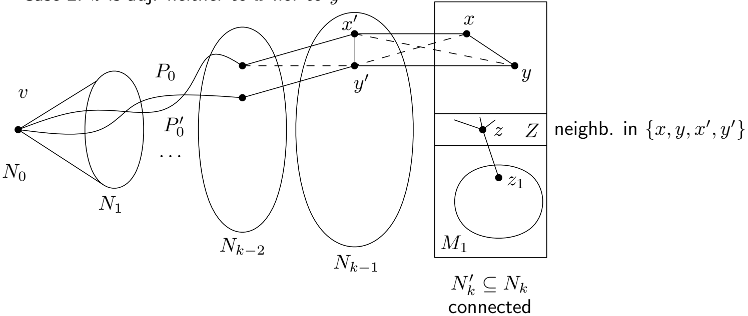

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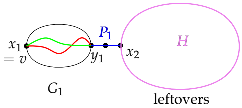

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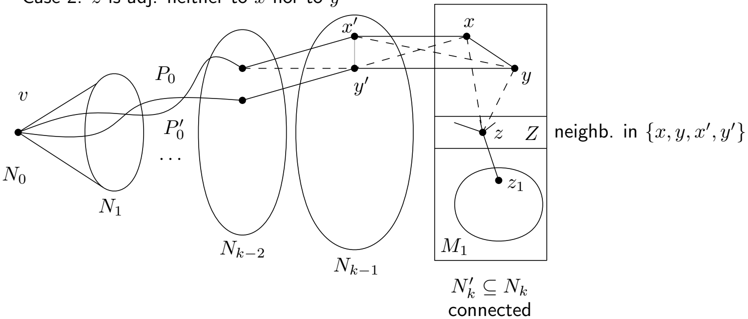

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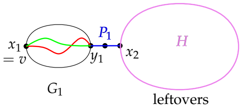

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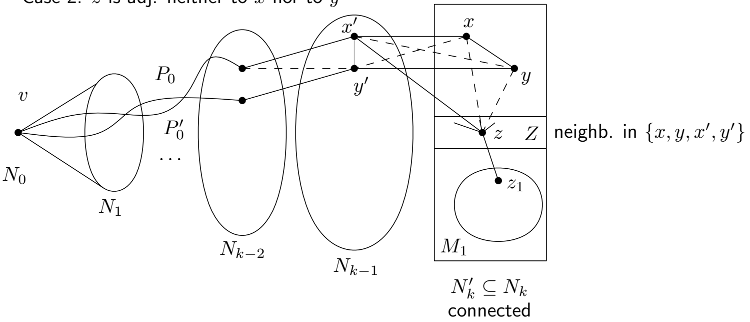

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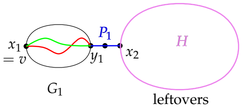

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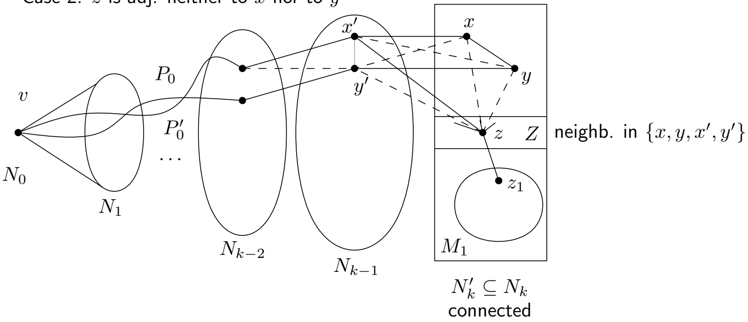

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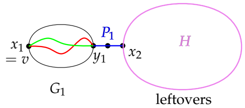

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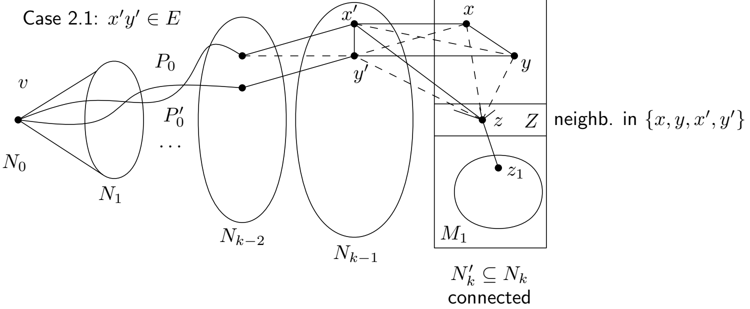
Case 2.1: $x'y' \in E$



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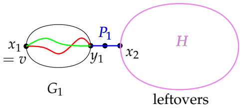

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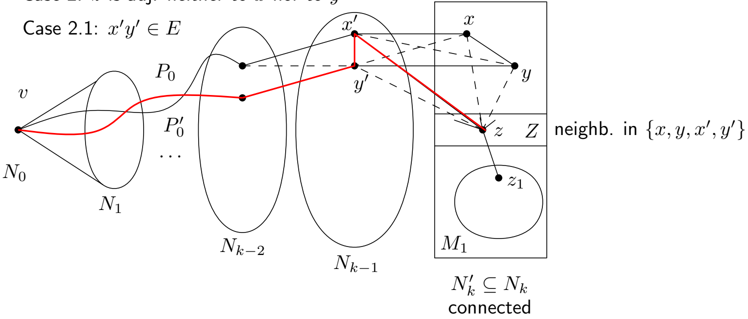

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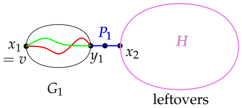

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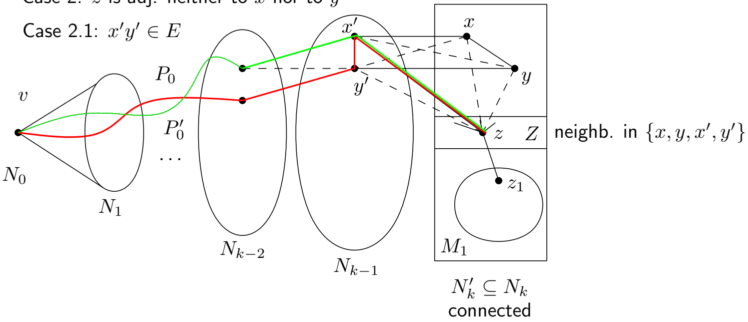

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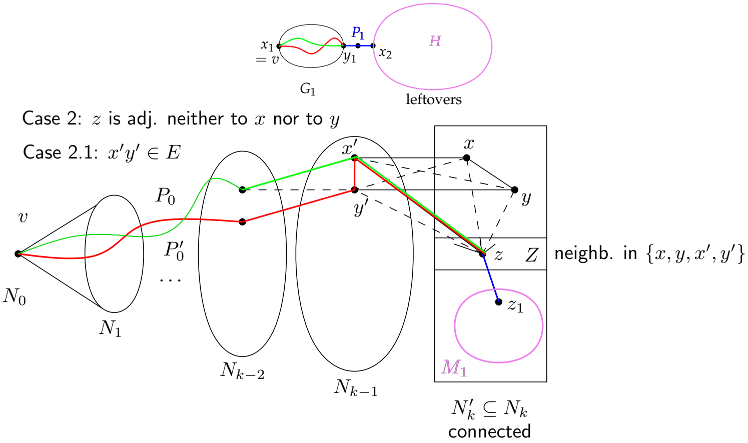

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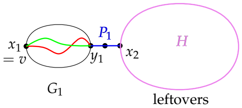

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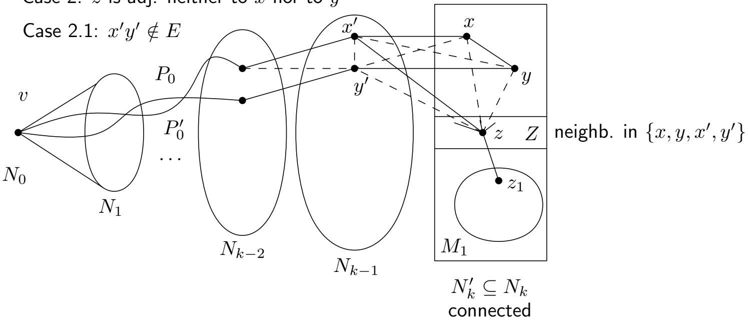

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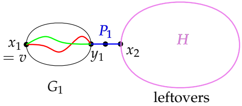

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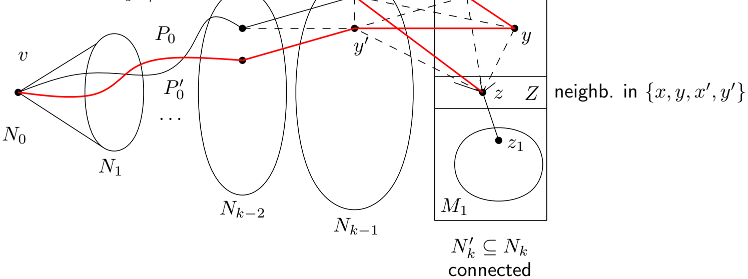

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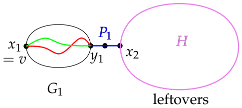

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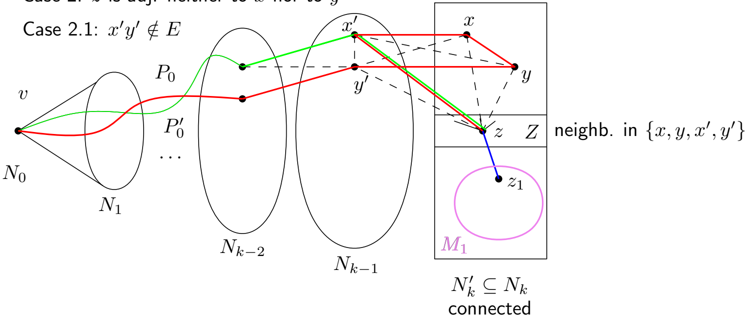
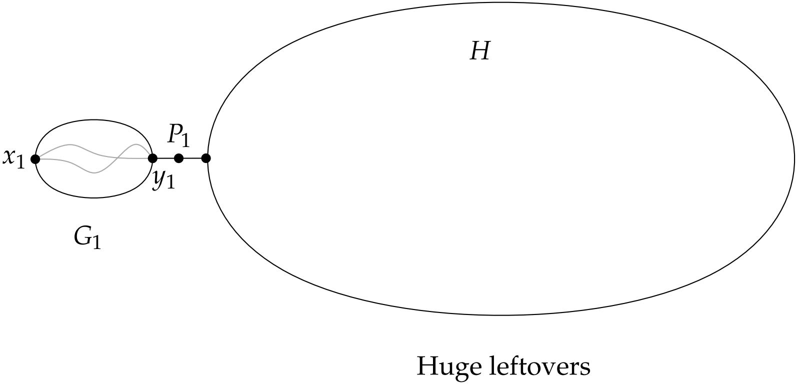


We can iterate the process:



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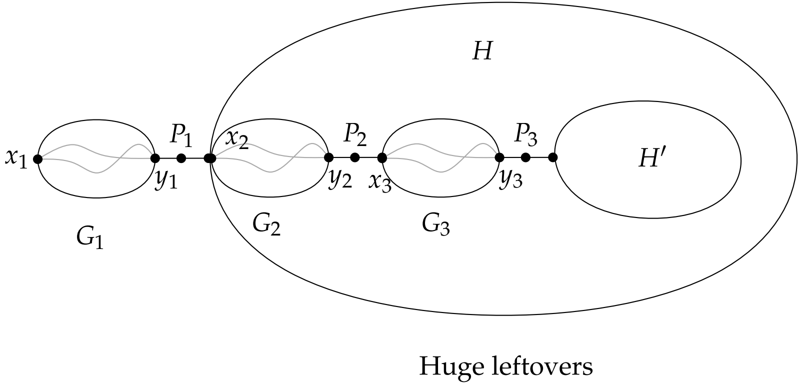

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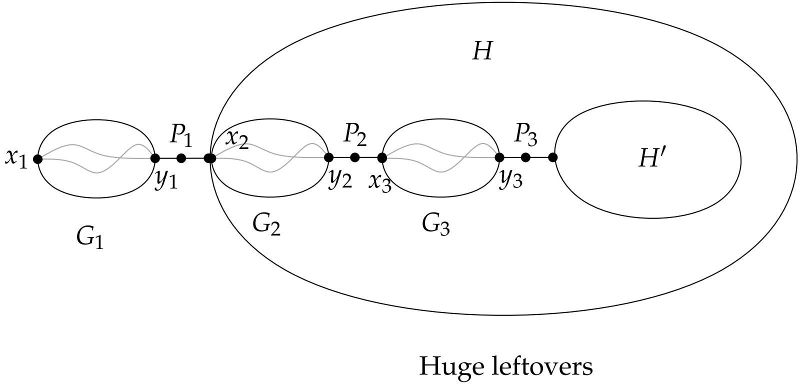
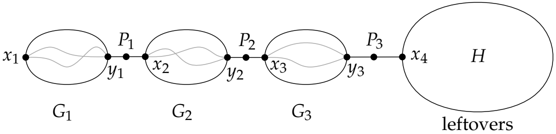


$\Rightarrow$ If χ is large enough, we can grow a PCP of order ℓ with large leftovers from any $v \in V(G)$.

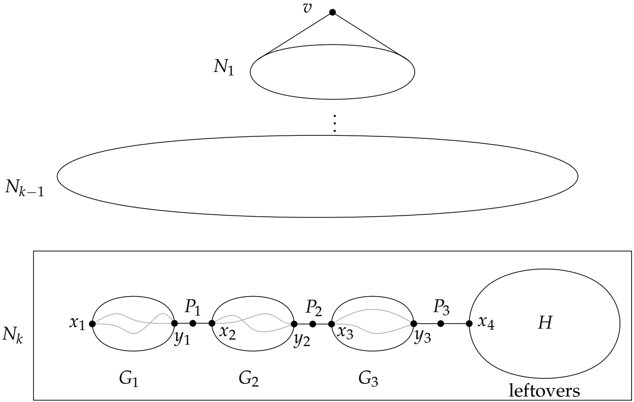
Sketch of proof

1. Big $\chi \Rightarrow$ Grow a PCP
2. Big $\chi \Rightarrow$ Grow a rooted PCP in N_k
3. Having a neighbor in $H \Rightarrow$ having neighbors everywhere
4. The active lift ($\sim$ parents of the PCP) has big χ.
5. Conclusion

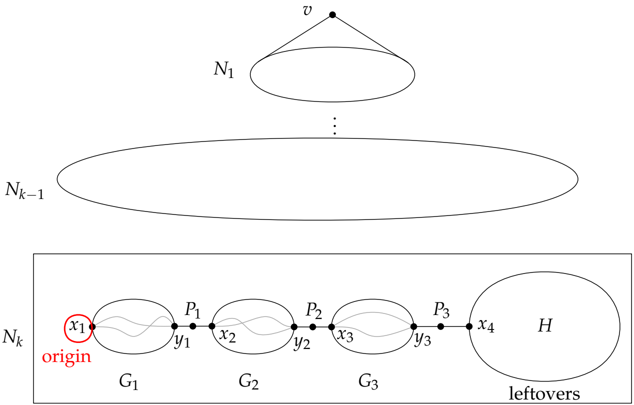
Rooted PCP



Rooted PCP

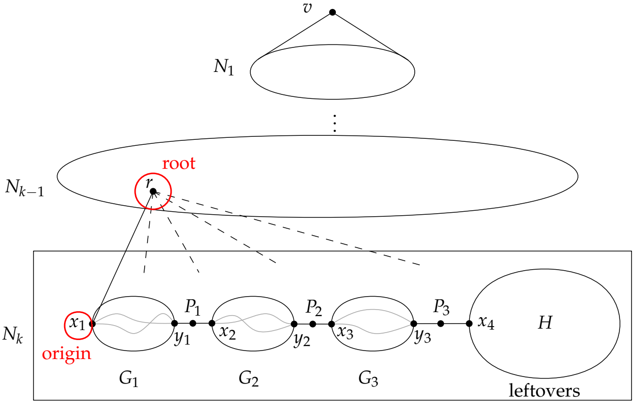


Rooted PCP



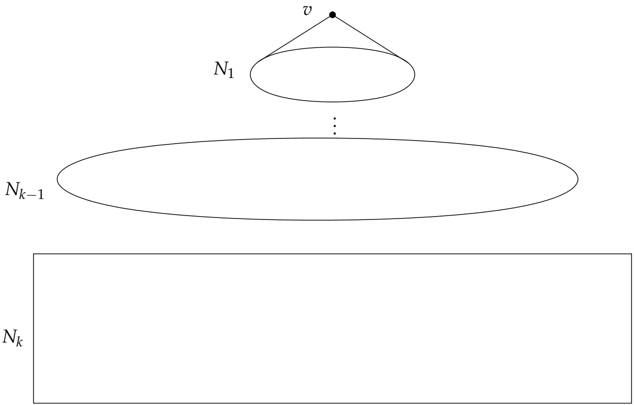
The *root* has exactly one neighbor in the PCP: the origin of the PCP.

Rooted PCP



The *root* has exactly one neighbor in the PCP: the origin of the PCP.

Rooted PCP: how to grow one?



Want: The *root* has exactly one neighbor in the PCP: the origin of the PCP.

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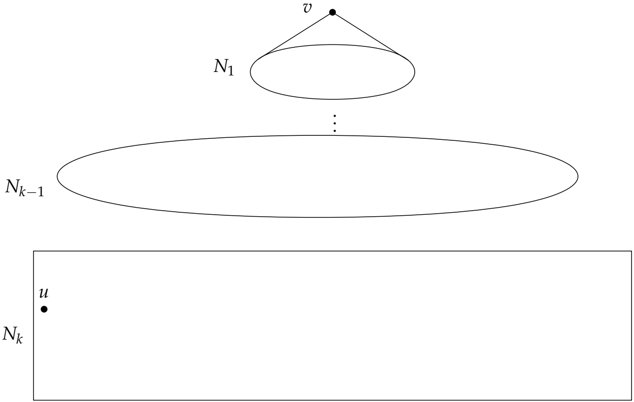

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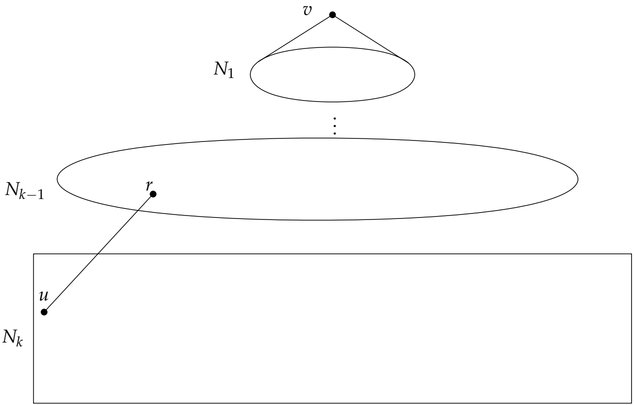

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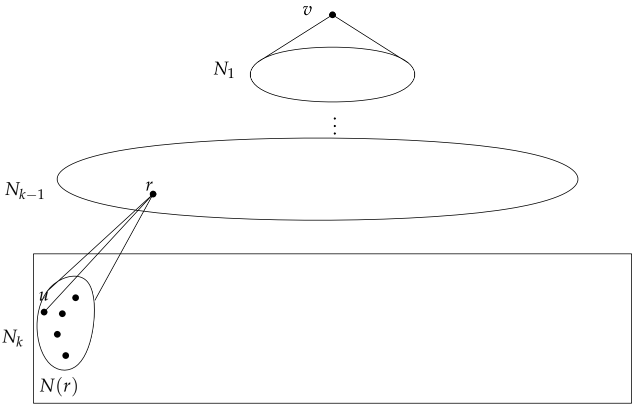

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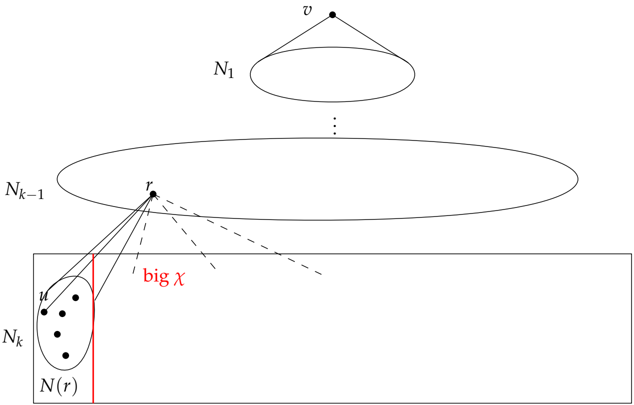

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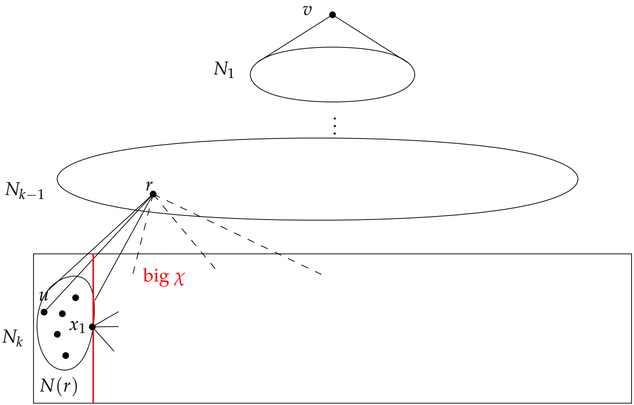

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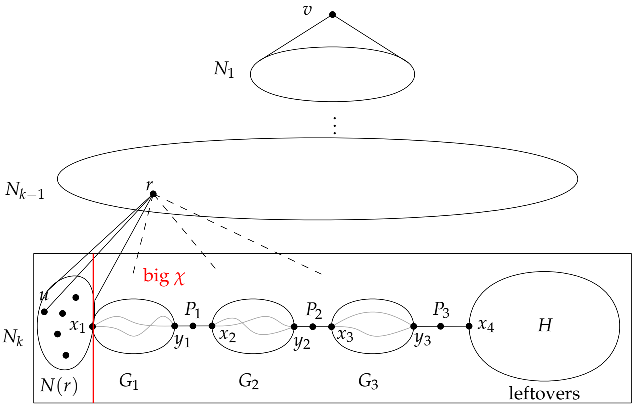

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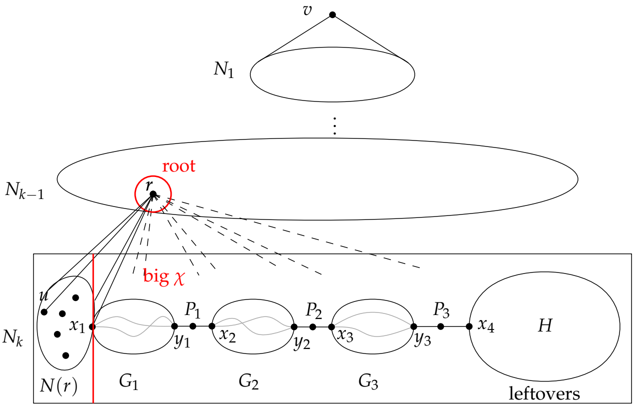
Rooted PCP: how to grow one?



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Sketch of proof

1. Big $\chi \Rightarrow$ Grow a PCP
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Neighbor in $H \Rightarrow$ Neighbors everywhere

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If u has a neighbor in H or G_ℓ, then u has a neighbor in every G_i.

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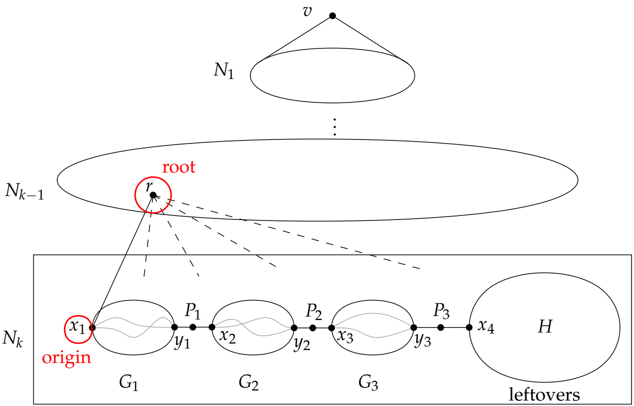

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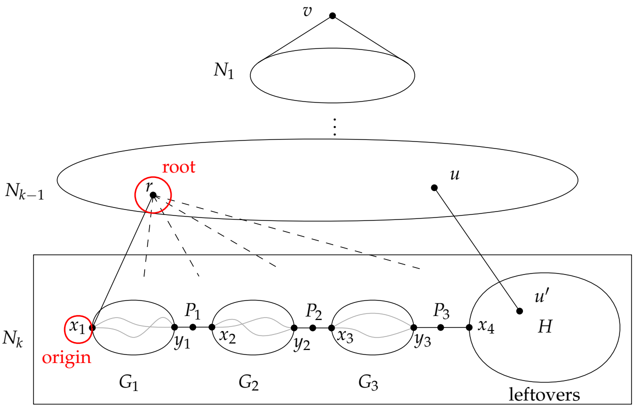

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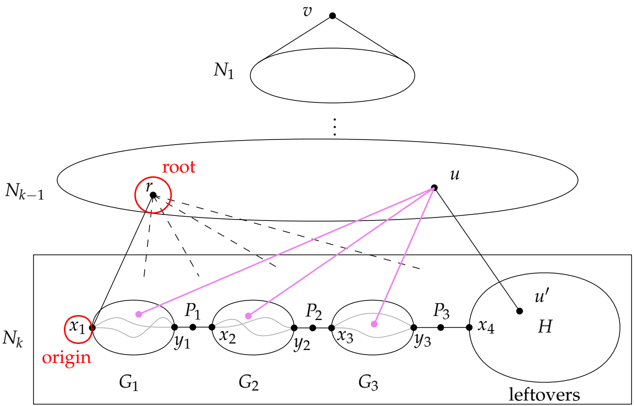

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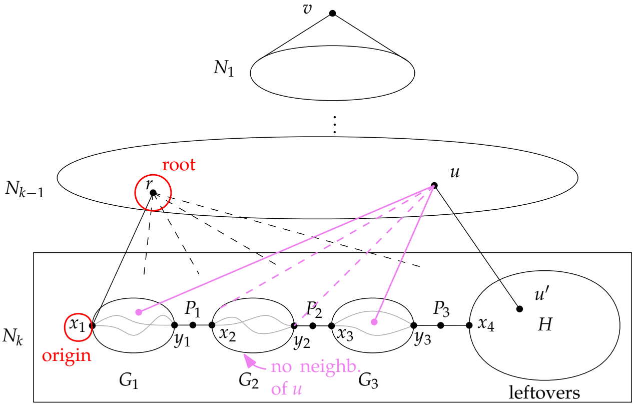

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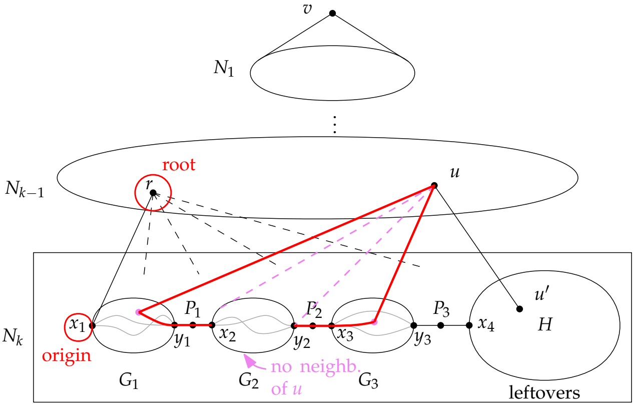

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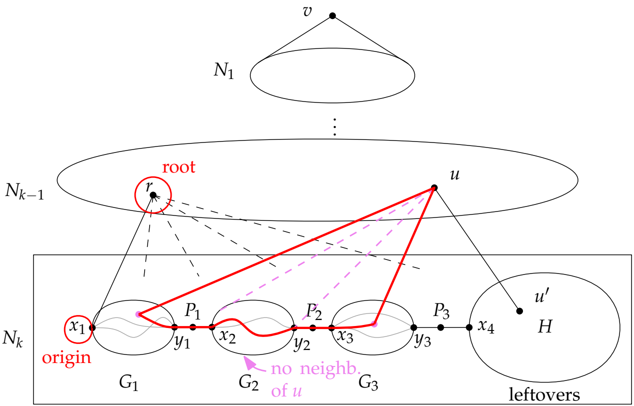

Neighbor in H ⇒ Neighbors everywhere

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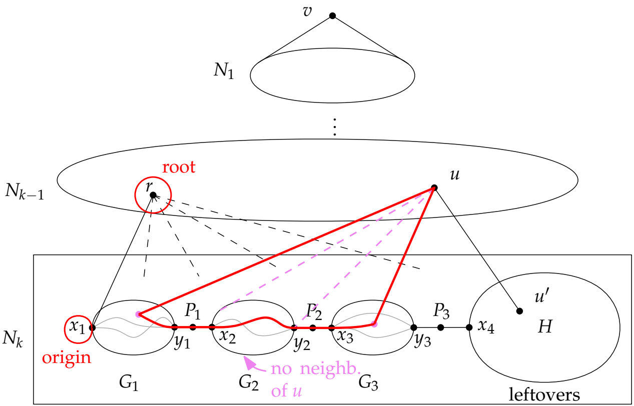

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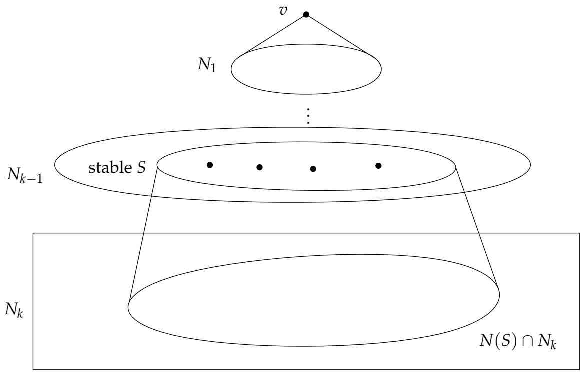
Stable set has children with small χ

Lemma

Let S be a stable set in N_{k-1}. Then $\chi(N(S) \cap N_k) \leq 52$.



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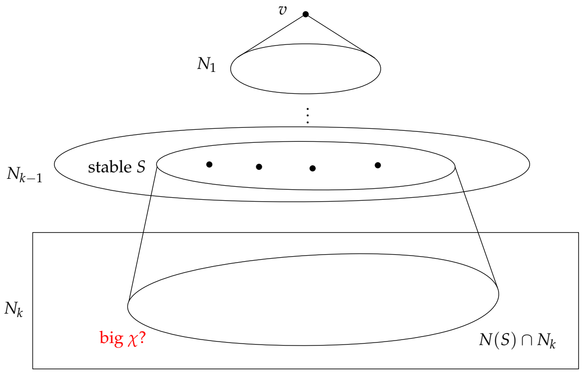

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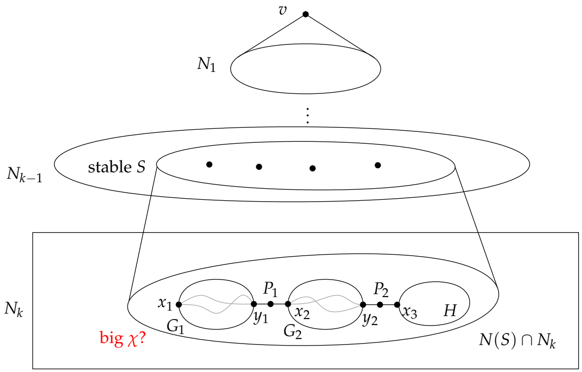

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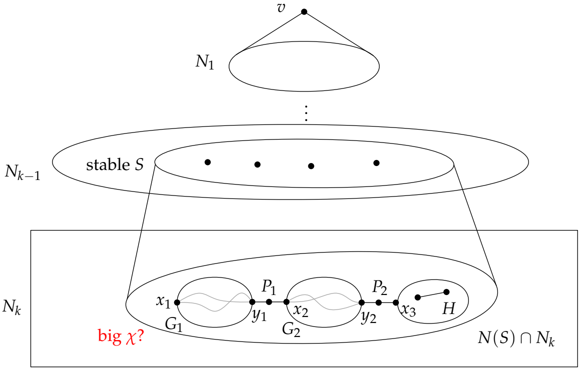

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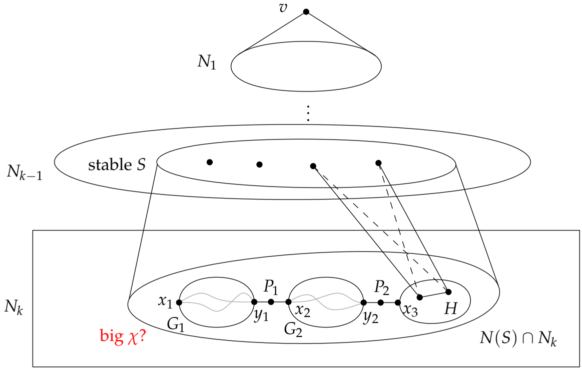

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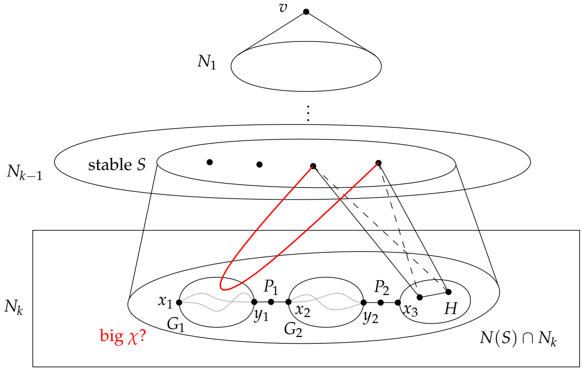

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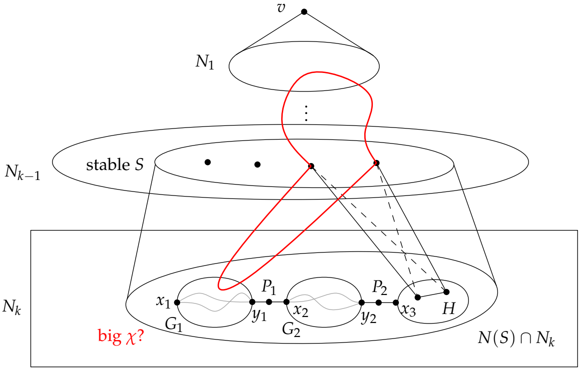
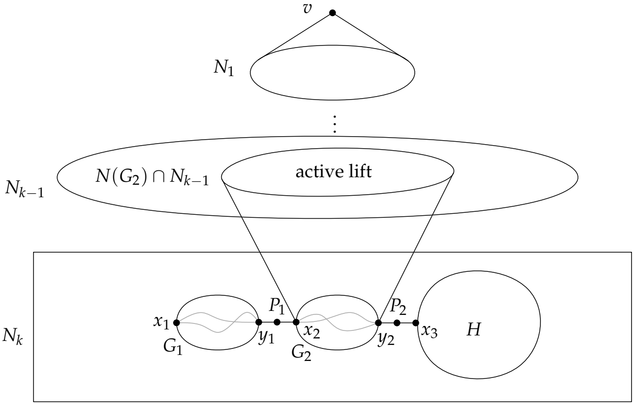


$\Rightarrow$ Creates a C_5 or a hole of even length ≥ 6.

Lemma

The active lift $N(G_2) \cap N_{k-1}$ has big χ.



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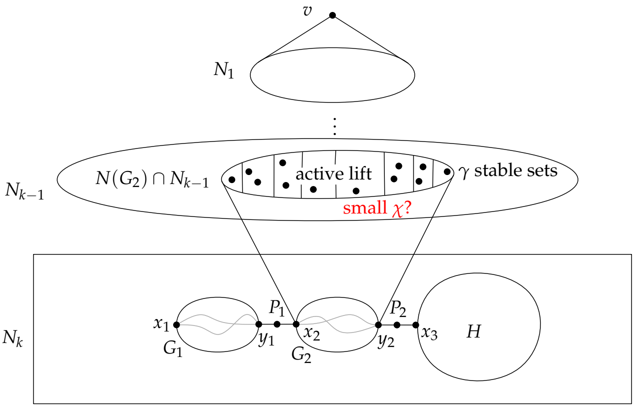

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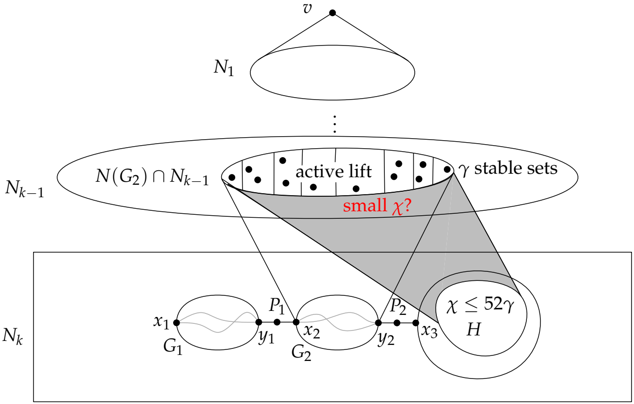

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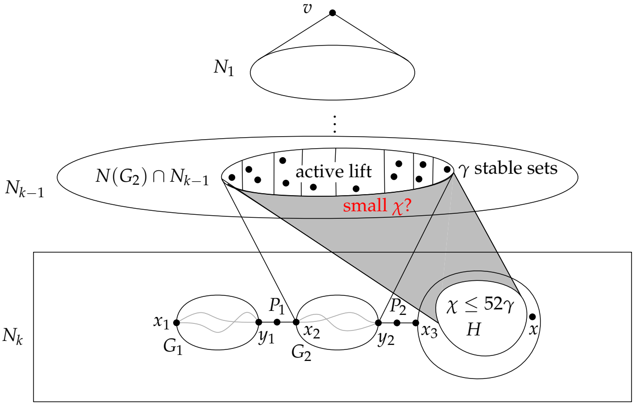

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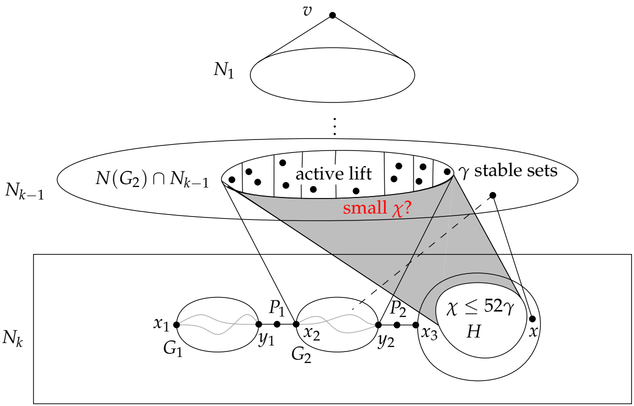

$\Rightarrow$ a contradiction with ③ .

Sketch of proof

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2. Big $\chi \Rightarrow$ Grow a rooted PCP in N_k
3. Having a neighbor in $H \Rightarrow$ having neighbors everywhere
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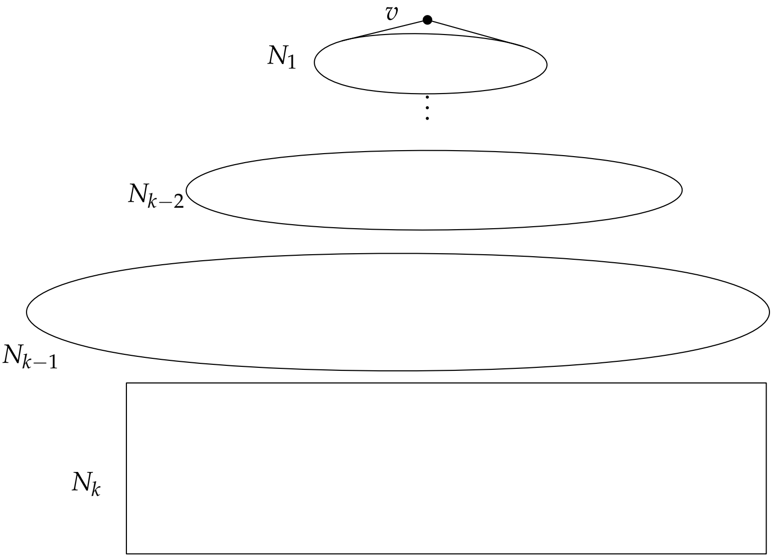
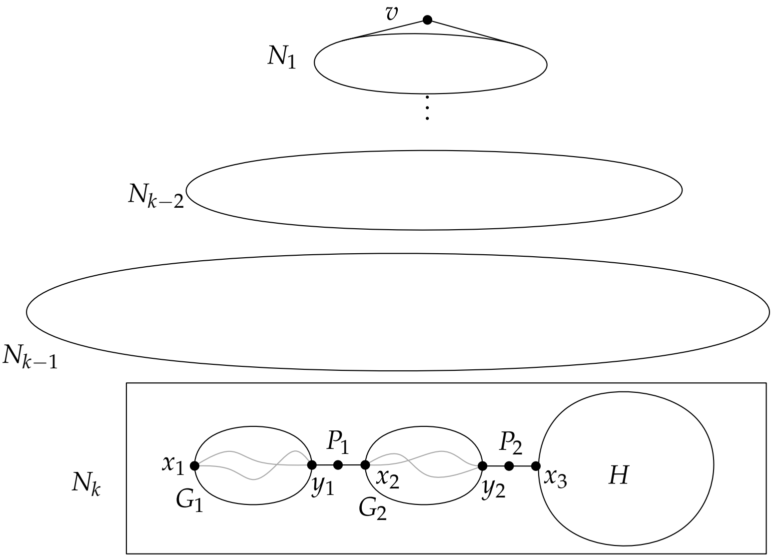
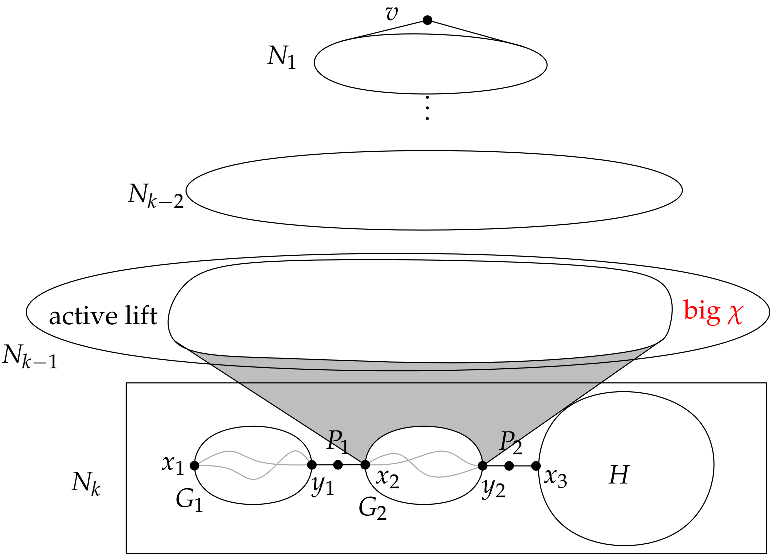
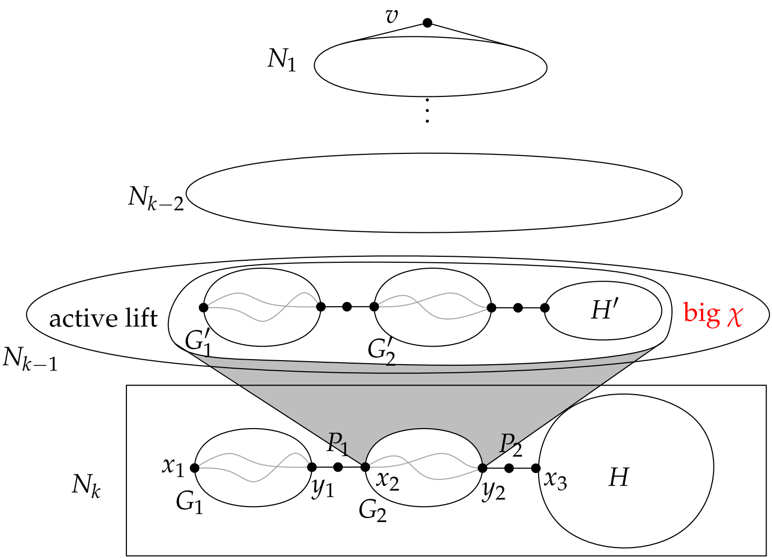
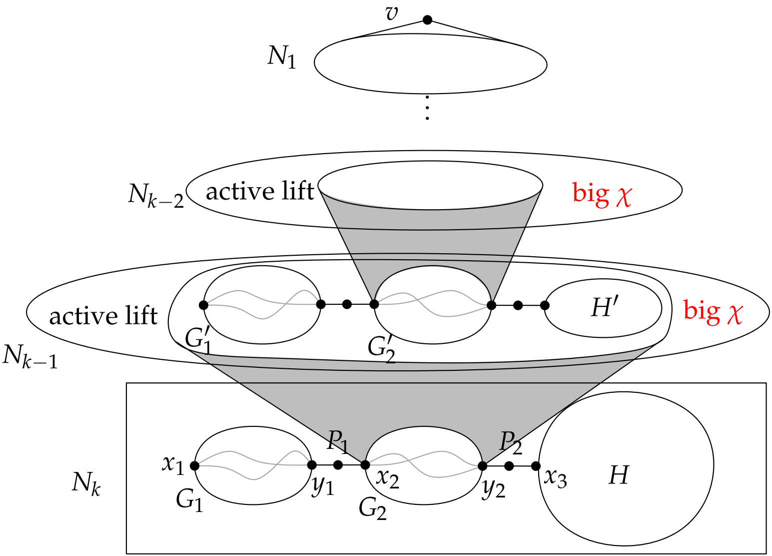
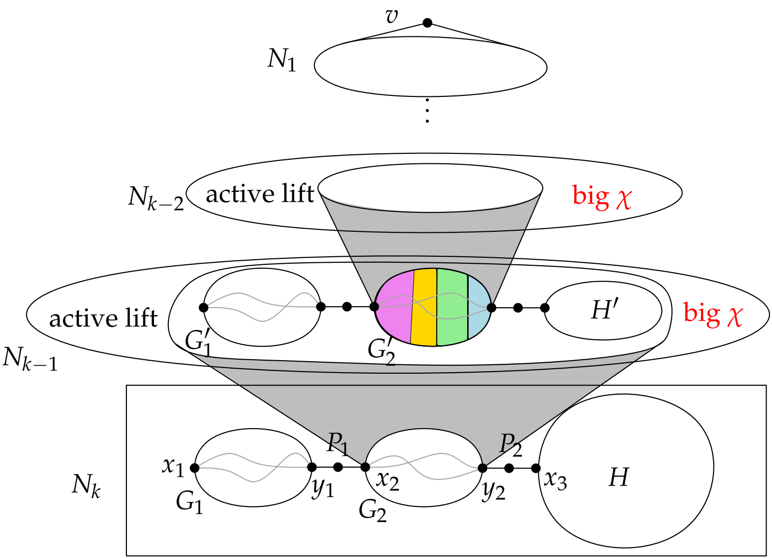

v
N1
Nk−2
Nk−1
Nk


v
N1
Nk−2
Nk−1
Nk
P1
P2
x1
y1
x2
y2
x3
G1
G2
H

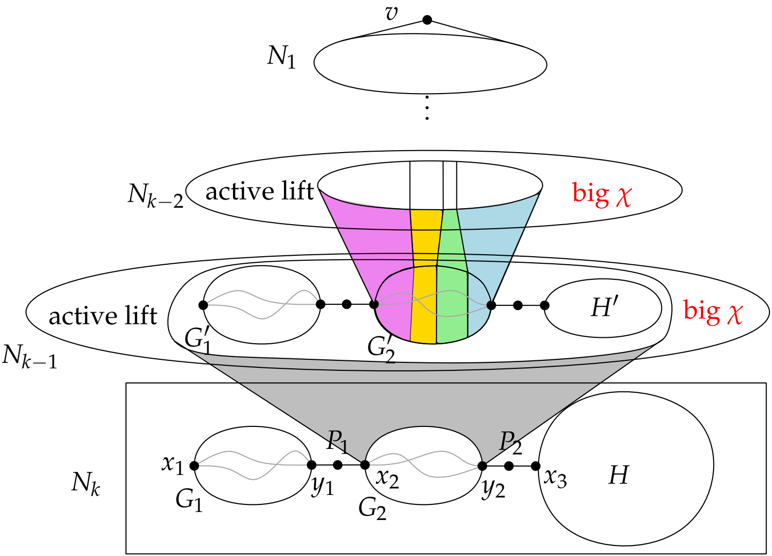

v
N1
Nk−2
active lift
big χ
Nk−1
P1
P2
x1
y1
x2
y2
x3
G1
G2
H
Nk


v
N_1
N_{k-2}
active lift
G_1'
G_2'
H'
big χ
N_{k-1}
N_k
x_1
G_1
y_1
P_1
x_2
G_2
P_2
y_2
x_3
H

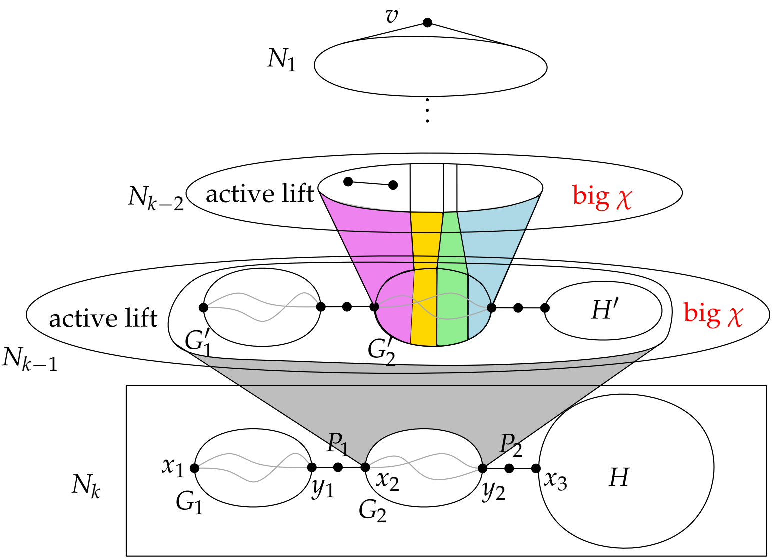

v
N_1
active lift
N_{k-2}
big χ
active lift
G'_1
G'_2
H'
big χ
N_{k-1}
P_1
P_2
x_1
G_1
y_1
x_2
G_2
y_2
x_3
H
N_k



v
N_1
N_{k-2}
active lift
big χ
N_{k-1}
active lift
G'_1
G'_2
H'
big χ
N_k
x_1
G_1
y_1
P_1
x_2
G_2
y_2
P_2
x_3
H



v
N1
Nk−2
active lift
big χ
Nk−1
active lift
G′1
G′2
H′
big χ
Nk
x1
y1
G1
P1
x2
y2
G2
P2
x3
H


v
N_1
N_{k-2}
active lift
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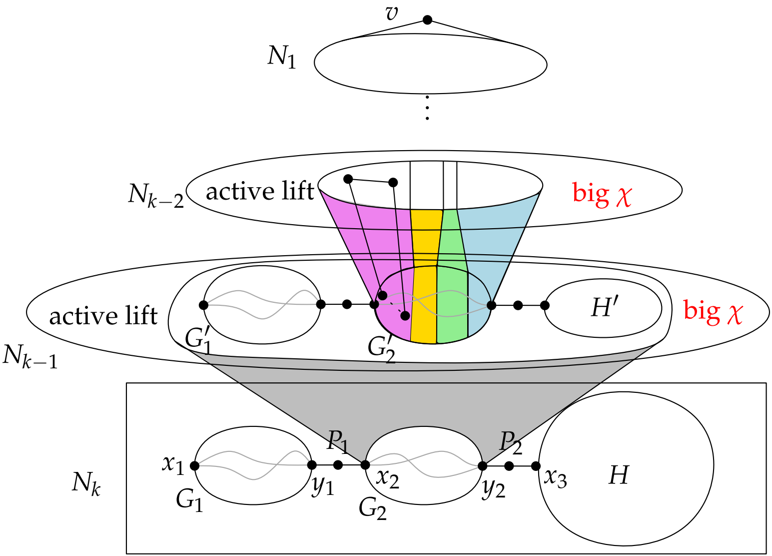
N_{k-1}
active lift
G_1'
G_2'
H'
big χ
N_k
x_1
G_1
y_1
P_1
x_2
G_2
y_2
P_2
x_3
H



v
N1
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G′1
G′2
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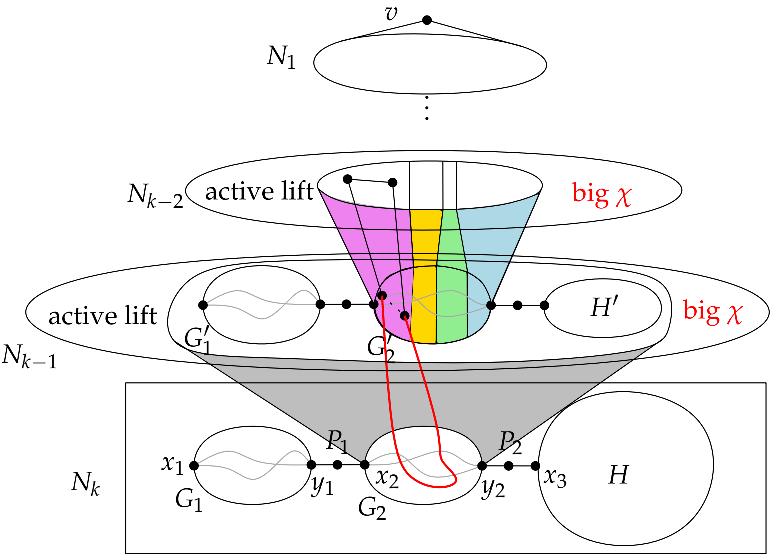
Nk
x1
y1
G1
P1
x2
y2
G2
P2
x3
H



v
N_1
N_{k-2}
active lift
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x_1
G_1
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P_1
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G_2
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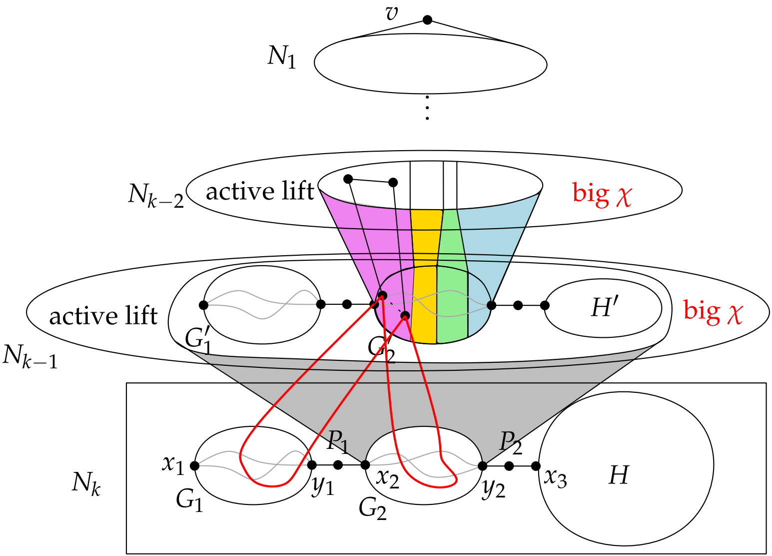
y_2
x_3
H



$\Rightarrow$ Creates a C_5 or a hole of even length ≥ 6.

We just proved:

Lemma

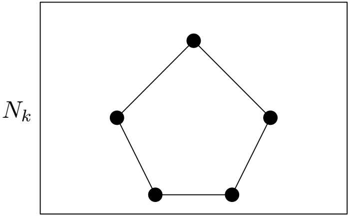
There exists $c' > 0$ such that for every graph $G \in \mathcal{C}_{3,5,2k\geq 6}$, $\chi(G) \leq c'$.

Where $\mathcal{C}_{3,5,2k\geq 6}$ is the class of graphs with no triangle, no C_5 and no hole of even length at least 6.

When C_5 is not forbidden

Key lemma

Let S be a stable set dominating a C_5.



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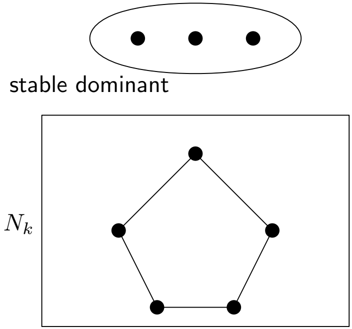

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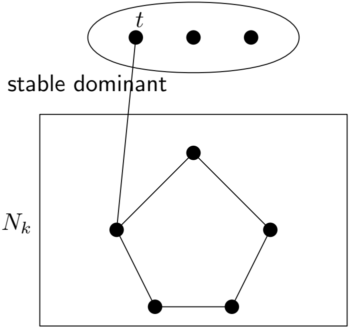

When C_5 is not forbidden

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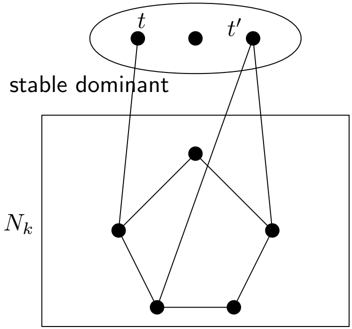

When C_5 is not forbidden

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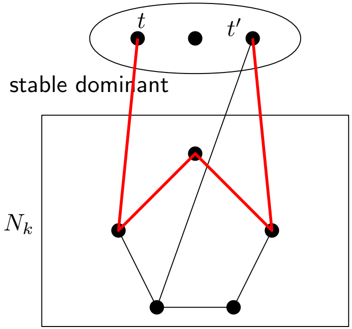
Let S be a stable set dominating a C_5. For every $t \in S$, there exists $t' \in S$ such that there is a tt'-path of length 4



When C_5 is not forbidden

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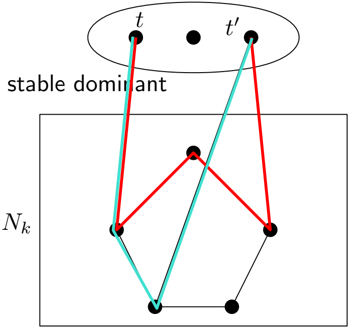
Let S be a stable set dominating a C_5. For every $t \in S$, there exists $t' \in S$ such that there is a tt'-path of length 4 and a tt'-path of length 3 or 5 (odd).



Conclusion

Theorem

The class of triangle-free graphs with no hole of even length ≥ 6 has bounded χ.

- Initial goal:

Conclusion

Theorem

The class of triangle-free graphs with no hole of even length ≥ 6 *has bounded* χ.

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  - Remove triangle-free hypothesis

Conclusion

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- Initial goal:
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  - Or change *no even hole* ≥ 6 into *no even hole* $\geq k$

Conclusion

Theorem

The class of triangle-free graphs with no hole of even length ≥ 6 *has bounded* χ*.*

- Initial goal:
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- If Scott & Seymour result gets confirmed:

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Theorem

For every k, there exists ℓ such that every triangle-free graph G with $\chi(G) \geq \ell$ has a sequence of holes of k consecutive lengths.

Only thing left: remove the triangle-free hypothesis.

Conclusion

Theorem

The class of triangle-free graphs with no hole of even length ≥ 6 *has bounded* χ.

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For every k*, there exists* ℓ *such that every triangle-free graph* G *with* $\chi(G) \geq \ell$ *has a sequence of holes of* k *consecutive lengths.*

Only thing left: remove the triangle-free hypothesis.

Thank you for your attention!