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# INTERACTIONS ENTRE LES CLIQUES ET LES STABLES DANS UN GRAPHE

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Merci à tous ceux qui, pour me faire plaisir, ont déjà pris la peine de me demander de quoi parle ma thèse et d'écouter la réponse (à ce sujet, lisez mon introduction de thèse et vous saurez une bonne fois pour toutes de quoi ça parle ; si vous tombez dans un ennui profond au bout de quelques pages, pas de panique, rendez-vous p. 152 pour un peu de détente).

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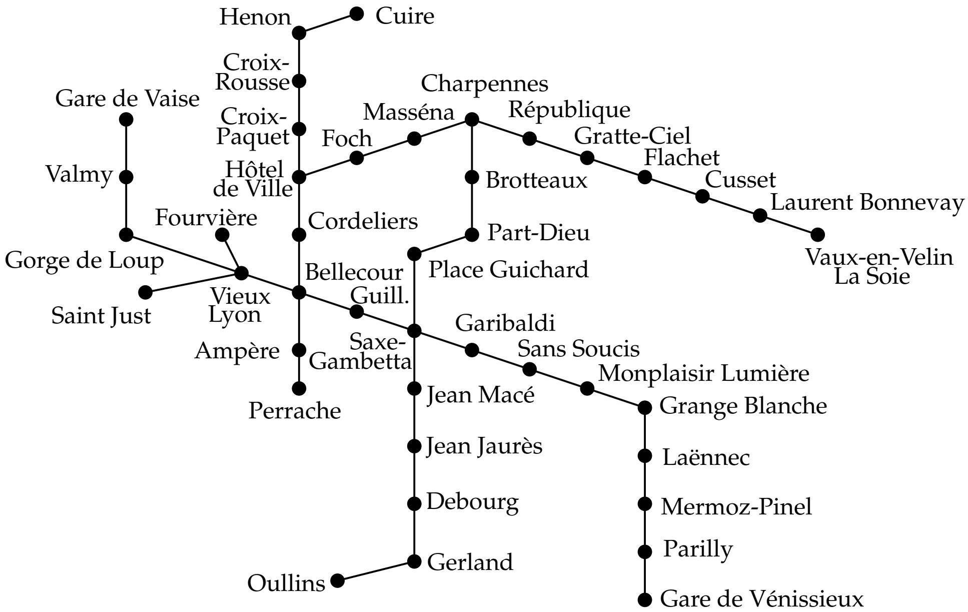
# Introduction ()

Sur *quoi travailles-tu ?* C'est une question qui m'est souvent posée, par ma famille ou mes amis, et l'on s'attend à ce que j'y réponde en quelques mots. Ce n'est pas une tâche facile que de répondre à cette question succinctement, surtout sans tableau ni papier, c'est pourquoi je vais saisir cette opportunité de quelques pages pour expliquer en termes non-spécialisés les bases du monde merveilleux de la théorie des graphes.

Commençons par un exemple de la vie quotidienne : le plan du métro. Supposons que vous vouliez aller de la station Lyon Part-Dieu à l'Hôtel de Ville, et que votre critère est de minimiser le nombre de stations intermédiaires auxquelles le métro va faire un arrêt (on négligera les correspondances pour la simplicité de l'exemple). *Comment allez-vous faire pour trouver le meilleur chemin ?* Vous allez vous procurer un plan du métro, où chaque station est représentée par un point, et où deux stations consécutives sur la même ligne sont reliées par un trait. Ce que vous avez dans les mains à ce moment-là, en plus d'être une carte, est *un graphe* (voir Figure 1.(a)) : un *graphe* est tout simplement un ensemble de *points* (ici, les stations de métro), que l'on nomme *sommets*, ainsi qu'un ensemble de *liens* reliant certains sommets deux à deux, que l'on nomme *arêtes*. Deux sommets reliés par une arête sont dits *adjacents* ou encore *voisins*. La Figure 1 donne quelques exemples de graphes. Revenons à notre plan du métro : le nombre minimal d'arrêts intermédiaires est le nombre minimal de sommets que l'on doit traverser sur le graphe pour aller du sommet initial *Part-Dieu* au sommet final *Hôtel de Ville*. Calculer ceci s'appelle *résoudre un problème de plus court chemin*.

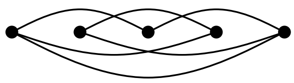
Les graphes sont des objets très puissants qui nous permettent de modéliser de nombreux problèmes provenant de la vie quotidienne ou du monde industriel. Dans ce contexte, *modéliser* une situation réelle consiste à déterminer quelles informations sont nécessaires pour résoudre le problème en question, puis à organiser ces informations sous la forme d'un objet mathématique connu (ici, un graphe). En se débarrassant ainsi des informations superflues, cela fait souvent apparaître le problème initial sous la forme d'un problème déjà étudié dans une autre situation, et l'on peut ainsi espérer utiliser des algorithmes connus pour le résoudre. Si le problème n'a pas encore été étudié, la modélisation nous permet alors de nous concentrer sur l'essentiel afin de chercher une solution. Il existe plusieurs variantes de graphes permettant de



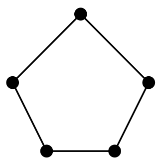

(a) Graphe qui décrit le réseau de métro de Lyon.



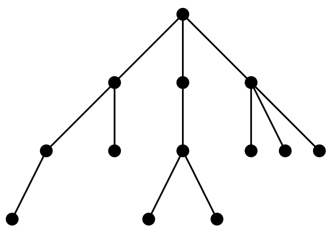
(b) Un chemin à 5 sommets, noté $P_5$.



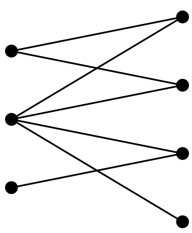
(c) Le complément de $P_5$, noté $\overline{P_5}$ : on remplace les arêtes de $P_5$ par des non-arêtes et vice-versa.



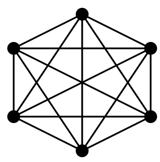
(d) Un cycle de longueur 5, noté $C_5$.



(e) Un arbre : chaque sommet est le *fils* de son *père*, ce dernier étant l'unique voisin au-dessus. Le sommet le plus en haut est appelé la *racine*, et les sommets sans fils sont appelés des *feuilles*.



(f) Un graphe qualifié de *biparti* : les sommets peuvent être divisés en deux groupes (ici, gauche et droite) de telle sorte que chaque arête ait une extrémité dans chaque groupe.



(g) Un graphe complet ou clique à 6 sommets, noté $K_6$.

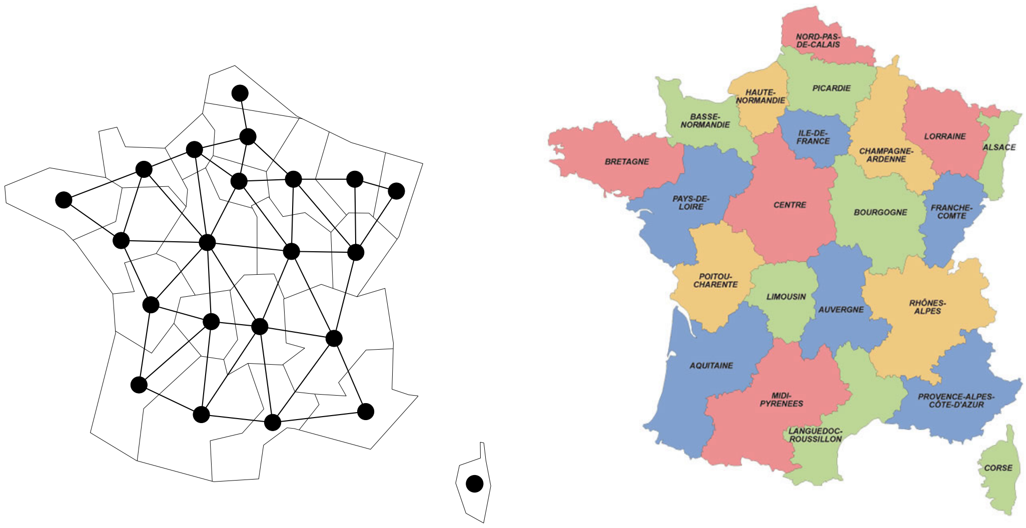
FIGURE 1 – Le plan du métro de Lyon ainsi que des exemples classiques de graphes.

conserver plus d'informations que le modèle simple présenté jusqu'ici, permettant ainsi de modéliser des problèmes plus sophistiqués. Par exemple, les graphes *orientés* ont la fonctionnalité supplémentaire suivante : chaque arête peut être orientée d'une extrémité à l'autre, permettant ainsi de modéliser les rues à sens unique. Les graphes *pondérés* nous permettent quant-à-eux d'associer à chaque sommet et/ou arête un nombre (que l'on appelle *poids*), et sont utilisés dans les GPS. En effet, la carte chargée dans un GPS fournit un immense graphe : il y a un sommet pour chaque point d'intersection entre plusieurs routes, et chaque portion directe de route donne une arête reliant ses deux extrémités. La distance entre les deux points devient alors le poids de l'arête (selon les réglages, il peut s'agir du temps de trajet nécessaire au lieu de la distance). Lorsque vous recherchez un itinéraire sur votre GPS, celui-ci effectue alors une résolution d'un *problème de plus court chemin*, comme dans l'exemple du métro, en prenant en compte cette fois le poids des arêtes.

Un nouvel exemple de graphe immense s'est développé récemment : les réseaux sociaux. Considérons par exemple le graphe de Facebook, qui est construit de la manière suivante : chaque personne ayant un compte Facebook est un sommet, et deux personnes sont reliées par une arête si elles sont amies sur Facebook. Un groupe d'amis va certainement créer une structure spéciale à l'intérieur de ce graphe puisqu'il va former un groupe de sommets tous adjacents deux à deux : un tel groupe de sommets est appelé une *clique*. D'autre part, si un institut de sondage sélectionne un groupe de personnes qui ne se connaissent pas entre elles, cela forme un groupe de sommets deux à deux non-adjacents : on appelle cela un *stable* ou encore un *ensemble indépendant*. Les cliques et les stables sont donc deux objets tout-à-fait naturels dans un graphe, et sont au cœur de cette thèse.

Il est intéressant de voir qu'une arête ne dénote pas toujours un lien à connotation positive entre ses deux extrémités, mais peut aussi dénoter un *conflit*. C'est souvent le cas dans les problèmes de *coloration*[1] de graphe. Le plus connu est la *conjecture des 4 couleurs*, question posée par Guthrie dans les années 1850 : imaginons une carte divisée en régions (les comtés d'Angleterre pour reprendre l'exemple historique, mais il peut également s'agir des pays du monde, des communes d'un pays, des terrains sur un cadastre, etc...) où l'on souhaite colorer toutes les régions de telle sorte que chacune reçoive une couleur différente de ses régions voisines (c'est-à-dire les régions avec qui elle partage une frontière commune). Pour des raisons de coûts d'impression, on veut utiliser le moins de couleurs différentes possible. *De combien de couleurs avons-nous besoin ?* Observons que l'on peut facilement modéliser la situation par un problème de graphe : construisons un sommet pour chaque région, et relions une paire de sommets si les régions correspondantes sont voisines (voir la Figure 2 pour un exemple). Le problème est maintenant de colorer les sommets du graphe de telle sorte que deux sommets adjacents n'aient jamais la même couleur. Guthrie a conjecturé qu'il suffirait de 4 couleurs pour n'importe quel graphe provenant d'une carte (les graphes ainsi construits sont qualifiés de *planaires* car ils peuvent être dessinés sans que les arêtes ne

[1] On notera que l'usage veut qu'on emploie le mot *coloration* et non *coloriage*, ce qui permet de distinguer notre travail de celui des enfants de maternelle...



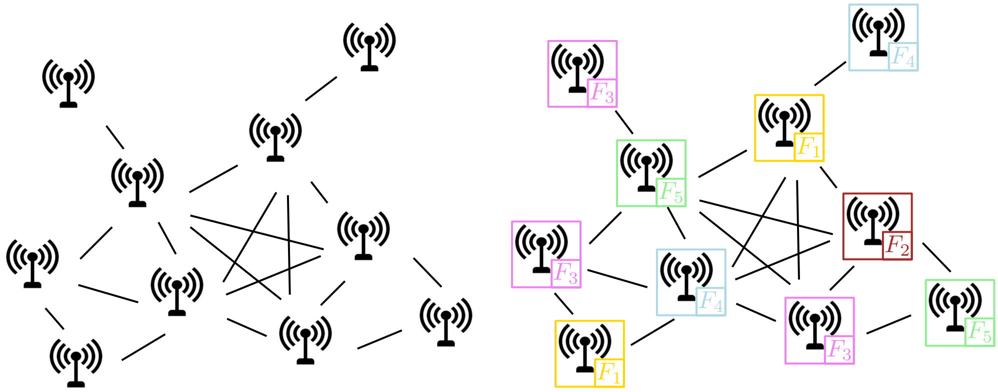

(a) Le graphe planaire obtenu à partir des régions de France.

(b) Une coloration avec 4 couleurs, conformément au théorème des 4 couleurs.

FIGURE 2 – Une illustration du problème de coloration des cartes. Quatre couleurs sont ici bel et bien nécessaires : le Limousin est entouré par cinq autres régions qui forment un cycle. Il faut trois couleurs pour le cycle car il est impair, puis une quatrième couleur différente pour le Limousin.

se croisent, en autorisant les arêtes à être des lignes courbes au lieu de lignes droites), mais n'a pas réussi à le prouver. La question a alors soulevé beaucoup d'intérêt dans la communauté mathématique jusqu'à ce que Kempe annonce en détenir une preuve en 1879. Cependant, une erreur dans cette preuve à été trouvée un an plus tard par Heawood, qui a néanmoins réussi à adapter la preuve de Kempe pour prouver que *cinq couleurs suffisent toujours pour les graphes planaires*. La question des quatre couleurs est ainsi restée ouverte jusqu'en 1976, date à laquelle Appel et Haken ont publié une preuve correcte. Le résultat s'appelle dès lors le *Théorème des 4 couleurs*. En plus de résoudre une conjecture vieille d'un siècle, cette preuve est également connue pour être la première preuve mathématique assistée par ordinateur. Plus précisément, Appel et Haken ont prouvé à la main que le théorème était vrai pour tous les graphes si et seulement si il était vrai pour une liste de 1 936 cas particuliers à vérifier. Cette liste étant beaucoup trop longue pour être vérifiée à la main, ils ont vérifié tous les cas par ordinateur. Pour cette raison, la preuve a été quelque peu controversée par certains mathématiciens. La liste de cas a ensuite été raccourcie à « seulement » 633 cas par Robertson, Sanders, Seymour et Thomas. Finalement, en 2005, Gonthier a établi une version formalisée de la preuve, en utilisant l'assistant de preuve Coq, ce qui est une sorte de « garantie » que la preuve est correcte.

Comme mentionné précédemment, le pouvoir des graphes est de modéliser plusieurs problématiques industrielles comme un seul et même problème. Illustrons-le




(a) Un graphe provenant d'un réseau d'antennes radio. Une arête entre deux antennes indique des perturbations éventuelles.

(b) Une allocation de fréquences, vue comme une coloration du graphe.

FIGURE 3 – Illustration de l'exemple des antennes radio.

avec un nouvel exemple : la diffusion d'ondes radio. Supposons qu'une compagnie de radio veuille diffuser une nouvelle station sur un réseau d'antennes existant. L'entreprise doit allouer à chaque antenne une fréquence radio pour sa nouvelle station, et, de manière à éviter les interférences, deux antennes ne peuvent pas se voir allouer la même fréquence si elles sont trop proches. L'entreprise doit payer un prix fixe pour chaque fréquence qui sera utilisée sur au moins une antenne, et évidemment elle souhaite minimiser les dépenses. *Quel est le graphe sous-jacent ?* Les sommets sont les antennes, et deux sommets sont adjacents si les antennes correspondantes présentent des risques d'interférences (voir Figure 3.(a) pour un exemple). Ici, nous pouvons voir que les arêtes représentent des conflits, comme dans l'exemple précédent. Maintenant, observons que le problème d'allocation des fréquences radio n'est en fait rien de plus qu'un problème de coloration de graphe : quel est le nombre minimal de couleurs (ici, de fréquences) nécessaire pour colorer chaque sommet (ici, allouer une fréquence à chaque antenne) de telle sorte que deux sommets adjacents reçoivent des couleurs différentes (ici, que les perturbations soit évitées) ?

De la même manière, la coloration de graphes peut modéliser de nombreux autres problèmes qui semblent, à première vue, complètement différents : allouer des salles de classes dans un collège où de nombreux cours ont lieu simultanément, conserver des produits chimiques dans des réfrigérateurs lorsque chaque produit doit être gardé dans un intervalle de température donné, etc. Plusieurs autres variantes du problème de coloration existent pour modéliser des problèmes légèrement différents : ainsi, on peut choisir de colorer les arêtes au lieu des sommets, voire même les deux ; on peut également choisir de changer les règles de coloration et déclarer, par exemple, que deux sommets ayant un voisin commun ne peuvent pas avoir la même couleur.

On pourrait être tenté d'appliquer le Théorème des 4 couleurs au graphe du réseau d'antennes décrit ci-dessus, et d'en conclure que quatre couleurs suffisent. Cependant, le Théorème des 4 couleurs ne peut être appliqué que si le graphe est *planaire*, ce qui n'est pas le cas des graphes d'antennes en général : par exemple, sur la Figure 3, le groupe des cinq antennes du milieu forme une clique de taille 5, qui ne peut pas être dessinée sans croiser les arêtes. De plus, cette clique prouve bien que cinq couleurs sont nécessaires : chaque antenne de ce groupe doit avoir une couleur différente des quatre autres. Cette observation donne la première borne sur le nombre de couleurs nécessaires pour colorer le graphe, ce nombre de couleurs étant appelé le *nombre chromatique* : il est *supérieur ou égal* à la taille de la plus grande clique présente dans le graphe. L'inégalité peut être stricte, comme on peut le voir par exemple sur le cycle de longueur 5, qui requiert trois couleurs mais qui ne contient aucun triangle. D'autre part, une borne *supérieure* est quant-à-elle obtenue de la manière suivante : le *degré* d'un sommet $v$ est simplement le nombre de voisins de $v$, et désignons par $\Delta$ le degré maximum parmi tous les sommet de $G$. Alors on peut affirmer que $\Delta + 1$ couleurs suffisent. En effet, on peut procéder de la manière suivante : commençons avec le graphe non coloré et répétons l'opération suivante tant que le graphe n'est pas complètement coloré. Choisissons arbitrairement un sommet $v$ non coloré. Il a au plus $\Delta$ voisins, donc au moins une couleur parmi les $\Delta + 1$ couleurs autorisées n'est utilisée sur aucun de ses voisins. Donnons cette couleur à $v$. À la fin du déroulement de l'algorithme, le graphe est coloré correctement avec seulement $\Delta + 1$ couleurs, comme affirmé.

On voit ainsi que les cliques donnent une borne inférieure sur le nombre chromatique. En fait, les stables jouent également un rôle naturel dans le problème de coloration : tous les sommets qui reçoivent une même couleur forment un stable. Ainsi, une coloration est une partition des sommets du graphe en stables. Le très célèbre problème de coloration est donc le premier type d'*interaction entre les cliques et les stables* qui sera étudiée dans cette thèse. Ces deux structures suscitent beaucoup d'intérêt en théorie des graphes : on s'interroge sur leur *taille*, on essaye de *trouver le plus grand*, on se demande *quelle structure induisent-ils* ou encore *comment sont-ils entremêlés*, comme dans le problème de coloration. Cette thèse se concentre sur différents types d'interaction entre les cliques et les stables, et les liens entre ces différents types d'interaction.

La borne supérieure de $\Delta + 1$ et le Théorème des 4 couleurs illustrent deux stratégies différentes de la théorie des graphes : dans le premier cas, il s'agit de prouver une borne qui est valable pour *tous* les graphes, alors que dans le deuxième cas, il s'agit de prouver une borne plus précise qui s'applique uniquement si le graphe satisfait certaines conditions. Dans ce dernier cas, on peut exploiter la structure particulière donnée par les hypothèses pour obtenir des bornes très puissantes, comme dans le Théorème des 4 couleurs, et qui sont souvent fausses si l'on enlève les hypothèses. La planarité est une condition assez courante, mais il en existe beaucoup d'autres. Si l'on appelle $H$ votre graphe préféré, on obtient une autre condition classique de la manière suivante : on dit qu'un graphe *$G$ contient $H$ comme sous-graphe* si on peut obtenir $H$ en partant de $G$ et en supprimant certains sommets et certaines arêtes (observons que si l'on supprime un sommet, on est obligé de supprimer toutes ses arêtes incidentes, car cela n'a pas de sens de garder une arête si l'une de ses extrémités n'est pas un sommet du graphe). Par exemple, le cycle $C_5$ contient le chemin $P_5$ comme sous-graphe, puisqu'il suffit de supprimer une arête de $C_5$ pour obtenir $P_5$. La clique de taille 6 contient le triangle $K_3$, comme en témoigne la suppression de trois sommets arbitraires. De cette manière, on peut ainsi s'intéresser aux *graphes qui ne contiennent pas H comme sous-graphe* et essayer d'en exploiter leur structure. Dans le contexte de cette thèse, une autre condition légèrement différente s'avère plus pertinente : il s'agit d'étudier les graphes qui ne contiennent pas $H$ comme sous-graphe *induit*, et non pas comme sous-graphe tout-court. On dit qu'un graphe $G$ contient $H$ comme *sous-graphe induit* si l'on peut obtenir $H$ en partant de $G$ et en supprimant un ou plusieurs sommets (et leurs arêtes incidentes). On notera que, contrairement à la définition précédente, il n'est ici pas autorisé de supprimer une arête de son choix sans supprimer au moins une de ses extrémités. Par exemple, le cycle $C_5$ ne contient *pas* le chemin $P_5$ comme sous-graphe induit. Par contre il contient le chemin à 4 sommets $P_4$ comme sous-graphe induit, il suffit pour cela de supprimer n'importe quel sommet. La clique $K_6$, quant-à-elle, contient bel et bien le triangle $K_3$ comme sous-graphe induit. Si un graphe $G$ ne contient pas $H$ comme sous-graphe induit, on dit que $G$ est *sans H*. On s'intéresse ainsi à prouver des résultats de la forme *tout graphe sans H se comporte bien vis-à-vis de la propriété que l'on est en train d'étudier*, cette propriété pouvant être par exemple la coloration, mais bien sûr, il en existe de nombreuses autres.

**Résumé détaillé** Survolons brièvement le contenu de cette thèse en termes plus techniques. Chaque chapitre commence par une introduction détaillée, présentant le sujet soumis à considération, donnant les définitions nécessaires à la bonne compréhension du chapitre, et fournissant une description des travaux antérieurs. Pour éviter la redondance, peu de définitions sont détaillées ici et le lecteur est invité à se référer au chapitre correspondant.

La coloration sera l'objet du Chapitre 1. Le chapitre commence par un état de l'art sur les graphes parfaits, qui sont les graphes pour lesquels le nombre chromatique est égal à la taille de la plus grande clique (et de même pour tous les sous-graphes induits). Les graphes parfaits ont été largement étudiés depuis leur introduction par Berge dans les années 1960 et ont des propriétés structurelles et polyédrales particulièrement intéressantes, dont certaines seront utilisées dans les Chapitres 3 et 4. La Section 1.2 présente le premier résultat de cette thèse, qui consiste à prouver l'existence d'une constante $c$ bornant supérieurement le nombre chromatique de tous les graphes sans triangle et sans cycle pair de longueur au moins 6.

Le Chapitre 2 s'intéresse à une autre forme bien connue d'interaction entre les cliques et les stables : la *conjecture d'Erdős-Hajnal*. En un mot, la conjecture affirme que, étant donné un graphe $H$ fixé, tout graphe sans $H$ contient une très grande clique ou un très grand stable. Plus précisément, la conjecture s'énonce ainsi : pour tout graphe $H$, il existe une constante $\varepsilon > 0$ telle que tout graphe $G$ sans $H$ contient une clique ou un stable de taille $|V(G)|^\varepsilon$. Notons que l'hypothèse *sans H* est essentielle puisque les graphes aléatoires contiennent seulement des cliques et des stables de taille logarithmique, ce qui est bien moindre que la borne linéaire avancée dans la conjecture. Après un rappel des résultats existants dans la Section 2.1, nous prouvons dans la Section 2.2 que la propriété d'Erdős-Hajnal est vraie pour les graphes où l'on interdit le chemin à $k$ sommets et son complémentaire comme sous-graphes induits ; plus exactement, on prouve même que la propriété *forte* d'Erdős-Hajnal est vraie pour ces graphes-là. Dans la Section 2.3, nous décrivons quelques outils qui peuvent s'avérer utiles pour prouver d'autre cas encore non résolus de la conjecture.

Introduite par Yannakakis en 1991, la question étudiée dans le Chapitre 3 n'est pas aussi connue dans la communauté des graphes que les deux précédentes, et nous l'appelons la *Séparation des Cliques et des Stables*. On peut la présenter ainsi[2] : une *coupe* dans un graphe $G$ est une partition des sommets de $G$ en deux parties $B$ et $W$. Une coupe *sépare* une clique $K$ et un stable $S$ donnés si $K$ est inclus dans la partie $B$ et si $S$ est inclus dans la partie $W$. Un *séparateur Clique-Stable* de taille $k$ est un ensemble de $k$ coupes tel que, pour toute clique $K$, et pour tout stable $S$ disjoint de $K$, il existe une coupe dans le séparateur qui sépare $K$ et $S$. La question est alors : étant donnée une classe de graphes $\mathcal{C}$, existe-t-il une constante $c$ telle que tout graphe $G$ de $\mathcal{C}$ admet un séparateur Clique-Stable de taille $\mathcal{O}(|V(G)|^c)$ ? Observons que la réponse est trivialement *oui* si l'on s'autorise un nombre de coupes exponentiel et non polynomial (il suffirait alors de prendre toutes les coupes possibles). La question originale ne concernait que le cas des graphes parfaits, mais est toujours ouverte. La question est également ouverte pour la classe de tous les graphes, et les meilleures bornes à ce jour sont $\Omega(n^{2-\varepsilon})$ comme borne inférieure et $\mathcal{O}(n^{\log n})$ comme borne supérieure, où $n$ désigne le nombre de sommets du graphe. La Section 3.1 donne une définition plus complète du problème et présente les cas les plus faciles, par exemple on peut y trouver un séparateur Clique-Stable de taille quadratique pour les graphes sans triangle. On prouve ensuite qu'un séparateur Clique-Stable de taille polynomial existe pour plusieurs classes de graphes : les graphes aléatoires dans la Section 3.2, pour lesquels le résultat est plutôt surprenant puisque l'on pourrait s'attendre à ce que les cliques et les stables y soient très entremêlés. Dans la Section 3.3, on s'intéresse aux graphes sans $H$ où $H$ est n'importe quel graphe split fixé, et l'on utilise ici un argument de VC-dimension. Dans la Section 3.4, on exploite des outils développés dans l'étude de la conjecture d'Erdős-Hajnal et on les adapte pour construire des séparateurs Clique-Stable. En particulier, la propriété forte d'Erdős-Hajnal, prouvée pour les graphes sans $(P_k, \overline{P_k})$, donne un séparateur Clique-Stable de taille polynomial pour ces graphes. Finalement, dans la Section 3.5, on utilise des propriétés structurelles des graphes parfaits pour prouver que les graphes parfaits sans skew partition équilibrée ont un séparateur Clique-Stable polynomial.

Le Chapitre 4 décrit la motivation initiale de Yannakakis qui l'a mené à s'intéresser à la Séparation Clique-Stable. Cette motivation vient de la combinatoire polyédrale, et plus précisément de l'étude du *polytope des stables* de $G$, qui est l'enveloppe convexe des vecteurs caractéristiques $\chi^S \in \{0,1\}^{|V(G)|}$ de tous les stables $S$. De manière intéressante, les cliques jouent un rôle capital dans la description du polytope par des inégalités linéaires, surtout si $G$ est parfait. En partant de rappels sur les bases de

[2]La définition brute semble un peu indigeste au premier abord. Pour une présentation plus intuitive du problème, mais trop longue pour apparaître dans ce résumé détaillé, se référer au début du Chapitre 3.

la programmation linéaire et entière en Sections 4.1 à 4.4, on expose ensuite dans la Section 4.5 des propriétés remarquables du polytope des stables dans les graphes parfaits, qui ont mené en particulier aux travaux célèbres de Grötchel, Lovász et Schrijver : la résolution en temps polynomial du problème du Stable Pondéré Maximum dans les graphes parfaits. On entre ensuite dans le monde des *formulations étendues* et *complexité d'extension d'un polytope*, dont le principe peut être résumé par la question suivante : étant donné un polytope $P \in \mathcal{R}^d$ ayant beaucoup de faces, existe-t-il un polytope $Q$ dans un espace $\mathcal{R}^{d+r}$ de dimension plus grande, qui aurait moins de faces et qui se projetterait sur $P$ ? De très beaux résultats ont été récemment prouvés dans ce domaine, on en présente les principaux dans la Section 4.6. La Section 4.7 décrit ensuite les outils les plus courants du domaine, dont le célèbre *Théorème de Factorisation* de Yannakakis. Enfin, les Sections 4.8 et 4.9 décrivent la relation très étroite entre combinatoire polyédrale, complexité de communication, et Séparation Clique-Stable.

Le cas général de la Séparation Clique-Stable peut en fait être reformulé en deux énoncés équivalents : le premier est étudié dans le Chapitre 5 et est une généralisation de la *conjecture d'Alon-Saks-Seymour*. Celle-ci donne une borne supérieure sur le nombre chromatique de $G$ en fonction de son *score de partition en bicliques*, c'est-à-dire le nombre minimum de bipartis complets permettant de partitionner les arêtes de $G$. La conjecture originale est inspirée du théorème de Graham-Pollak, qui traite le cas des graphes complets. Cependant, cette conjecture a finalement été réfutée en 2012 par Huang et Sudakov. Néanmoins, sa généralisation est encore ouverte, et la Section 5.2 s'attache à prouver son équivalence avec la Séparation Clique-Stable. La Section 5.3 étend l'équivalence au cas où les bipartis complets sont autorisés à se chevaucher de manière contrôlée.

Le deuxième énoncé équivalent vient du monde des *Problèmes de Satisfaction de Contraintes* (abrégé CSP, pour *Constraint Satisfaction Problems* en anglais), et est étudié au Chapitre 6. Plus précisément, on s'intéresse au Problème Têtu et au Problème de 3-Coloration Compatible, et l'on prouve que résoudre ces problèmes en utilisant une méthode spécifique (appelée *couverture par listes de taille* 2) est équivalente à la construction de séparateurs Clique-Stable. Après une présentation du contexte autour des Problèmes de Satisfaction de Contraintes, le chapitre est consacré à la preuve de cette équivalence.

**Notation** Décrivons ici quelques notations classiques qui seront utilisées tout au long de ce manuscrit. Tous les logarithmes considérés seront en base 2 et on utilisera la notation $\log x$. Un graphe fini simple $G = (V, E)$ est un couple constitué d'un ensemble fini $V$, dont les éléments sont appelés sommets, et d'un ensemble d'arêtes $E$, où chaque arête est une paire non-ordonnée de sommets distincts. Une paire $\{u, v\}$ de sommets sera notée indifféremment $uv$ ou $vu$. Lorsque $V$ et $E$ ne sont pas explicitement nommés lors de la définition de $G$, on leur fera référence grâce à la notation $V(G)$ et $E(G)$, respectivement. Dans cette thèse, tous les graphes considérés seront supposés finis et simples, c'est pourquoi nous omettrons désormais de préciser ces hypothèses. En particulier, il ne peut pas y avoir plusieurs arêtes entre les deux mêmes sommets. Deux sommets $u$ et $v$ qui vérifient que $uv \in E$ sont dits *adjacents* ou encore *voisins* l'un de l'autre. Le *voisinage (ouvert)* de $v$ est noté $N(v)$ et désigne l'ensemble des voisins de $v$. Le *voisinage fermé* de $v$ est noté $N[v]$ et désigne $N(v) \cup v$. Le *complément* de $G$ est le graphe $\overline{G} = (V, \overline{E})$ où $\overline{E} = \{uv \mid u \neq v,\ uv \notin E\}$. Une *clique* dans $G$ est un ensemble de sommets deux à deux adjacents. Un *stable* dans $G$ est un ensemble de sommets deux à deux non-adjacents. Le *graphe complet* $K_n$ est le graphe à $n$ sommets contenant toutes les arêtes possibles, c'est-à-dire tous les $uv$ tels que $u \neq v$. Le *chemin* à $k$ sommets est noté $P_k$ et est le graphe dont l'ensemble des sommets est $\{1, \ldots, k\}$ et dont les arêtes sont les paires de la forme $i(i+1)$ pour $i \in \{1, \ldots, k-1\}$. Sa *longueur* est $k-1$.

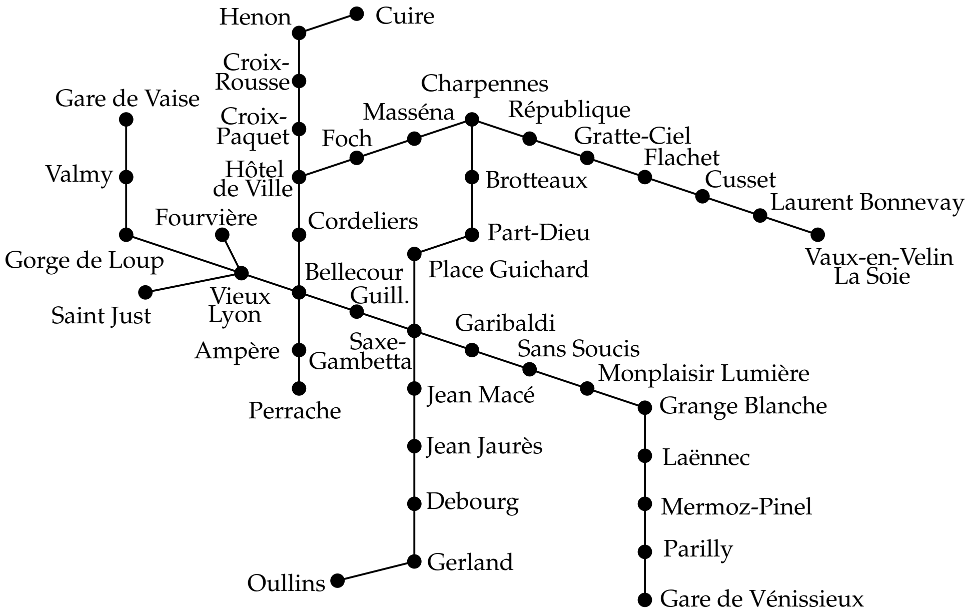
Si $X \subseteq V$, le *sous-graphe induit par $X$*, est le graphe $G[X]$ dont les sommets sont les éléments de $X$ et dont les arêtes sont $\{uv \in E \mid u, v \in X\}$. On dit que $G[X]$ est un *sous-graphe induit* de $G$. Il est *strict* si $X \neq V$. Pour simplifier les notations, le graphe induit par $V \setminus X$ est souvent noté $G \setminus X$ au lieu de $G[V \setminus X]$. Pour $x \in V$, le singleton $\{x\}$ sera parfois simplement noté $x$, en particulier si $Y \subseteq V$, alors on notera $Y \setminus x$ au lieu de $Y \setminus \{x\}$ et $G \setminus v$ au lieu de $G \setminus \{x\}$. Une classe de graphe est *héréditaire* si elle est close par sous-graphe induit. On dit que $G$ est *isomorphe* à $G' = (V', E')$ s'il existe une bijection $\phi$ entre $V$ et $V'$ telle que $uv \in E \Leftrightarrow \phi(u)\phi(v) \in E'$. Soit $H$ un graphe, on dit que *$G$ contient $H$ comme sous-graphe induit* s'il existe $X \subseteq V$ tel quel $G[X]$ soit isomorphe à $H$. Un graphe est *sans $H$* s'il ne contient pas $H$ comme sous-graphe induit. Étant donné une famille de graphes $\mathcal{H}$, on dit que $G$ est sans $\mathcal{H}$ s'il est sans $H$ pour tout $H \in \mathcal{H}$. Si $\mathcal{H} = \{H_1, H_2\}$, on écrira par abus de notation que $G$ est sans $(H_1, H_2)$ au lieu de la notation avec accolades.

# Introduction

WHAT *are you working on?* This is a question that I often face, from family and friends, when one expects you to answer with no more than two sentences. It is not an easy task to answer this question succinctly, especially with no paper and no blackboard nearby, so let me take this opportunity to explain from scratch the basis of the beautiful world of graph theory.

Let me start with an example taken from everyday life: the subway network. Imagine that you want to go by subway from the station Lyon Part-Dieu to the city hall, and for some reason you want to choose the way with the minimum number of stops, no matter the number of connections. *How are you going to proceed?* You will get a map from the subway network, each station being depicted by a point, and two consecutive stations on the same line being linked by a straight line. What you have in your hands at this moment *is* a graph (see Figure 4.(a)), and you are about to solve a *shortest-path problem* on it: a *graph* is anything that you can describe with a set of *points* (here, the stations), which we call *vertices*, together with some *links* (here, the direct lines), which we call *edges*, each of them connecting two points. Two vertices linked by an edge are said to be *adjacent*, and we also say that they are *neighbors* one from each other. Figure 4 gives examples of graphs. Coming back to our subway network, the minimal number of stops is the minimal number of vertices you have to go through when moving along the edges of the graph, starting from the vertex *Part-Dieu* to the target vertex *Hôtel de Ville* (*i.e.* City Hall).

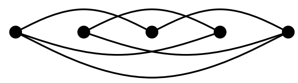
Graphs are very powerful objects that enable us to model a lot of problems that occur in everyday life or in industrial applications. A lot of variants exist to model more sophisticated problems, for example *directed graphs* allow us to orient each edge from one vertex to the other and can model one-way streets; *weighted graphs* allow us to put some weight on the vertices and/or the edges, and have a famous application to GPS. Indeed, the map loaded on your GPS provides a huge graph: there is a vertex for each intersection point between several roads, and each portion of road corresponds to an edge linking its two extremities. This edge is weighted by its distance (or by the travel time needed to go along it, depending on the settings). A route query in your GPS asks for a solution of the shortest-path problem, as in the previous example, except that the distance is now computed with the edge weight.



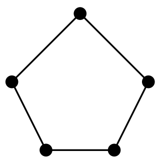

(a) The graph describing Lyon subway network.



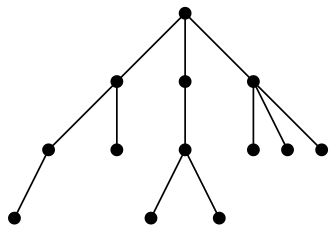
(b) A path on five vertices, called $P_5$.



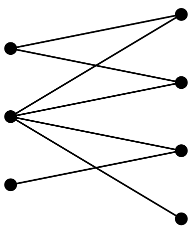
(c) The complement of $P_5$, called $\overline{P_5}$: edges of $P_5$ become non-edges and *vice-versa*.



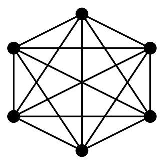
(d) A cycle of length five, called $C_5$.



(e) A tree: each vertex is the *child* of its *father*, *i.e.* its unique neighbor upwards. The top vertex is the *root*, and vertices with no child are *leaves*.



(f) A bipartite graph: the vertices are divided into two parts (here, left and right), each of them containing no edges.



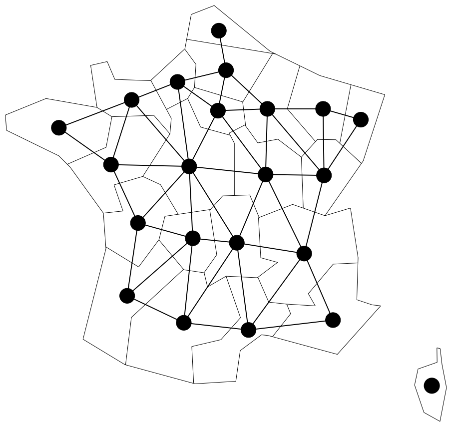
(g) A complete graph on 6 vertices, called $K_6$.

FIGURE 4: Lyon subway network together with some examples of classical types of graphs.

Social networks provide a new example of huge graphs. Let us consider the Facebook graph, which is given as follows: each person with a Facebook account is a vertex, and two people are linked by an edge if they are friends on Facebook. A group of friends will certainly create a special structure inside this graph because it forms a group of vertices that are all pairwise adjacent: such a group is called a *clique*. On the other hand, if a poll institute selects a group of people who do not know each other, it forms a group of pairwise non-adjacent vertices in the graph: this is called a *stable set* or else an *independent set*. Those two objects are thus very natural structures to study in a graph, and are at the core of this thesis.

Interestingly, an edge is not always meant to express a *positive* link between the two vertices it connects: an edge can also express a *conflict*. This is often the case when dealing with *coloring problems*, the most famous of which is the *Four-Color Conjecture*, which was stated by Guthrie in the 1850's: imagine that you have a map divided into regions (counties of England for the historical example of Guthrie, but it can also be countries of the world, cities in a country, plots in a land register and so on), and you want to color every region so that no two regions that share a common border get the same color. *How many colors do you need?* Observe that this is just a graph problem: construct one vertex for each region, and link two regions by an edge if they share a common border (see Figure 5 for an example). The question is now how to color the vertices with the condition that any two adjacent vertices get different colors. Guthrie conjectured that four colors were always enough for graphs built from a map (called *planar graphs*, that is to say graphs that we can draw in the plane with no crossing edges), but he did not manage to prove it. The question then attracted a lot of attention from mathematicians until a proof was announced by Kempe in 1879 [133]. However, the proof was proved incorrect one year later by Heawood, who managed to adapt Kempe's proof to show that *five colors are always enough for planar graphs* [118]. Still, the question with only four colors remained wide open until Appel and Haken finally proved it in 1976 [9, 10]. The statement was then called the *Four-Color Theorem*. Apart from resolving a century old conjecture, the proof is also famous for another reason: it was the first computer-assisted proof. More precisely, Appel and Haken proved by hand that the theorem is true if and only if it is true for a list of 1,936 particular cases that needed to be checked. This list is way too long to be checked by hand, so they checked it with a computer. For this reason, the proof remained controversial to some mathematicians. The list was thereafter shortened to 633 cases by Robertson, Sanders, Seymour and Thomas [174]. Finally in 2005, Gonthier completed a formalized version of the proof, using the Coq proof assistant [106].

The power of graphs is to model several real-life problems as the same graph problem. By eliminating all superfluous information, the modeling focuses on what is important. Let us go on with another example: radio broadcasting. Imagine that a radio company wants to broadcast a channel on a given antenna network. The company has to allocate to each antenna a frequency on which the channel will be broadcast, and in order to avoid interference, two antennas that are too close must be allocated different frequencies. The company must buy every frequency that will be



(a) The planar graph obtained from regions of France.

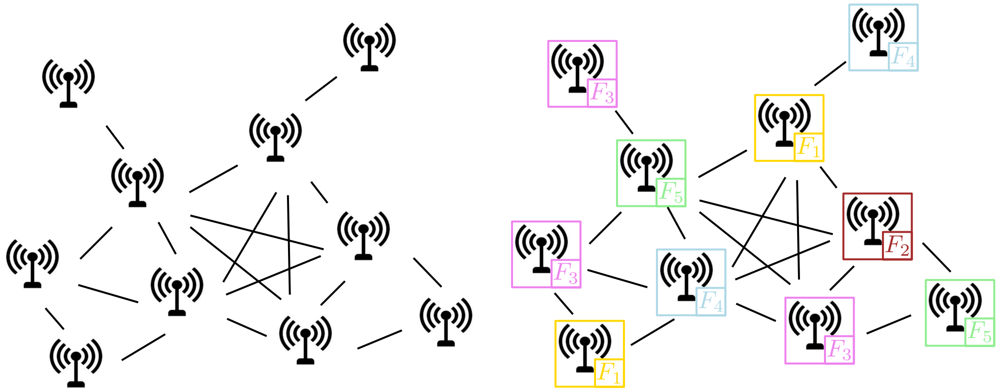

(b) A coloring with four colors, possible according to the Four-Color Theorem.

FIGURE 5: An illustration of the map coloring problem. Four colors are indeed needed: the Limousin is surrounded by 5 other regions forming a cycle. Three colors are needed for the odd cycle, and a fourth one for the Limousin.

used, and of course wants to minimize expenses. *What is the underlying graph?* The vertices are the antennas, and two antennas are linked by an edge if they are close enough to create perturbations. Here, we can see as before that each edge stands for a conflict. Now, we can observe that the frequency allocation is just a coloring problem (see Figure 6): what is the minimum number of colors (here, frequencies) needed in order to color every vertex (here, allocate a frequency to each antenna) so that two adjacent vertices get different colors?

In the same fashion, graph coloring can model several other problems that, at first sight, seem completely different: allocating classrooms in a school with several parallel lessons, keeping chemical products in the minimum number of different fridges when each product should be kept in a given temperature interval, and so on. Several other variants of coloring exist to model more sophisticated problems: edges can be colored instead of vertices, or even both; one can also change the coloring condition and stipulate, for example, that two vertices with a common neighbor cannot get the same color.

One might be tempted to apply the Four-Color Theorem to the antenna graph described above and to conclude that four colors are always enough. However, the Four-Color Theorem can be applied only if the graph is *planar*, which is not the case here: for example in Figure 6, the central group of five antennas forms a clique of size 5, which cannot be drawn with no crossing edges, even if we allow curved edges. Moreover, this clique proves that at least five colors are necessary: every antenna of this group should get a color different from the other four. This observation gives the first bound on the minimum number of colors needed, called the *chromatic number*




(a) An antenna network graph. Edges stand for possible perturbations.

(b) A frequency allocation, seen as a coloring.

FIGURE 6: Illustration for the antenna network example.

of the graph: it is at least the size of the largest clique of the graph. Equality does not always hold, as seen with the 5-cycle $C_5$, which needs three colors but contains no triangle. On the other hand, an upper bound is obtained as follows: let $\Delta$ be the maximum degree over all the vertices of the graph $G$, where the degree of a vertex $v$ is the number of neighbors of $v$. Then $\Delta + 1$ colors are always enough: indeed, we can proceed as follows. As long as the graph is not completely colored, choose an uncolored vertex; it has at most $\Delta$ neighbors, so at least one color is used on none of its neighbors; color the vertex with this color.

One can see that cliques give a lower bound on the chromatic number. In fact, stable sets also play a natural role in the coloring problem: vertices that get the same color form a stable set. Hence, a coloring is a partition of the vertices of the graph into stable sets. The well-known matter of *coloring* thus led us to the first kind of *interaction between cliques and stable sets* studied in this thesis. These two structures attract a lot of attention in graph theory, whether it is about their *size*, about *how to find the largest one*, about *what structure they enforce* or, as in coloring, about *how intertwined they are*. This thesis is concerned with several kinds of interactions between cliques and stable sets, and with the links between them.

The $(\Delta + 1)$ upper bound and the Four-Color Theorem illustrate two different strategies in graph theory: either proving results that apply to *all* graphs, or proving results that apply only to graphs fulfilling some condition. The latter seems to lack generality, but exploiting the structure of the graphs under consideration can give very strong bounds, as illustrated by the Four-Color Theorem, which are often false for the general case. Planarity is a very common hypothesis in graph theory, but many other types of conditions exist. If $H$ is any given graph, we say that a graph $G$ *contains $H$ as a subgraph* if, by deleting some vertices and some edges from $G$, we can obtain $H$ (observe that deleting a vertex implies deleting all its incident edges as well, because it makes no sense to have an edge whose extremity is not a vertex).

For example, the cycle $C_5$ contains the path $P_5$ as a subgraph because we can obtain $P_5$ by deleting one edge of $C_5$. The complete graph $K_6$ contains the triangle $K_3$ as a subgraph, as witnessed by deleting 3 vertices and their incident edges. However, $C_5$ does not contain the triangle as a subgraph. Considering graphs that do not contain some given graph $H$ as a subgraph can be interesting, but in the context of this thesis it is more relevant to forbid $H$ as an *induced* subgraph: *$G$ contains $H$ as an induced subgraph* if $H$ can be obtained from $G$ by deleting vertices (and their incident edges). So this time, we are not allowed to delete an edge if we keep both of its extremities. For example, $C_5$ does *not* contain $P_5$ as an induced subgraph. However, it contains $P_4$, the path on 4 vertices, as an induced subgraph, as showed by deleting one vertex. If $G$ does not contain $H$ as an induced subgraph, we say that $G$ is *$H$-free*. Hence, we are interested in proving results of type *every $H$-free graph behaves well with respect to the property under study*, where the property in question can be *coloring*, for example.

**Outline** Let us briefly overview the content of this thesis in more technical terms. Each chapter starts with a detailed introduction, presenting the subject under study, giving the necessary definitions and providing a description of the state of the art. To avoid redundancy, few detailed definitions are provided here, and the reader is referred to the corresponding chapter.

Coloring will be the subject of Chapter 1. We first start with a survey on perfect graphs, which are graphs for which the chromatic number is equal to the size of the largest clique (and if the same holds for every induced subgraph). Perfect graphs have been widely studied since their introduction by Berge in the 1960's and have very interesting structural and polyhedral properties which will be used in Chapters 3 and 4. Section 1.2 is devoted to the first result of this thesis, namely proving that there exists a constant $c$ bounding the chromatic number for graphs containing no triangle and no cycle of even length at least 6 as induced subgraphs.

Chapter 2 is concerned with another quite well-known interaction between cliques and stable sets: the so-called *Erdős-Hajnal conjecture*. In a nutshell, it states that every $H$-free graph has a large clique or a large stable set. More precisely, it asserts that for every graph $H$, there exists a constant $\varepsilon > 0$ such that every $H$-free graph $G$ has a clique or a stable set of size $|V(G)|^\varepsilon$. Note that forbidding some fixed graph $H$ is necessary since random graphs get only logarithmic-size clique or stable sets, which is much less than the polynomial lower bound in the conjecture. After a brief survey of existing results in Section 2.1, we prove in Section 2.2 that the Erdős-Hajnal property holds when we forbid the path on $k$ vertices and its complement as induced subgraphs; more precisely, we prove that the *Strong* Erdős-Hajnal property holds. Section 2.3 describes some tools that may help to prove the conjecture in some more cases.

Introduced by Yannakakis in 1991 [211], the question studied in Chapter 3, which we call the *Clique-Stable Set Separation*, is not as well-known in the graph community as the previous ones. It can be stated as follows[3] : a *cut* of a graph $G$ is a partition of

[3]The definition may be a bit difficult to understand without appropriate context. At the beginning of

the vertices into two parts $B$ and $W$, and the cut *separates* a clique $K$ and a stable set $S$ if $K$ is included in $B$ and $S$ is included in $W$. A *Clique-Stable Set separator* of size $k$ is a set of $k$ cuts such that for every clique $K$ and for every stable set $S$ disjoint from $K$, there exists a cut that separates $K$ and $S$. Given a class of graphs $\mathcal{C}$, does there exists $c > 0$ such that every $G \in \mathcal{C}$ admits a Clique-Stable Set separator of size $\mathcal{O}(|V(G)|^c)$? Observe that the question is trivial if we allow an exponential number of cuts, since we can then take all possible cuts of $G$. The original question was concerned only with perfect graphs, and is still open. During this thesis, the question was also open for the class of all graphs, and the best bounds so far were $\Omega(n^{2-\varepsilon})$ for the lower bound and $\mathcal{O}(n^{\log n})$ for the upper bound, where $n = |V(G)|$. A very recent result by Göös [108], which appeared when writing this manuscript, exhibits the first example of a family of graphs that require a superpolynomial-size Clique-Stable set separator. Section 3.1 gives a more complete definition and also points out the trivial cases, for example one can easily build a Clique-Stable Set separator of quadratic size for triangle-free graphs. We then provide a positive answer for several classes of graphs: Section 3.2 is concerned with random graphs, for which the result is quite surprising since one can expect the cliques and the stable sets to be very intertwined. Section 3.3 is concerned with $H$-free graphs where $H$ is any fixed split graph, and here we use VC-dimension. In Section 3.4, we exploit tools used in the study of the Erdős-Hajnal property and we adapt them for building Clique-Stable Set separators. In particular, the *Strong Erdős-Hajnal property* proved for $(P_k, \overline{P_k})$-free graphs gives a polynomial Clique-Stable Set separator in those graphs. Finally, in Section 3.5 we use structural properties of perfect graphs to prove that perfect graphs with no balanced skew partition have polynomial CS-Separators.

Chapter 4 describes Yannakakis' initial motivation for the Clique-Stable Set Separation. It comes from polyhedral combinatorics and more precisely from the study of the *stable set polytope* of $G$, which is the convex hull over all stable sets $S$ of the characteristic vectors $\chi^S \in \{0,1\}^{|V(G)|}$. Interestingly, cliques appear to play a special role when trying to describe the polytope with linear inequalities, especially if $G$ is perfect. Starting from basic recalls on linear and integer programming in Sections 4.1 to 4.4, we then highlight in Section 4.5 some properties of the stable set polytope in perfect graphs, which in particular led to the celebrated result of Grötchel, Lovász and Schrijver proving the polynomial tractability of the Maximum Weighted Stable Set problem in perfect graphs [110]. We then enter the world of *extended formulations* and *extension complexity*, whose essence is given by the following question: given a polytope $P$ in $\mathbb{R}^d$ with many facets, does there exist a polytope $Q$ in a higher dimensional space $\mathbb{R}^{d+r}$ with fewer facets such that $P$ is the projection of $Q$? Beautiful results have recently been proved in this field, and we survey the most important of them in Section 4.6. We then describe in Section 4.7 the most common tools used to obtain these results, in particular Yannakakis' well-known *Factorization Theorem*. Finally Sections 4.8 and 4.9 describe the close connection of polyhedral combinatorics

Chapter 3, we give more explanations that may help the reader gain a more intuitive understanding of the problem.

with communication complexity and with the Clique-Stable Set Separation.

The Clique-Stable Set separation in the class of all graphs can in fact be stated in two equivalent ways: the first one is studied in Chapter 5 and is a generalization of the *Alon-Saks-Seymour conjecture*, which gives an upper bound for the chromatic number of $G$ by in terms of its *biclique partition number*, that is the minimum number of complete bipartite graphs needed to partition the edges of $G$. The original conjecture was inspired from the Graham-Pollak theorem [109], which proves it for $G = K_n$; however, the conjecture was finally disproved by Huang and Sudakov in 2012 [122]. Nonetheless its generalization is still open, and Section 5.2 is devoted to proving the equivalence with the Clique-Stable Set Separation. Section 5.3 extends the equivalence to the case where the complete bipartite graphs may slightly overlap.

The second equivalent statement comes from the world of *Constraint Satisfaction Problems* (CSP), and is dealt with in Chapter 6. More precisely, the two problems under study are the Stubborn Problem and the 3-Compatible Coloring Problem, and we show that solving them with a particular method (so-called *2-list covering*) is equivalent to building Clique-Stable Set separators. After giving some context about CSP, the chapter is devoted to proving this equivalence.

**Notation** By $\log x$ we denote the logarithm to base 2. Following the usual notation, a finite simple graph $G = (V, E)$ is a pair consisting of a finite vertex set $V$ and an edge set $E \subseteq \binom{V}{2}$, where $\binom{V}{2}$ stands for the set of unordered pairs of distinct vertices. For brevity of notation, an element $\{u, v\}$ of $\binom{V}{2}$ is also denoted by $uv$ or $vu$. In this thesis, all graphs are supposed to be finite and simple, so we simply say *graph*; in particular there cannot be several edges between the same pair of vertices. We often refer to $V$ and $E$ as $V(G)$ and $E(G)$, respectively. Two vertices $u$ and $v$ such that $uv \in E$ are said to be *adjacent*, and we also say that $u$ is a *neighbor* of $v$. The *(open) neighborhood* of $v$, denoted by $N(v)$, is the set of neighbors of $v$. The *closed neighborhood* of $v$, denoted by $N[v]$, is defined as $\{v\} \cup N(v)$. Furthermore, for $X \subseteq V(G)$, we define $N(X) = (\cup_{v \in X} N(v)) \setminus X$ and $N[X] = N(X) \cup X$. The *complement* of $G$ is the graph $\overline{G} = (V, \overline{E})$ where $\overline{E} = \binom{V}{2} \setminus E$. A *clique* in $G$ is a set of vertices that are all pairwise adjacent. A *stable set* in $G$ is a set of vertices that are all pairwise non-adjacent. The *complete graph* $K_n$ is the graph on $n$ vertices containing all possible edges $uv$ with $u \neq v$. The *path* on $k$ vertices, denoted by $P_k$, is the graph with vertex set $\{1, \dots, k\}$ and whose edges are $i(i+1)$ for $i \in \{1, \dots, k-1\}$. Its *length* is $k - 1$.

Let $G = (V, E)$. A graph $G' = (V', E')$ is said to be a *subgraph* of $G$ if $V' \subseteq V$ and $E' \subseteq E$. If $X \subseteq V$, the *subgraph induced by X*, denoted by $G[X]$, is $(X, E_X)$ where $E_X = \{uv \in E \mid u, v \in X\}$. A subgraph $G'$ of $G$ is said to be an *induced subgraph* of $G$ if there exists $X \subseteq V$ such that $G' = G[X]$. It is a *proper* induced subgraph if $X \neq V$. For convenience, the graph induced by $V \setminus X$ is often denoted $G \setminus X$ instead of $G[V \setminus X]$. For $x \in V$, the singleton $\{x\}$ may sometimes be denoted just by $x$, for example for $Y \subseteq V$, we denote $Y \setminus x$ for $Y \setminus \{x\}$ and $G \setminus x$ for $G \setminus \{x\}$. A class of graphs is *hereditary* if it is closed under taking induced subgraphs. We say that $G$ is *isomorphic* to $G' = (V', E')$ if there exists a bijection $\phi$ between $V$ and $V'$ such that $uv \in E \Leftrightarrow \phi(u)\phi(v) \in E'$. The graph $G$ is said to *contain $H$ as an induced subgraph* if there exists $X \subseteq V$ such that $G[X]$ is isomorphic to $H$. A graph is *$H$-free* if it does not contain $H$ as an induced subgraph. Given a family of graphs $\mathcal{H}$, a graph $G$ is said to be *$\mathcal{H}$-free* if it is $H$-free for every $H \in \mathcal{H}$. If $\mathcal{H} = \{H_1, H_2\}$, we abuse notation and say that $G$ is $(H_1, H_2)$-free instead of $\{H_1, H_2\}$-free.

# Chapter 1

# Perfect graphs and Coloring

As described in the Introduction, the most well-known type of interaction between cliques and stable sets comes from *graph coloring* and the very first question that comes to mind is *how many colors do we need to color a graph properly?* or more precisely *what guarantee on the number of the colors can we have, depending on the other graph parameters?* In Section 1.1, we survey historical results on perfect graphs, graphs for which the number of colors is the best we can hope, and their relationship with Berge graphs. We carry on with $\chi$-bounded classes of graphs, that is to say graphs which are not perfect, but for which the number of colors still has a good behavior with respect to another graph parameter, called the clique number. We focus in particular on forbidding some cycle lengths, which leads us in Section 1.2 to bound the number of colors needed in graphs with no even cycle length except 4, and no triangles.

Note that the results of this chapter (Section 1.2) are covered in:

[I] **Coloring graphs with no even holes $\geqslant 6$: the triangle-free case**, preprint on *ArXiv*, *Computer Science* > *Discrete Mathematics*, abs/1503.08057.

## 1.1 Context and Motivations

### 1.1.1 Perfect graphs

The three following graph parameters appeared to be of great importance in many graph theory problems, and are well-known to be NP-hard to compute in the general case [132, 99]:

- The *chromatic number* of $G$, denoted by $\chi(G)$: it is the minimum number $k$ of colors such that one can assign a color to every vertex by means of a *coloring* $f : V \to \{1, \ldots, k\}$ with the condition that any two adjacent vertices are assigned different colors (such a coloring is called *proper*).
- The *stability number* of $G$, also called the *independence number* of $G$, denoted by $\alpha(G)$: it is the maximum cardinality of a *stable set* (or *independent set*) of $G$, that

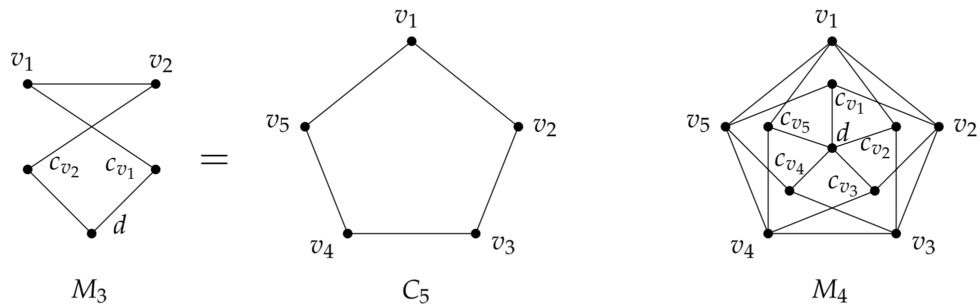



FIGURE 1.1: Some Micielski graphs. The first Micielski graph $M_2$ is just an edge $v_1v_2$. The second one $M_3$ is isomorphic to a 5-cycle.

is to say a subset $S \subseteq V(G)$ of vertices such that any two vertices of $S$ are non-adjacent.

- The *clique number* of $G$, denoted by $\omega(G)$: it is in some sense the complement notion of the stability number, since it is the maximum cardinality of a *clique*, that is to say a subset $K \subseteq V(G)$ of vertices such that any two vertices of $K$ are adjacent.

Since in a coloring, any two vertices of a clique need two different colors, we obviously have $\chi(G) \geqslant \omega(G)$. One could hope that we always have the equality $\chi(G) = \omega(G)$, but it is well-known to be false: consider any cycle of odd length, then $\chi(G) = 3$ but $\omega(G) = 2$. In fact this gap between $\chi(G)$ and $\omega(G)$ can be made arbitrarily large, as we can observe for instance on Micielski's nice recursive construction [162] (see Figure 1.1): the first Micielski graphs $M_2$ is the graph on two vertices with an edge joining those two vertices. The $k$-th Micielski graph $M_k$ is constructed from $M_{k-1}$ by:

*(i)* Duplicating every vertex $v$ into a copy $c_v$ of $v$. The new vertex $c_v$ is not adjacent to $v$, but is adjacent to every neighbor of $v$ (the vertex $c_v$ is called a *clone* of $v$ since they have the same set of neighbors in $M_{k-1}$).

*(ii)* Adding an extra vertex $d$, adjacent to every vertex $c_v$ created at step (i).

Then Micielski graphs have interesting properties:

**THEOREM 1.1** [162]

*Every Micielski graph $M_k$ is triangle-free and satisfies $\chi(M_k) = k$.*

*Proof.* We prove it by induction on $k$. This is trivially true for $k = 2$. Assume that the statement holds for $M_{k-1}$ and let us prove it for $M_k$. If $M_k$ has a triangle, then it can use at most one vertex $c_v$ from step (i), and it does not contain the vertex $d$ from step (ii) since $\{c_v \mid v \in V(M_{k-1})\}$ is a stable set. Now if $u$ and $w$ are the two others vertices of the triangle, then $uvw$ is a triangle in $M_{k-1}$, a contradiction with the induction hypothesis. It is also easy to prove that $\chi(M_k) \leqslant k$: by induction hypothesis, color $M_{k-1}$ with $k-1$ colors. Color every clone $c_v$ with the same color as $v$ in $M_{k-1}$, and add a new color for $d$. The interesting part is to prove that $\chi(M_k) \geqslant k$: consider an optimal coloring of $M_k$ with only $k-1$ colors. To get a contradiction, it is enough to prove that $k-1$ different colors appear on the vertices created at step (i), since the vertex $d$ created at step (ii) needs one extra color. By induction hypothesis, the subgraph $M$ isomorphic to $M_{k-1}$ contains all $k-1$ different colors. Moreover, for each color $i$, there exists a vertex $v_i$ whose neighborhood in $M$ contains all $k-2$ other colors (otherwise we could recolor all vertices of color $i$ and save one color). Since $c_{v_i}$ and $v_i$ have the same neighborhood in $M$, color $i$ must be given to $c_{v_i}$. We reach a contradiction. □

Thus $M_k$ is a testifier of the unbounded gap between $\chi$ and $\omega$[1]. That is why we wonder *what kind of property would ensure that $\chi(G) = \omega(G)$?* The question is in fact not very interesting as it is. Consider for example the graph $G$ obtained by the disjoint union of the $k$-th Mycielski graph $M_k$ and the complete graph on $k$ vertices $K_k$. Then $\chi(G) = \omega(G) = k$, thus it fulfills the above condition, although one part of it, $M_k$, does not have a nice coloring behaviour. To avoid such an artificial construction, we adjust our requirements: we say that a graph is *perfect* if $\chi(H) = \omega(H)$ for all of its induced subgraphs $H$.

The concept of perfect graphs appeared in the 1960's in Shannon's work on *zero-error capacity for a noisy channel* [189]. Moreover, several earlier min-max theorems (Kőnig's [136, 137], Dilworth's [64], Erdős-Szekeres' [70], Vizing's [207] and Mirsky's [160] theorems) can be restated in terms of perfectness of some class of graphs, which supports the importance of perfect graphs and shows that they are a very natural object to study. We introduce here some well-known subclasses of perfect graphs:

- **bipartite graphs:** a graph $G$ is *bipartite* if $V(G)$ can be partitioned into two stable sets $A$ and $B$. Thus, for every induced subgraph $H$ of $G$ which is not a not a stable set, $\chi(H) = \omega(H) = 2$ and $G$ is trivially perfect. Equivalently, a bipartite graph is a graph that contains no (induced)[2] odd cycles. Knowing that $G$ is bipartite, we often write $G = (A \cup B, E(G))$ to explicitly give names to the two stable sets the graph is made from. A *complete bipartite graph* is a bipartite graph $G = (A \cup B, E(G))$ such that $E(G)$ contains all the possible edges having one endpoint in $A$ and the other endpoint in $B$.

- **complements of bipartite graphs:** the perfectness of these graphs is equivalent to the well-known Kőnig's theorem [137] stating that the size of the maximum matching in any bipartite graph is the same as a minimum vertex cover (a *matching* is a set of edges that pairwise do not share an endpoint; a *vertex cover* is a

[1] In fact, if one is interested only in finding a gap, there is a much simpler construction (see *e.g.* [114]): consider $k$ disjoint 5-cycles, and add all possible edges between any two copies. Then $\chi(G) = 3k$ but $\omega(G) = 2k$. Micielski's construction is stronger since it provides triangle-free graphs.

[2] We can drop the *induced* here: indeed a graph has no odd cycles if and only it has no induced odd cycles, since the shortest odd cycle in a graph has to be induced.

set of vertices that contains at least one endpoint of every edge). Indeed, on the one hand a bipartite graph $G$ admits a matching $M$ of size $\nu_M$ if and only if $\overline{G}$ can be colored with at most $|V(G)| - \nu_M$ colors (every edge of the matching gives a pair of vertices that get the same color in $\overline{G}$, and all the other vertices get different colors). On the other hand, $C$ is a vertex cover of $G$ if and only if the complement of $C$ is a clique in $\overline{G}$, thus if $\nu$ is the size of the minimum vertex cover (equivalently, maximum matching), then $\omega(\overline{G}) = |V(G)| - \nu = \chi(\overline{G})$. Since the class of complements of bipartite graphs is hereditary, this proves their perfectness.

- **line graphs of bipartite graphs:** the *line graph* $L(G)$ of a graph $G = (V, E)$ is defined as follows: the set of vertices of $L(G)$ is the set $E$ of edges of $G$, and $e, e' \in E$ are adjacent in $L(G)$ if and only if $e$ and $e'$ share a common endpoint in $G$ (see Figure 1.2 for an example). Not every graph is the line graph of some graph, in particular the so-called *claw* depicted on Figure 1.2 is forbidden in any line graph as an induced subgraph. $G$ is called the *root* of $L(G)$. The perfectness of line graphs of bipartite graphs is equivalent to Kőnig's Line Coloring theorem [136], stating that the edge chromatic number (the minimum number of colors needed to color the edges such that two edges sharing an endpoint are given different colors) of a bipartite graph $G$ is equal to the maximum degree $\Delta$ of $G$ (indeed there is a one-to-one correspondence between cliques of $L(G)$ and neighborhoods of a vertex of $G$, and between colorings of $L(G)$ and edge colorings of $G$).

- **complements of line graphs of bipartite graphs:** interestingly, their perfectness is also equivalent to Kőnig's theorem [137]: a matching in a bipartite graph $G$ is exactly a stable set in $L(G)$, and thus exactly a clique in $\overline{L(G)}$. Similarly, a vertex cover of $G$ is exactly a set of cliques in $L(G)$ that covers every vertex of $L(G)$, and thus a coloring in $\overline{L(G)}$.

- **chordal graphs[3]:** a graph $G$ is *chordal* (also called *triangulated*) if it has no *hole*, *i.e.* no induced cycle of length 4 or more. It is well-know [95] that every chordal graph has a *simplicial* vertex, that is a vertex whose neighborhood is a clique. Now it is easy to prove by induction on $|V(G)|$ that every chordal graph is perfect: let $G = (V, E)$ be a chordal graph and $v$ be a simplicial vertex of $G$. Since chordal graphs are hereditary, we just have to prove that $\chi(G) = \omega(G)$. Color by induction $G' = G[V \setminus \{v\}]$ with $k = \omega(G')$ colors. Now all we need to do is to properly color $v$: either $v$ has less than $k$ neighbors, then one of the $k$ colors is not used on its neighborhood and can be given to $v$. Or $v$ has exactly $k$ neighbors, but then $v$ and its neighbors form a clique of size $k+1$ and thus we are allowed to choose a new color for $v$ to get $\chi(G) = \omega(G) = k+1$. Note that chordal graphs also have structural properties in terms of tree decomposition [100], which are not dealt with here.

[3]Historical note: in the 1950-60's, they were known as Gallai graphs.

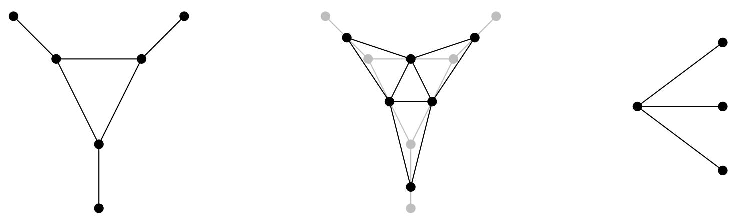


FIGURE 1.2: From left to right: the *net*, its line graph (superimposed on a gray copy of the net, in the background) and the *claw*.

- **comparability graphs**: a graph $G = (V, E)$ is a *comparability* graph if one can orient the edges into arcs (meaning that for every edge $uv \in E$, one has to choose if the edge goes from $u$ to $v$ or from $v$ to $u$) in order to fulfill the following requirement: if there is an arc from $u$ to $v$ and an arc from $v$ to $w$, then there should be an arc from $u$ to $w$ (the resulting relation is *transitive*). The most well-known graphs that are not comparability graphs are odd holes, and the so-called *net*, depicted on Figure 1.2. Mirsky's theorem states that comparability graphs are perfect [160], and Dilworth's theorem states that their complements are perfect [64].

Claude Berge is commonly acknowledged as the founding father of the study of perfect graphs. He observed that all non-perfect graphs the community could produce either contain a hole of odd length (these are called *odd holes*, for short), or, as first observed by Berge's student Ghouila-Houri [104], the complement of an odd hole (called *odd antihole*), which are themselves not perfect. This led him to conjecture that these were the only possible obstructions for a graph to be perfect. Because of his strong interest for those graphs, there were later on named after him: a *Berge graph* is a graph which does not contain any odd hole or any odd antihole as an induced subgraph. Observe that any perfect graph is thus Berge. The conjecture, restated as *a graph is perfect if and only if it is Berge* was quickly known as the *Strong Perfect Graph Conjecture*. However, he estimated this conjecture to be a hard goal to reach, so he produced a weaker statement (soon referred to as the *Weak Perfect Graph Conjecture*), which would be obviously implied by the stronger one: *a graph is perfect if and only if its complement is perfect*.

Those two questions generated a lot of interest from the researchers in discrete mathematics and are the foundations of a whole field of graph theory. After an attempt by Fulkerson in 1971, which failed to reach the goal but gave rise to his theory on anti-blocking pairs of polyhedra [94], the Weak Perfect Graph Conjecture was solved in 1972 by Lovász [150][4]. The proof technique is of interest in itself, so let us provide it here the details: let $G$ be a graph and $v \in V(G)$, we say that $G'$ is obtained

[4]Historical note [167]: Fulkerson observed that proving the Replication Lemma would be enough to prove the Weak Perfect Graph Conjecture, but this would also have implied a statement which he found too strong to be true. When he learned that Lovász had proved it, he managed to fill the missing lines in a few hours.

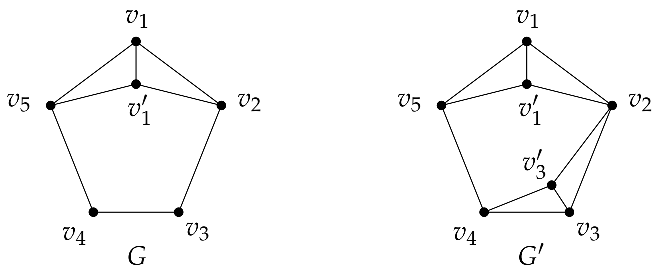



FIGURE 1.3: The graph $G$ is obtained from a 5-cycle by replicating $v_1$. The graph $G'$ is obtained from $G$ by replicating $v_3$. Moreover $\chi(G) = \omega(G)$ whereas $\chi(G') > \omega(G')$.

from $G$ *by replication of $v$* if $G'$ is obtained from $G$ by adding a new vertex $v'$, adjacent to $v$ and to all the neighbors of $v$ in $G$.

**LEMMA 1.2** [150] (*Replication Lemma*)

*If $G'$ is obtained from a perfect graph $G$ by replicating a vertex, then $G'$ is perfect.*

*Proof.* Let $v$ be the replicated vertex and $v'$ be the new vertex. Observe that any induced subgraph of $G'$ that does not contain both $v$ and $v'$ is isomorphic to an induced subgraph of $G$ and thus is perfect. So let $H'$ be an induced subgraph of $G'$ such that $v, v' \in V(H')$, and let $H$ be the perfect graph obtained from $H'$ by deleting $v'$.

Color $H$ with $k = \omega(H)$ colors, and let us try to find a suitable color for $v'$: either $v$ belongs to some maximum clique $K$ of $H$, and then $K \cup \{v'\}$ is a clique of $H'$ (by definition of replication, $v'$ is adjacent to all the neighbors of $v$). In that case, $\omega(H') = k + 1$ so we can use a new color for $v'$.

Otherwise, let $S \subseteq V(H)$ be the set of vertices given the same color as $v$. Since $H$ is colored with $k = \omega(H)$ colors, every maximum clique of $H$ has a vertex in $S$, and even in $S \setminus \{v\}$ since $v$ is in no maximum clique. Consequently, $H \setminus (S \setminus \{v\})$ has clique number at most $k - 1$ and is perfect so it has a coloring with $k - 1$ colors. Use one extra color for $S$. Since $v$ has no neighbors in $S$, neither does $v'$, so color $v'$ with the same extra color. This is a coloring of $H'$ with $k = \omega(H')$ colors. □

To provide evidence that the Replication Lemma is nevertheless slightly unexpected, one can observe the following behaviour, depicted on Figure 1.3 (this example appears *e.g.* in [199]): let $G$ be the graph obtained from a 5-cycle denoted $v_1v_2v_3v_4v_5$, by replicating $v_1$ into $v_1'$. Then $\omega(G) = \chi(G) = 3$. Now let $G'$ be the graph obtained from $G$ by replicating $v_3$. Now $\chi(G') = 4 > \omega(G') = 3$. So the property $\chi = \omega$ is not preserved by replication.

Using the Replication Lemma, Lovász was thus able to solve the weak conjecture:

**THEOREM 1.3** [150] (*Weak Perfect Graph Theorem*)

*If a graph $G$ is perfect, then so is its complement $\overline{G}$.*

*Proof.* Since perfectness is hereditary, it is enough to show that $\chi(\overline{G}) = \omega(\overline{G})$, *i.e.* that $G$ can be partitioned into $\alpha(G)$ cliques. Let us prove this by induction on $\alpha(G)$, the base case $\alpha(G) = 1$ being trivial.

Let $S_1, \ldots, S_t$ be all the maximum stable sets of $G$, thus all of size $\alpha(G)$. For every vertex $v \in V(G)$, let $\mathcal{S}(v) = \{S_{i_1}, \ldots, S_{i_k}\}$ be the set of maximum stable sets that contain $v$. We construct $G'$ as follows: for very vertex $v$, if $\mathcal{S}(v)$ is empty, $v$ is deleted, otherwise $v$ is duplicated $|\mathcal{S}(v)| - 1$ times. Call $v_{i_2}, \ldots, v_{i_k}$ the new vertices and rename $v$ into $v_{i_1}$. Now for each maximum stable set $S_i$ of $G$, one can construct a corresponding stable set $\{v_i \mid v \in S_i\}$ in $G'$. This gives a partition of $V(G')$ into $t$ stable sets of size $\alpha(G)$, thus $|V(G')| = t\alpha(G)$ and $\chi(G') \leqslant t$. This is in fact optimal: on the one hand $\alpha(G') = \alpha(G)$, and on the other hand $\chi(G') \geqslant |V(G')|/\alpha(G')$ (since a coloring is a partition of $V(G')$ into $\chi(G')$ stable sets of size at most $\alpha(G')$), which gives $\chi(G') \geqslant t$.

By the Replication Lemma (Lemma 1.2), $G'$ is perfect so there is a clique $K'$ in $G'$ of size $t$. Construct a clique $K$ of $G$ by setting $K = \{v \in V(G) | \exists i \; v_i \in K'\}$. Since $K'$ cannot contain any two vertices $u_i, v_i$ such that $u$ and $v$ belong to the same maximum stable set of $G$, then for every $i$, $K'$ contains a vertex $v_i$ such that $v \in S_i$. Consequently $K$ intersects every maximum stable set of $G$. So $\alpha(G \setminus K) \leqslant \alpha(G) - 1$, so by induction hypothesis, $V(G) \setminus K$ can be partitioned into $\alpha(G) - 1$ cliques, hence $V(G)$ can be partitioned into $\alpha(G)$ cliques. □

Subsequently to the successful approach of the Replication Lemma, a lot of attention was drawn on *how to build a bigger perfect graph out of two (or several) smaller perfect graphs?* We give here some historically important examples of such operations (illustrations can be found on Figure 1.4). Observe that it is sometimes more convenient to consider the situation the other way around: *can we cut a perfect graph into smaller pieces in order to have a nice decomposition?* Here, by nice decomposition we mean a decomposition that cannot appear in a *minimally imperfect graph* $G$, *i.e.* a non-perfect graph any proper induced subgraph of which is perfect.

- **Clique-cutset** (introduced by Gallai in 1962 [96]): a graph $G$ has a *clique-cutset* if $V(G)$ can be partitioned into three non-empty parts $(A, B, C)$ such that $C$ is a clique and $A$ is *anticomplete* to $B$, meaning that there is no edge with one endpoint in $A$ and the other in $B$.

- **Substitution** (introduced by Gallai in 1967 [97]): Given two graphs $G_1$ and $G_2$ on disjoint vertex sets $V_1$ and $V_2$ respectively, each with at least two vertices, and $v \in V_1$, we say that $G$ is *obtained from $G_1$ by substituting $G_2$ for $v$*, or *obtained from $G_1$ and $G_2$ by substitution* in a less detailed manner, if

  *(i)* $V(G) = (V_1 \cup V_2) \setminus \{v\}$

  *(ii)* $G[V_2] = G_2$

  *(iii)* $G[V_1 \setminus \{v\}] = G_1[V_1 \setminus \{v\}]$

  *(iv)* $v_1 \in V_1$ is adjacent in $G$ to $v_2 \in V_2$ if and only if $v_1$ is adjacent to $v$ in $G_1$.

Observe in particular that replicating $t$ times a vertex $v$ in $G$ gives the same graph as substituting a clique of size $t$ for $v$ in $G$. Observe also that all vertices from $V_2$ have exactly the same neighbors in $V_1$, for this reason $V_2$ is also called a *homogeneous set* or a *module* in $G$.

- **1-join** (introduced by Cunningham in 1982 [58]): a graph $G$ has a 1-join if $V(G)$ can be partitioned into $(B_1, A_1, A_2, B_2)$ such that $A_1$ is *complete* to $A_2$ (*i.e.* for every $a_1 \in A_1$, $a_2 \in A_2$, the pair $a_1a_2$ forms an edge in $E(G)$), and these are the only edges between $A_1 \cup B_1$ and $A_2 \cup B_2$. Moreover it is required that $|A_1 \cup B_1|, |A_2 \cup B_2| \geqslant 2$.

- **2-join** (introduced by Cornuéjols and Cunningham in 1985 [57]): a graph $G$ has a *2-join* if $V(G)$ can be partitioned into $(A_1, B_1, C_1, A_2, B_2, C_2)$ with the following conditions, where $X_1 = A_1 \cup B_1 \cup C_1$ and $X_2 = A_2 \cup B_2 \cup C_2$ (the most important one is (ii), the other ones are rather technical):

  *(i)* $A_1, A_2, B_1$ and $B_2$ are non-empty.

  *(ii)* $A_1$ is complete to $A_2$, $B_1$ is complete to $B_2$, and there are no other edges between $X_1$ and $X_2$.

  *(iii)* for $i = 1, 2$, $|X_i| \geqslant 3$.

  *(iv)* for $i = 1, 2$, every component of $G[X_i]$ intersects both $A_i$ and $B_i$.

  *(v)* for $i = 1, 2$, if $A_i$, $B_i$ and $C_i$ are all singletons ($\{a\}$, $\{b\}$, $\{c\}$, respectively) then $G[X_i]$ is different from the path $acb$.

- **Star-cutset** (introduced by Chvátal in 1985 [49]): a graph $G$ has a *star-cutset* if $V(G)$ can be partitioned into non-empty sets $(A, B, S)$ such that $A$ is anticomplete to $B$ and $S$ contains a vertex $v$ which is complete to $S \setminus \{v\}$ (but $S \setminus \{v\}$ is not required to be a stable set, contrary to the case of an induced *star*). Observe that if an *anticonnected* graph $G$ (*i.e.* $\overline{G}$ is connected, which can most of the time be assumed for otherwise one can easily color $G$ by induction) has a clique-cutset, a 1-join or a homogeneous set, then it has a star-cutset.

**Lemma 1.4** [96, 97, 58, 57, 49]

*Let $G$ be a minimally imperfect graph (in particular $\overline{G}$ is connected). Then $G$ has no star-cutset, and hence no clique-cutset, no homogeneous set, no 1-join. Moreover if $G$ is not an odd hole, $G$ has no 2-join.*

We give here the proof for the star-cutset case (more specific proofs exist for the special cases of clique-cutset, homogeneous set and 1-join, with better algorithmic properties) :

*Proof.* Let $G$ be a minimally imperfect graph and suppose it has a star-cutset $(A, B, S)$ and call $v_1$ the (maybe not unique) vertex of $S$ adjacent to all other vertices of $S$. Then

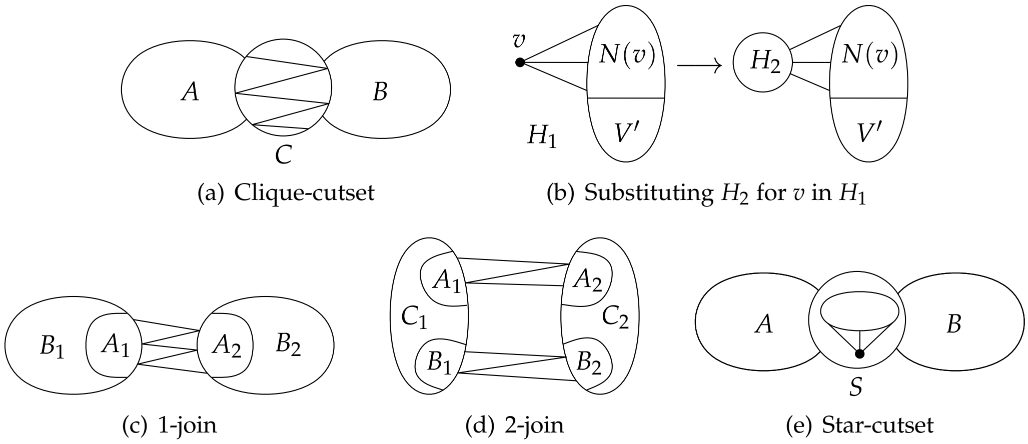
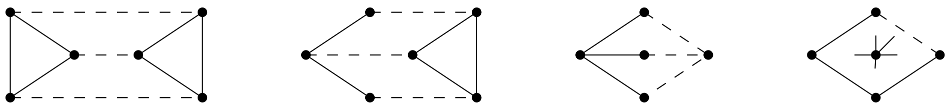



FIGURE 1.4: Diagrams for the described decompositions.

by minimality of $G$, the graphs $G[A \cup S]$ and $G[B \cup S]$ are both perfect. Moreover, $\omega(G) = \max(\omega(G[A \cup S]), \omega(G[B \cup S]))$ so $G[A \cup S]$ and $G[B \cup S]$ can both be colored with $\omega(G)$ colors. Up to renaming to colors, we can assume that $v_1$ is given color 1 in both colorings. Moreover no other vertex of $S$ is colored 1, and the set $S_1$ of vertices of $G$ colored by 1 in any of those two colorings is indeed a stable set since no edge crosses from $A$ to $B$. Now observe that both $G[(A \cup S) \setminus S_1]$ and $G[(B \cup S) \setminus S_1]$ are colored with $\omega(G) - 1$ colors. Since any clique of $G \setminus S_1$ is a clique in one of these two graphs, we have the inequality $\omega(G \setminus S_1) \leqslant \omega(G) - 1$. But $G \setminus S_1$ is perfect, so it can be colored by $\omega(G) - 1$ colors. Adding $S_1$ to this coloring shows that $\chi(G) \leqslant \omega(G)$, a contradiction. □

Each of these results was a new step forward the Strong Perfect Graph Conjecture, but Chvátal believed that a self-complementary version of the star-cutset should come into play. That is why he introduced the skew-partition in 1985 [49]: a graph $G$ has a *skew-partition* if $V(G)$ can be partitioned into two parts $(X, Y)$ such that $G[X]$ is not connected and $\overline{G[Y]}$ is not connected. Observe that any star-cutset $(A, B, S)$ provides a skew-partition since $G[A \cup B]$ and $\overline{G[S]}$ are not connected (provided $|S| \geqslant 2$, which can be ensured if $|V(G)| \geqslant 5$ and $G$ is not a stable set). However, he could not prove that a minimally imperfect graph has no skew-partition. Several attempts were made and succeeded for particular types of skew-partitions (see Reed [172] for a survey).

In the meantime, a new ingredient became of great importance: the *Truemper configurations*, namely the prism, the theta, the pyramid and the wheel (see Figure 1.5; formal definitions are not provided here). Around 2000, Conforti, Cornuéjols and Vušković designed a new fruitful approach for using them in a thorough study of even-hole-free graphs (with Kapoor [51, 52]). This led them to the solving of special cases of the Strong Perfect Graph conjecture, among which a noteworthy decomposition theorem for square-free perfect graphs [54], and to a decomposition theorem for odd-hole-free graphs [53] (which do not imply the Strong Perfect Graph Conjecture).



(a) A *prism* (b) A *pyramid* (c) A *theta* (d) A *wheel*

FIGURE 1.5: Truemper configurations. Dashed edges stand for induced paths with no edge (chord) from one to each other. The center of the wheel has at least 3 neighbors on the hole.

We refer the reader to Vušković's survey on Trumper configurations [209] and to the devoted Section in Trotignon's survey [199] for further details.

Finally, using this approach, Chudnovsky, Robertson, Seymour and Thomas managed to prove the Strong Perfect Graph Conjecture in 2002:

**THEOREM 1.5** [40] (*Strong Perfect Graph Theorem*)

*A graph is perfect if and only if it is Berge.*

One of the breakthrough that finally made the proof works is an additional technical condition in the definition of a skew-partition: a (skew or not) partition $(A, B)$ of $V(G)$ is *balanced* if every path in $G$ of length at least 3, with ends in $B$ and interior in $A$, and every path in $\overline{G}$ of length at least 3, with ends in $A$ and interior in $B$, has even length. Such a definition is designed to ensure that a partition $(A, B)$ of a Berge graph is balanced if and only if adding a vertex complete to $B$ and anticomplete to $A$ yields a Berge graph. With this new definition in mind, they were able to find a decomposition theorem for Berge graphs, which has been simplified the year after by Chudnovsky using so-called trigraphs (to be defined and used in Section 3.5). In the following, a graph G is a *double split graph* if $V(G)$ can be partitioned into $(A, B)$ such that both $G[A]$ and $\overline{G[B]}$ are disjoint unions of edges, and for every edge $a_1a_2 \in G[A]$ and non-edge $b_1b_2 \in G[B]$, both $a_1$ and $a_2$ have exactly one neighbor among $\{b_1, b_2\}$ and this neighbor is not the same for both of them. It is not difficult to prove that double split graphs are perfect.

**THEOREM 1.6** [40, 32] (*Decomposition Theorem for Berge graphs*)

*Let G be a Berge graph. Then one of the following holds:*

(i) *$G$ or $\overline{G}$ is a bipartite graph, or*

(ii) *$G$ or $\overline{G}$ is the line graph of a bipartite graph, or*

(iii) *$G$ is a double split graph, or*

(iv) *$G$ or $\overline{G}$ has a 2-join, or*

(v) *$G$ has a balanced skew-partition.*

When Case (i), (ii) or (iii) occurs, $G$ is called *basic* (because it lies in some well understood subclasses of perfect graphs). Otherwise $G$ is said to have a decomposition. The hard part of the proof of the Strong Perfect Graph Theorem is the decomposition theorem, and the easier part is to prove that a minimally imperfect Berge graph cannot have such a decomposition.

For all but two aforementioned decompositions, it is possible to find a polynomial-time algorithm to recursively color the graph (quite directly from the proof for the clique-cutset, substitution and 1-join, but for 2-joins it is the object of a full paper [200]). Unfortunately, for the star-cutset and its generalization, the (balanced or not) skew-partition, the only known algorithm (following the outline of the proof) does not run in polynomial time, if one has no control on the size of the blocks in the partition (a more detailed discussion can be found in Trotignon's survey [199]). We finally mention here that clique-cutset, homogeneous set, 1-join, 2-join and skew-partition can all be detected in polynomial time [195, 115, 30, 31, 134], whereas balanced skew-partitions are NP-complete to detect [198]. However, in the same paper Trotignon proved that balanced skew-partitions can be detected in polynomial time when restricted to Berge graphs, which is the interesting case anyway.

The Strong Perfect Graph Theorem solved the 40-year-old conjecture, but did not closed one other big open question about perfect graphs: *does there exist a combinatorial algorithm to optimally color a perfect graph in polynomial time?* It is crucial to stress the importance of *combinatorial* in the previous question, since a polynomial-time algorithm has been discovered by Grötchel, Lovász and Schrijver in 1985 [110] (based on Lovász $\vartheta$-function introduced in 1979 [151]). It is not combinatorial in the sense that it relies on semi-definite programming and more precisely on the *ellipsoid method*, a powerful tool in combinatorial optimization that enables in particular to compute in polynomial time a *maximum weighted stable set* in any perfect graph (every vertex is given a weight, and the weight of a stable set is the sum of the weight over all its vertices; instead of asking for a stable set of maximum cardinality, we now ask for one of maximum weight. It is equivalent to the unweighted case if one assigns the same weight to all vertices). Then they designed a polynomial time algorithm that optimally colors a perfect graph, using the maximum weighted stable set algorithm as a *black box*. Note that, besides the black box calls, this algorithm is combinatorial.

Chudnovsky, Trotignon, Trunck and Vušković [46] still managed to achieve a partial result by designing a polynomial-time algorithm to color perfect graphs with no balanced skew-partitions. In a word, the difficulty lies in proving that after decomposing a perfect graph with no balanced skew-partition, the two smaller graphs we get still have no balanced skew-partition. Note that their proof uses trigraphs as well, and Section 3.5 deeply relies on their results.

Since the Strong Perfect Graph Theorem was proved, the research in this field has continued and some other open problems have been closed, for instance the polynomial-time recognition of Berge graphs [36]. Some others remain widely open, such as the 3-clique-coloration of perfect graphs: *can we (non-properly) color all perfect graphs with three colors such that every inclusion-wise maximal clique gets at least two different colors?* Note that Berge graphs with no balanced skew-partition are 2-clique-colorable [166], so once more the hard case is concerned with balanced skew-partition. Another range of questions came from the world surrounding the concept of *even pair*, which is a pair of vertices between which all induced paths have even length. It is another kind of structural feature which cannot appear in a minimally imperfect graph, as proved by Meyniel in 1987 [158]. Furthermore it was shown by Fonlupt and Uhry [85] that an even pair can be contracted into a single vertex without modifying the chromatic number (nor the clique number), which is a nice property for algorithm purposes. Moreover, Chudnovsky and Seymour [44] were able to shorten 50 pages of the proof of the Strong Perfect Graph Theorem with the help of this structural feature, and several subclasses of perfect graphs have been shown to have an even pair. This shows that even pairs can be of great importance (see [74] for a survey). Ideally, one could dream of a decomposition theorem for Berge graphs that got rid of the balanced skew-partition for the benefit of even pairs: *is it true that every Berge graph satisfies the following: either $G$ or $\overline{G}$ is basic, or $G$ or $\overline{G}$ has a 2-join, or $G$ has an even pair?* Such a theorem would provide a combinatorial polynomial-time algorithm to color perfect graphs. However, there is a counterexample to this statement (discovered by Chudnovsky and Seymour and then nicknamed "The Worst Berge Graph Known So Far" in [199]), so the most reasonable remaining statements are the following (unfortunately, one does not know how to handle an even pair in the complement for coloring algorithm purposes):

**CONJECTURE 1.7** [Thomas, 2002, unpublished]

*If a Berge graph $G$ is uniquely decomposable by a balanced skew-partition (so $G$ is not basic, has no 2-join and no 2-join in the complement), then one of $G$ or $\overline{G}$ has an even pair.*

**QUESTION 1.8** (Restated from [199])

*Let $G$ be a Berge graph with no even pair, no 2-join, no 2-join in the complement, and not basic. Is there an easy way to algorithmically color $G$ by introducing a new type of decomposition or a new basic class?*

For further information about perfect graphs, we recommend the two devoted books (*Perfect Graphs* [167] and *Topics on Perfect Graphs* [12]). One can also refer to Trotignon's recent survey [199], or to Lovaśz survey from the 1980's [152]. For an outline of the proof of the Strong Perfect Graph Theorem, see [39], and for a survey on the different attempts, see [178]. For chronological details, one can read [187] where Seymour sums up the story of how and when the Strong Perfect Graph Theorem was proved.

As the initiated reader may observe, this short survey focuses on structural properties of perfect graphs and Berge graphs. However, perfect graphs gave also rise to a fruitful branch of research in the field of combinatorial optimization (see Schrijver's very complete series of books *Combinatorial Optimization* [183]). Some of these aspects are dealt with in Chapters 3 and 4. In particular, they are at the origin of the Clique-Stable Set separation problem, to which Chapter 3 is dedicated.

### 1.1.2 $\chi$-boundedness

In the turmoil around perfect graphs, Gyárfás tried to generalize them in order to have a more global view. He hence introduced the concept of $\chi$-bounded classes [113]: a family $\mathcal{C}$ of graphs is called *$\chi$-bounded* if there exists a function $f$ (called the *$\chi$-bounding function*[5]) such that $\chi(G') \leqslant f(\omega(G'))$ holds whenever $G'$ is an induced subgraph of $G \in \mathcal{C}$. The class of all graphs is not $\chi$-bounded, as proved for instance by Mycielski graphs (described in Subsection 1.1.1), Tutte graphs[6] [62] and Zykov graphs [214] which provide families of triangle-free graphs with arbitrarily high chromatic number. Moreover, perfect graphs can be defined as the class of graphs with $\chi$-bounding function $f(x) = x$. Surprisingly, besides the long and complicated proof of the Strong Perfect Graph Theorem, for over 10 years there has been no other proof of the $\chi$-boundedness of Berge graphs, even with a much larger $\chi$-bounding function[7].

This notion has been widely studied since, in particular in hereditary classes for which it is enough to prove $\chi(G) \leqslant f(\omega(G))$ for every $G \in \mathcal{C}$. A classical result of Erdős [71] asserts *via* a probabilistic argument that there exist graphs with arbitrarily large *girth* (that is, the length of the shortest induced cycle) and arbitrarily large chromatic number. Thus forbidding only one induced subgraph $H$ may lead to a $\chi$-bounded class only if $H$ is acyclic. Gyárfás conjectured that this condition is also sufficient:

**CONJECTURE 1.9** [112]

*Let $H$ be a forest. The class of $H$-free graphs is $\chi$-bounded.*

He proved Conjecture 1.9 for paths:

**THEOREM 1.10** [113]

*For all $k \geqslant 2$, every $P_k$-free graph $G$ satisfies $\chi(G) \leqslant (k-1)^{\omega(G)-1}$.*

Since the proof technique for the triangle-free case became quite classical and will be useful later on (Lemma 2.20, see Chapter 2), we provide here a sketch of proof of Theorem 1.10 for this special case.

*Sketch of proof.* We prove by induction on $k$ the following statement: for every connected triangle-free graph $G$, either $\chi(G) \leqslant k$ or for every vertex $v$ of $G$ there exists an induced path of length $k$ starting at $v$ (*i.e.* $v$ is one extremity of the path). The case $k = 1$ is trivial. Let $G$ be a connected triangle-free graph and $v$ be a vertex of $G$. Partition $V(G)$ into three parts: the first one is $\{v\}$; the second one is $N(v)$, which is a stable set, thus it has *small chromatic number*; and the last one is $V'$, the rest of the graph. If $\chi(V') \leqslant k-1$, then we color $v$ with any of the already existing color, and use

[5]Historical note: in Gyárfás' paper, it was introduced as *$\chi$-binding* function, but in most recent papers, it is called a *$\chi$-bounding* function.

[6]Historical note: as several other papers in the 1940-50's, Brooks, Stone, Smith and Tutte published it under the pseudonym *Blanche Descartes*.

[7]Now, such a proof has been provided by Scott and Seymour for odd-hole free graphs, see [185] and discussion on Conjecture 1.11 below.

one extra color for $N(v)$ to get the first outcome. Otherwise, there exists a connected component $C'$ of $V'$ with *big chromatic number*, *i.e.* $\chi(C') \geqslant k$. Let $w \in N(v)$ be a vertex having a neighbor in $C'$, which exists since $G$ is connected. Apply the induction hypothesis with parameter $k-1$ to $G[\{w\} \cup C']$. By assumption on $C'$, the first outcome cannot occur, so there is an induced path of length $k-1$ starting at $w$ and with interior in $C'$. Adding the edge $vw$ gives an induced path of length $k$ starting at $v$. □

Some other evidence support Conjecture 1.9: it has also been proved if $H$ is a star [113], a tree of radius two [138], or a tree of radius three with additional technicial conditions [139]. The most important step so far was provided by Scott [184]: he answered positively to this question for any tree $H$ if we forbid every induced subdivision of $H$, instead of just $H$ itself (a *subdivision* of $H$ is a graph obtained from $H$ by replacing the edges by induced paths; in particular an edge can be preserved as it is, since it is an induced path of length 1).

Because of Erdős' result, forbidding holes in order to get a $\chi$-bounded class is conceivable only if we forbid infinitely many hole lengths. Two parameters should be taken into account: in the first place, the length of the holes, and in the second place, the parity of their length. In this respect, Gyárfás made the following three conjectures:

**Conjecture 1.11** [113] (solved in [185])

*The class of graphs with no odd holes is $\chi$-bounded.*

**Conjecture 1.12** [113]

*For every $k \geqslant 4$, the class of graphs with no holes of length $\geqslant k$ is $\chi$-bounded.*

It is further conjectured [120] that if $G$ has no holes of length $\geqslant k$ and is moreover triangle-free, then $\chi(G) \leqslant \max(k,4) - 2$. Gyárfás' last conjecture is even stronger than the two previous ones:

**Conjecture 1.13** [113]

*For every $k \geqslant 4$, the class of graphs with no odd holes of length greater or equal to $k$ is $\chi$-bounded.*

Conjecture 1.11 has recently been solved by Scott and Seymour [185] with a quite elegant proof. Together with Chudnovsky, they carried on with the other two conjectures, and get some partial answers described below. Observe that when triangles are forbidden, the maximum clique has size 2 and thus the upper bound on $\chi(G)$ obtained from the definition of $\chi$-boundedness is constant. Thus in this case, it makes more sense to say that the class of graph has *bounded chromatic number* rather than to say that the class is $\chi$-bounded.

**THEOREM 1.14** [42]

- *For every $k$, the class of graphs with no 5-hole and no hole of length at least $k$ is $\chi$-bounded.*
- *For every $k$, the class of graphs with no triangle, no 5-hole and no odd hole of length at least $k$ has bounded chromatic number.*
- *The class of graphs with no triangle, and no odd hole of length at least 7 has bounded chromatic number.*

Furthermore, the class of even-hole-free graphs has been extensively studied from a structural point of view. As already mentioned, a decomposition theorem together with a recognition algorithm have been found by Conforti, Cornuéjols, Kapoor and Vušković [51, 52, 28]. Reed conjectured that every even-hole-free graph has a vertex whose neighborhood is the union of two cliques (called a *bisimplicial* vertex) [117], which he and his co-authors proved a few years later [2]. As a consequence, they obtained that every even-hole-free graph $G$ satisfies $\chi(G) \leqslant 2\omega(G) - 1$.

Forbidding $C_4$ is in fact a strong restriction since $C_4$ can also be seen as the complete bipartite graph $K_{2,2}$: Kühn and Osthus [143] proved that for every graph $\Gamma$ and for every integer $s$, every graph of large average degree (with respect to $\Gamma$ and $s$) with no $K_{s,s}$ as a (not necessarily induced) subgraph contains an induced subdivision of $\Gamma$, where each edge is subdivided at least once. This strong result implies that $\chi$ is bounded in any class $\mathcal{C}$ defined as graphs with no triangles, no induced $C_4$ and no induced cycles of length 0 modulo $k$, for any fixed integer $k$. Indeed, let $G$ be a graph of minimum size such that $G \in \mathcal{C}$ and $\chi(G) > c_k$ (for some well-chosen $c_k$ depending on $k$ and *very large*), and let us reach a contradiction. Observe that $G$ has minimum degree at least $c_k$, otherwise we can remove a vertex $v$ of small degree, color $G \setminus \{v\}$ with $c_k$ colors by minimality of $G$, and reintroduce $v$ without needing an extra color. Consequently, $G$ has average degree at least $c_k$. Moreover it has neither induced $C_4$ nor triangles, so it has no $C_4$ subgraphs. By Kühn and Osthus' theorem (with $s = 2$ and $\Gamma = K_\ell$ for some well-chosen integer $\ell$ depending on $k$), there exists an induced subdivision $H$ of $K_\ell$ in $G$, where each edge of $K_\ell$ is subdivided at least once. Consider $K_\ell$ as an auxiliary graph where we color each edge with $c \in \{1, \ldots, k\}$ if this edge is subdivided $c$ times modulo $k$ in $H$. By Ramsey's theorem [168], if $\ell$ is large enough we can find a monochromatic clique $K$ of size $k$. Let $C_0$ be a Hamiltonian cycle through $K$ and call $C$ the corresponding cycle in the subdivided edges in $H$. Since $K$ was monochromatic in $K_\ell$, then the edges used in $C_0$ have been subdivided the same number of times modulo $k$, consequently $C$ has length 0 modulo $k$. Moreover, it is an induced cycle since each edge is subdivided at least once in $H$.

This is why we are interested in Section 1.2 in finding a $\chi$-boundedness result when every even holes except $C_4$ are forbidden, which has been conjectured by Bruce Reed [171]. We achieve a partial result by forbidding also triangles. This is a classical step towards $\chi$-boundedness, and Thomassé, Trotignon and Vušković even asked whether this could always be sufficient:

**QUESTION 1.15** [196]

*Given a graph $G$, denote by $\chi_T(G)$ the maximum chromatic number of a triangle-free induced subgraph of $G$. Does there exists a function $f$ such that for every graph $G$, we have $\chi(G) \leqslant f(\chi_T(G), \omega(G))$?*

# 1.2 Excluding even holes except $C_4$

The goal of this section is to prove the following theorem, whose motivation has been explained in the previous section:

**THEOREM 1.16**

*There exists a constant $c$ such that every graph $G$ with no triangles and no holes of even length at least 6 satisfies $\chi(G) < c$.*

This result is closely related to the following recent one, by Bonamy, Charbit and Thomassé, answering to a question by Kalai and Meshulam [129] (it was asked in homology terms, more precisely about turning a bound on the sum of the Betti numbers of the stable set complex to a bound on the chromatic number- see the book dedicated to simplicial complexes of graphs [124] for definitions):

**THEOREM 1.17** [17]

*There exists a constant $c$ such that every graph $G$ with no induced cycle of length 0 modulo 3 satisfies $\chi(G) < c$.*

Indeed, the so-called Parity Changing Path tool (to be defined below) is directly inspired by their Trinity Changing Path. The structure of the proofs also have several similarities.

To begin with, let us introduce and recall some notations: the class under study, namely graphs with no triangles and no induced $C_{2k}$ with $k \geqslant 3$ (meaning that every even hole is forbidden except $C_4$) will be called $\mathcal{C}_{3,2k\geqslant 6}$ for short. Moreover, we will consider in Subsection 1.2.1 the subclass $\mathcal{C}_{3,5,2k\geqslant 6}$ of $\mathcal{C}_{3,2k\geqslant 6}$ in which the 5-hole is also forbidden. For two disjoint subsets of vertices $A, B \subseteq V$, we say that $A$ *dominates* $B$ if $B \subseteq N(A)$. A *major connected component* of $G$ is a connected component $C$ of $G$ for which $\chi(C) = \chi(G)$. Note that such a component always exists. For any induced path $P = x_1 x_2 \cdots x_\ell$ we say that $P$ is a path from its *origin* $x_1$ to its *end* $x_\ell$ or an *$x_1 x_\ell$-path*. Its *interior* is $\{x_2, \ldots, x_{\ell-1}\}$ and its *length* is $\ell - 1$.

Moreover, we use a rather common technique called *levelling* [185, 42] : given a vertex $v$, the *$v$-levelling* is the partition $(N_0, N_1, \ldots, N_k, \ldots)$ of the vertices according to their distance to $v$: $N_k$ is the set of vertices at distance exactly $k$ from $v$ and is called the *$k$-th level*. In particular, $N_0 = \{v\}$ and $N_1 = N(v)$. We need two more facts about levellings: if $x$ and $y$ are in the same part $N_k$ of a $v$-levelling, we call an *upper $xy$-path* any shortest path from $x$ to $y$ among those with interior in $N_0 \cup \cdots \cup N_{k-1}$. Observe

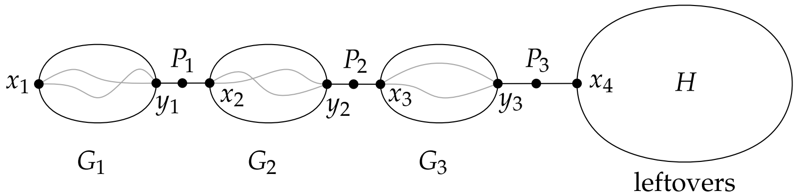



FIGURE 1.6: An unformal diagram for a PCP of order 3. Grey curved lines stand for the even and odd length $x_iy_i$-paths.

that it always exists since there is an $xv$-path and a $vy$-path (but it may take shortcuts; in particular, it may be just one edge). Moreover, in any $v$-levelling, there exists $k$ such that $\chi(N_k) \geqslant \chi(G)/2$: indeed if $t$ is the highest chromatic number of a level, one can color $G$ using $2t$ colors by coloring $G[N_i]$ with the set of colors $\{1, \ldots, t\}$ if $i$ is odd, and with the set of colors $\{t+1, \ldots, 2t\}$ if $i$ is even. Such a level with chromatic number at least $\chi(G)/2$ is called a *colorful* level.

Let us now introduce the main tool of the proof, called *Parity Changing Path* (*PCP* for short[8]) which, as already mentioned, is inspired by the *Trinity Changing Path (TCP)* appearing in [17]: intuitively (see Figure 1.6 for an unformal diagram), a PCP is an induced sequence of induced subgraphs and paths denoted $(G_1, P_1, \ldots, G_\ell, P_\ell, H)$ such that each block $G_i$ can be crossed by two possible paths of different parities, and the last block $H$ typically is a "reservoir" of big chromatic number. Formally, a *PCP of order $\ell$* in $G$ is a sequence of induced subgraphs $G_1, \ldots, G_\ell, H$ (called *blocks*; the $G_i$ are the *regular* blocks) and induced paths $P_1, \ldots, P_\ell$ such that the origin of $P_i$ is some vertex $y_i$ in $G_i$, and the end of $P_i$ is some vertex $x_{i+1}$ of $G_{i+1}$ (or of $H$ if $i = \ell$). Apart from these special vertices which belong to exactly two subgraphs of the PCP, the blocks and paths $G_1, \ldots, G_\ell, H, P_1, \ldots, P_\ell$ composing the PCP are pairwise disjoint. The only possible edges have both endpoints belonging to the same block or path. We also have one extra vertex $x_1 \in G_1$ called the *origin* of the PCP. Moreover in each block $G_i$, there exists one induced $x_iy_i$-path of odd length, and one induced $x_iy_i$-path of even length (these paths are not required to be disjoint one from each other). In particular $x_i \neq y_i$ and $x_iy_i$ is not an edge. For technical reasons that will appear later, we also require that $H$ is connected, every $G_i$ has chromatic number at most 4 and every $P_i$ has length at least 2. Finally the chromatic number of $H$ is called the *leftovers*.

In fact in Subsection 1.2.2, we need a slightly stronger definition of PCP: a *strong PCP* is a PCP for which every $G_i$ contains an induced $C_5$.

We first bound the chromatic number in $\mathcal{C}_{3,5,2k\geqslant 6}$ (see Lemma 1.18 below), which is easier because we forbid one more cycle length, and then deduce the theorem for $\mathcal{C}_{3,2k\geqslant 6}$. The proofs for $\mathcal{C}_{3,2k\geqslant 6}$ and $\mathcal{C}_{3,5,2k\geqslant 6}$ follow the same outline, which we informally describe here:

[8] Not to be confused with other concepts having the same acronym (Probabilistically Checkable Proofs, Post Correspondence Problem, and maybe some more).

*(i)* If $\chi(G)$ is large enough, then for every vertex $v$ we can grow a PCP whose origin is $v$ and whose leftovers are large (Lemmas 1.19, 1.20 and then Lemma 1.27).

*(ii)* Using (i), if $\chi(G)$ is large enough and $(N_0, N_1, \ldots)$ is a $v$-levelling, we can grow a *rooted* PCP: it is a PCP in a level $N_k$, which has a *root*, *i.e.* a vertex in the previous level $N_{k-1}$ whose unique neighbor in the PCP is the origin (Lemma 1.21 and then Lemma 1.28).

*(iii)* Given a rooted PCP in a level $N_k$, if a vertex $x \in N_{k-1}$ has a neighbor in the last block $H$, then it has a neighbor in every regular block $G_i$ (Lemma 1.22).

*(iv)* Given a rooted PCP of order $\ell$ in a level $N_k$ and a stable set $S$ in $N_{k-1}$, the set of neighbors of $S$ inside $N_k$ cannot have a big chromatic number. Consequently, the *active lift* of the PCP, defined as $N(G_\ell) \cap N_{k-1}$, has high chromatic number (Lemmas 1.23, 1.24 and 1.25 and then Lemmas 1.24, 1.29, 1.30).

*(v)* The final proofs put everything together: consider a graph of $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ (resp. $\mathcal{C}_{3,2k\geqslant 6}$) with chromatic number large enough. Then pick a vertex $v$, let $(N_0, N_1, \ldots)$ be a $v$-levelling and $N_k$ be a colorful level. By (ii), grow inside $N_k$ a rooted PCP $P$. Then by (iv), get an active lift $A$ of $P$ inside $N_{k-1}$ with big chromatic number. Grow a rooted PCP $P'$ inside $A$, and get an active lift $A'$ of $P'$ inside $N_{k-2}$ with chromatic number big enough to find an edge $xy$ (resp. a 5-hole C) in $A'$ . Then *clean* $P'$ in order to get a stable set $S$ inside the last regular block of $P'$, dominating this edge (resp. hole). Now find an even hole of length at least 6 in $\{x, y\} \cup S \cup P$ (resp. $C \cup S \cup P$), a contradiction.

### 1.2.1 Forbidding 5-holes

This section is devoted to the proof of the following lemma :

**LEMMA 1.18**

*There exists a constant $c'$ such that every graph $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ satisfies $\chi(G) < c'$.*

We follow the above described outline. Let us start with step (i):

**LEMMA 1.19**

*Let $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ be a connected graph and $v$ be any vertex of $G$. For every $\delta$ such that $\chi(G) \geqslant \delta \geqslant 18$, there exists a PCP of order 1 with origin $v$ and leftovers at least $h(\delta) = \delta/2 - 8$.*

*Proof.* The proof is illustrated on Figure 1.7(a). Let $(N_0, N_1, \ldots)$ be the $v$-levelling and $N_k$ be a colorful level. Let $N'_k$ be a major connected component of $G[N_k]$, so we have $\chi(N'_k) \geqslant \delta/2$. Let $xy$ be an edge of $N'_k$, and $x'$ (resp. $y'$) be a neighbor of $x$ (resp. $y$) in $N_{k-1}$. Let $Z' = N[\{x', y', x, y\}] \cap N'_k$ and $Z = Z' \setminus \{x, y\}$. Let $z \in Z$ be a vertex having a neighbor $z_1$ in a major connected component $M_1$ of $N'_k \setminus Z'$. Observe that $N'_k \setminus Z'$ is not empty since $\chi(Z') \leqslant 6$ (indeed we can observe that $\chi(\{x', y', x, y\}) \leqslant 2$ and, since the neighborhood of any vertex is a stable set, we have that $\chi(N(\{x', y', x, y\})) \leqslant 4$). The goal is now to find two $vz$-paths $P$ and $P'$ of different parities with interior in $G[N_0 \cup \ldots \cup \{x', y'\} \cup \{x, y\}]$. Then we can set $G_1 = G[P \cup P']$, $P_1 = G[\{z, z_1\}]$ and $H = G[M_1]$ as parts of the wanted PCP. In practice, we need to be a little more careful to ensure the condition on the length of $P_1$ and the non-adjacency between $z$ and $H$, which is described after finding such a $P$ and a $P'$.

Let $P_0$ (resp. $P_0'$) be a $vx'$-path (resp. $vy'$-path) of length $k-1$ (with exactly one vertex in each level). By definition of $Z$, $z$ is connected to $\{x', y', x, y\}$.

*(i)* (see Figure 1.7(b)) If $z$ is connected to $x$ or $y$, say $x$, then $z$ is connected neither to $x'$ nor to $y$, otherwise it creates a triangle. We add the path $x'xz$ to $P_0$ to form $P$. Similarly, we add either the edge $y'z$ if it exists, or else the path $y'yxz$ to $P_0'$ to form $P'$. Observe that $P'$ is indeed an induced path since there is no triangle.

*(ii)* (see Figure 1.7(c)) If $z$ is not connected to $\{x, y\}$, then $z$ is connected to exactly one of $x'$ and $y'$, since otherwise it would either create a triangle $x', y', z$ or a 5-hole $zx'xyy'$. Without loss of generality, assume that $zx' \in E$ and $zy' \notin E$. We add the edge $x'z$ to $P_0$ to form $P$. We add the path $y'x'z$ if $y'x' \in E$, otherwise add the path $y'yxx'z$ to $P_0'$ to form $P'$. Observe that this is an induced path since $G$ has no triangle and no 5-hole.

Now comes the fine tuning. Choose in fact $z_1 \in M_1 \cap N(z)$ so that $z_1$ is connected to a major connected component $M_2$ of $M_1 \setminus N(z)$. Choose $z_2$ a neighbor of $z_1$ in $M_2$ such that $z_2$ is connected to a major connected component $M_3$ of $M_2 \setminus N(z_1)$. We redefine $H = G[\{z_2 \cup M_3\}]$ and $P_1 = G[\{z, z_1, z_2\}]$. Then $P_1$ is a path of length 2, $G_1$ is colorable with 4 colors as the union of two induced paths, and $H$ is connected. Moreover $H$ has chromatic number at least $\chi(N_k') - \chi(Z') - \chi(N(z)) - \chi(N(z_1))$. Since the neighborhood of any vertex is a stable set, $\chi(Z') \leqslant 6$ and $\chi(N(z)), \chi(N(z_1)) \leqslant 1$. Thus $\chi(H) \geqslant \delta/2 - 8$.

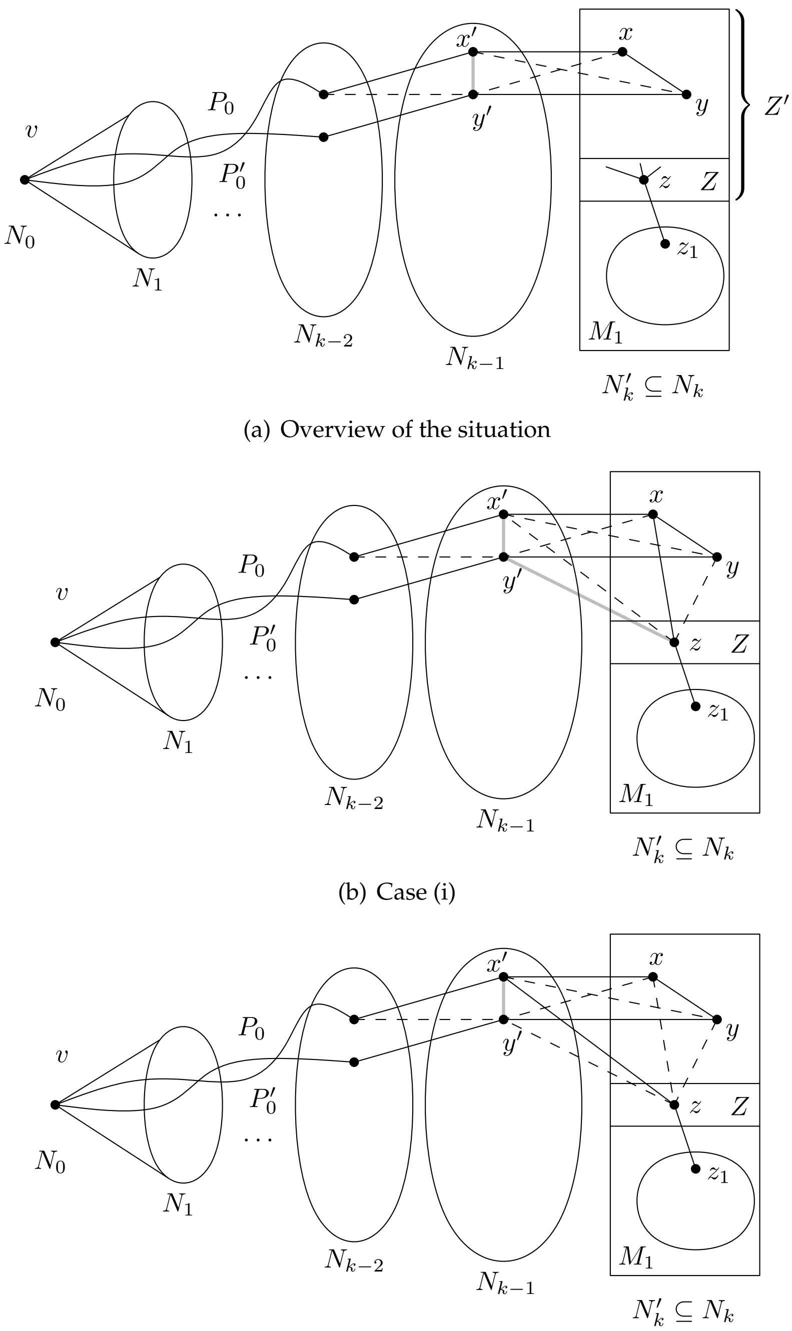
□

We can iterate the previous process to grow some longer PCP. In the following, for a function $f$ and an integer $k$, $f^{(k)}$ denotes the $k$-th iterate of $f$, that is to say that $f^{(k)}(x) = \underbrace{(f \circ \ldots \circ f)}_{k \text{ times}}(x)$.

**LEMMA 1.20**

*Let $h(x) = x/2 - 8$ be the function defined in Lemma 1.19. For every positive integers $\ell, \delta \in \mathbb{Z}_+$, if $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ is connected and satisfies $\chi(G) \geqslant \delta$ and $h^{(\ell-1)}(\delta) \geqslant 18$, then from any vertex $x_1$ of $G$, one can grow a PCP of order $\ell$ with leftovers at least $h^{(\ell)}(\delta)$.*

*Proof.* We prove the result by induction on $\ell$. For $\ell = 1$, the result follows directly from Lemma 1.19. Now suppose it is true for $\ell - 1$, and let $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ be such that $\chi(G) \geqslant \delta$ and $h^{(\ell-1)}(\delta) \geqslant 18$. Then $\delta \geqslant h^{(\ell-1)}(\delta) \geqslant 18$, so we can apply Lemma 1.19




(a) Overview of the situation

(b) Case (i)

(c) Case (ii)

FIGURE 1.7: Illustrations for the proof of Lemma 1.19. Dashed edges stand for non-edges, and grey edges stand for edges that may or may not exist.

to get a PCP of order 1 and leftovers at least $h(\delta)$ from any vertex $x_1$. Let $x_2$ be the common vertex between the last block $H$ and the first path $P_1$ of the PCP (as in the definition). Now apply the induction hypothesis to $H$, knowing that $H$ is connected, $\chi(H) \geqslant h(\delta) = \delta'$ and $h^{(\ell-2)}(\delta') \geqslant 18$. Then we obtain a PCP of order $\ell - 1$ with origin $x_2$ and leftovers at least $h^{(\ell-2)}(\delta')$, which finishes the proof by gluing the two PCP together. □

Now we grow the PCP in a level $N_k$ of high chromatic number, and we want the PCP to be rooted (*i.e.* there exists a root $u' \in N_{k-1}$ that is adjacent to the origin $u$ of the PCP, but to no other vertex of the PCP). This is step (ii).

**LEMMA 1.21**

*Let $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ be a connected graph. Let $v \in V(G)$ and $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $h$ be the function defined in Lemma 1.19. For every $k, \delta$ such that $\chi(N_k) \geqslant \delta + 1$ and $h^{(\ell-1)}(\delta) \geqslant 18$, there exists a rooted PCP of order $\ell$ in $N_k$ with leftovers at least $h^{(\ell)}(\delta)$.*

*Proof.* Let $N'_k$ be a major connected component of $N_k$ and $u \in N'_k$. Consider a neighbor $u'$ of $u$ in $N_{k-1}$. Since there is no triangle, $N'_k \setminus N(u')$ still has big chromatic number (at least $\delta$), and let $N''_k$ be a major connected component of $N'_k \setminus N(u')$. Let $z$ be a vertex of $N(u) \cap N'_k$ having a neighbor in $N''_k$. Then we apply Lemma 1.20 in $\{z\} \cup N''_k$ to grow a PCP of order $\ell$ from $z$ with leftovers at least $h^{(\ell)}(\delta)$. Now $u'$ has an only neighbor $z$ on the PCP, which is the origin. □

Let us observe the properties of such a rooted PCP. We start with step (iii):

**LEMMA 1.22**

*Let $v$ be a vertex of a graph $G \in \mathcal{C}_{3,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $P$ be a rooted PCP $(G_1, P_1, \ldots, G_\ell, P_\ell, H)$ of order $\ell$ in a level $N_k$ for some $k$. If $x' \in N_{k-1}$ has a neighbor $x$ in some regular block $G_{i_0}$ (resp. in $H$), then $x'$ has a neighbor in every $G_i$ for $1 \leqslant i \leqslant i_0$ (resp. for $1 \leqslant i \leqslant \ell$).*

*Proof.* If $x'$ has a neighbor in $H$, we set $i_0 = \ell + 1$. Let $u$ be the origin of the PCP and $u'$ its root. Since $x' \neq u'$ by definition of the root, there exists an upper $x'u'$-path $P_{up}$ of length at least one. Consider a $ux$-path $P$ inside the PCP. Let $v_1, \ldots, v_r$ be the neighbors of $x'$ on this path, different from $x$ (if any), in this order (from $u$ to $x$). Now we can show that any regular block $G_i$ with $1 \leqslant i \leqslant i_0 - 1$ contains at least one $v_j$: suppose not for some index $i$, let $j$ be the greatest index such that $v_j$ is *before* $x_i$, *i.e.* $v_j \in G_1 \cup P_1 \cup \cdots \cup G_{i-1} \cup P_{i-1}$.

If such an index does not exist (*i.e.* all the $v_j$ are after $G_i$), then there is an odd and an even path from $u$ to $v_1$ of length at least 3 by definition of a regular block, and this path does not contain any neighbor of $x'$. Close them to build two induced cycles by going through $x'$, $P_{up}$ and $u'$: one of them is an even cycle, and its length is at least 6.

If $j = r$ (*i.e.* all the $v_j$ are before $G_i$), then we can use the same argument with a path of well-chosen parity from $v_r$ to $x$, crossing $G_i$.

Otherwise, there is an odd and an even path in the PCP between $v_j$ and $v_{j+1}$, crossing $G_i$, and its length is at least 4 because $x_i$ and $y_i$ are at distance at least 2 one from each other. We can close the even path path by going back and forth to $x$: this gives an even hole of length at least 6. □

Note that, in the above lemma, $G$ is taken in $\mathcal{C}_{3,2k\geqslant 6}$ and not in $\mathcal{C}_{3,5,2k\geqslant 6}$. In particular, we will use Lemma 1.22 in the next subsection as well. Let us now continue with step (iv):

**LEMMA 1.23**

*Let $v$ be a vertex of a graph $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $S \subseteq N_{k-1}$ be a stable set. Then $\chi(N(S) \cap N_k) \leqslant 52$.*

*Proof.* Let $\delta = \chi(N(S) \cap N_k) - 1$. Suppose by contradiction that $\delta \geqslant 52$, then we have $h(\delta) \geqslant 18$, hence by Lemma 1.21, we can grow a rooted PCP of order 2 inside $N(S) \cap N_k$. Let $u$ be the origin of the PCP and $u'$ its root. Observe in particular that $S$ dominates $G_2$. Let $xy$ be an edge of $G_2$, and let $x'$ (resp. $y'$) be a neighbor of $x$ (resp. $y$) in $S$. By Lemma 1.22, both $x'$ and $y'$ have a neighbor in $G_1$. This gives an $x'y'$-path $P_{down}$ with interior in $G_1$. In order not to create an even hole nor a 5-hole by closing it with $x'xyy'$, we can ensure that $P_{down}$ is an even path of length at least 4. Moreover, there exists an upper $x'y'$-path $P_{up}$. Then either the hole formed by the concatenation of $P_{up}$ and $x'xyy'$, or the one formed by the concatenation of $P_{up}$ and $P_{down}$ is an even hole of length at least 6, a contradiction. □

The previous lemma allows us to prove that one can *lift* the PCP up into $N_{k-1}$ to get a subset of vertices with high chromatic number. We state a lemma that will be re-used in the next subsection:

**LEMMA 1.24**

*Let $v$ be a vertex of a graph $G \in \mathcal{C}_{3,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $P$ be a rooted PCP of order $\ell$ in a level $N_k$ (for some $k \geqslant 2$) with leftovers at least $\delta$. Let $A = N(G_\ell) \cap N_{k-1}$ (called the active lift of the PCP). Suppose that for every stable set $S \subseteq A$, we have $\chi(N(S) \cap N_k) \leqslant \gamma$, then $\chi(A) \geqslant \delta/\gamma$.*

*Proof.* Let $r = \chi(A)$, suppose by contradiction that $r < \delta/\gamma$ and decompose $A$ into $r$ stable sets $S_1, \ldots, S_r$. Then $N(A) \cap N_k$ is the (not necessarily disjoint) union of $r$ sets $N(S_1) \cap N_k, \ldots, N(S_r) \cap N_k$, each of which has chromatic number at most $\gamma$ by assumption. Consequently $\chi(N(A) \cap N_k) \leqslant r\gamma < \delta$ and we can deduce that $\chi(H \setminus N(A)) \geqslant \chi(H) - \chi(N(A) \cap N_k) \geqslant 1$. Let $x$ be any vertex of $H \setminus N(A)$ and $x'$ be a neighbor of $x$ in $N_{k-1}$. By construction, $x' \notin A$ so $x'$ has no neighbor in $G_\ell$. This is a contradiction with Lemma 1.22. □

By Lemmas 1.23 and 1.24 with $\gamma = 52$, we can directly deduce the following:

**LEMMA 1.25**

*Let $v$ be a vertex of a graph $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $P$ be a rooted PCP of order $\ell$ in a level $N_k$ (for some $k \geqslant 2$) with leftovers at least $\delta$. Let $A = N(G_\ell) \cap N_{k-1}$ be the active lift of the PCP, then we have $\chi(A) \geqslant g(\delta) = \delta/52$.*

We can now finish the proof, this is step (v). Recall that a sketch was provided, and it may help to understand the following proof.

*Proof of Lemma 1.18.* Let $c'$ be a constant big enough so that

$$g\left(h^{(2)}\left(g\left(h^{(2)}\left(\frac{c'}{2}-1\right)\right)-1\right)\right) \geqslant 5\,.$$

Suppose that $\chi(G) \geqslant c'$. Pick a vertex $v$, let $(N_0, N_1, \ldots)$ be the $v$-levelling and $N_k$ be a colorful level, so $\chi(N_k) \geqslant \chi(G)/2 \geqslant c_1 + 1$ where $c_1 = c'/2 - 1$. By Lemma 1.21, grow a rooted PCP $P = (G_1, P_1, G_2, P_2, H)$ inside $N_k$ of order 2 with leftovers at least $c_2 = h^{(2)}(c_1)$. Then apply Lemma 1.25 and get an active lift $A$ of $P$ inside $N_{k-1}$ with chromatic number at least $c_3 = g(c_2)$. Since $h(c_3 - 1) \geqslant 18$, apply again Lemma 1.21 to get a rooted PCP $P' = (G'_1, P'1, G'_2, P'_2, H')$ of order 2 inside $N_{k-1}$ with leftovers at least $c_4 = h^{(2)}(c_3 - 1)$. Now apply Lemma 1.25 to get an active lift $A'$ of $P'$ inside $N_{k-2}$ with chromatic number at least $c_5 = g(c_4)$.

Because of the chromatic restriction in the definition of the PCP, one can color $G'_2$ with 4 colors. Moreover, $G'_2$ dominates $A'$ by definition. Thus there exists a stable set $S \subseteq G'_2$ such that $\chi(N(S) \cap A') \geqslant c_6 = c_5/4$ (since $A'$ is the union of the $N(S') \cap A'$ for the four stable sets $S'$ that partition $G'_2$).

Now $c_6 > 1$ so there is an edge $xy$ inside $N(S) \cap A'$. Call $x'$ (resp. $y'$) a vertex of $S$ dominating $x$ (resp. $y$). Both $x'$ and $y'$ have a neighbor in $G_2$ by definition of $A$ and, by Lemma 1.22, both $x'$ and $y'$ also have a neighbor in $G_1$. This gives a $x'y'$-path $P_1$ (resp. $P_2$) with interior in $G_1$ (resp. $G_2$). Due to the path $x'xyy'$ of length 3, $P_1$ and $P_2$ must be even paths of length at least 4. Thus the concatenation of $P_1$ and $P_2$ is an even hole of length at least 6, a contradiction. □

### 1.2.2 Dealing with 5-holes

This subsection aims at proving Theorem 1.16, using the result of the previous subsection. As already mentioned, we follow the same outline, except that we now need the existence of a $C_5$ several times. Let us start by a technical lemma to find both an even and an odd path out of a 5-hole and its dominating set:

**LEMMA 1.26**

*Let $G$ be a triangle-free graph containing a 5-hole $C$. Let $S \subseteq V(G)$ be a minimal dominating set of $C$, assumed to be disjoint from $C$. If we delete the edges with both endpoints in $S$, then for every vertex $t \in S$, there exists a vertex $t' \in S$ such that one can find an induced $tt'$-path of length 4 and an induced $tt'$-path of length 3 or 5, both with interior in $C$.*

*Proof.* Let $t \in S$, call $v_1$ a neighbor of $t$ on the cycle and number the others vertices of $C$ with $v_2, \ldots, v_5$ (following the adjacency on the cycle). Since $G$ is triangle-free, $t$ cannot be adjacent to both $v_3$ and $v_4$, so up to relabeling the cycle in the other direction we assume that $t$ is not adjacent to $v_3$. Let $t' \in S$ be a vertex dominating $v_3$. Then $tv_1v_2v_3t'$ is an induced path of length 4 between $t$ and $t'$. Moreover, $tv_1v_5v_4t'$ is a (not necessarily induced) path of length 5 between $t$ and $t'$. If this path is not induced, the only possible chords are $tv_4$ and $t'v_5$ since $G$ is triangle-free, which in any case gives an induced $tt'$-path of length 3. □

Recall that in this subsection, we are interesting in strong PCP, *i.e.* PCP, all regular blocks $G_i$ of which contain an induced $C_5$. We start with step (i):

**LEMMA 1.27**

*Let $c'$ be the constant of Lemma 1.18, let $G \in \mathcal{C}_{3,2k \geqslant 6}$ and $v$ be any vertex of $G$. For every $\delta \in \mathbb{N}$ such that $\chi(G) \geqslant \delta \geqslant 2c'$, there exists a strong PCP of order 1 with origin $v$ and leftovers at least $f(\delta) = \delta/2 - 15$.*

*Proof.* Let $(N_0, N_1, \ldots)$ be the $v$-levelling, $N_k$ be a colorful level and let $N'_k$ be a major connected component of $G[N_k]$, so $\chi(N'_k) \geqslant c'$. Using Lemma 1.18, there exists a 5-hole $C$ in $G[N'_k]$. Consider a minimum dominating set $D$ of $C$ inside $N_{k-1}$.

From now on, the proof is very similar to the one of Lemma 1.19. Similarly, we define $Z' = N[D \cup C] \cap N'_k$ and $Z = Z' \setminus C$. Let $z \in Z$ be a vertex having a neighbor $z_1$ in a major connected component $M_1$ in $N'_k \setminus Z'$. The goal is now to find two $vz$-paths $P$ and $P'$ of different parity with interior in $N_0 \cup \ldots \cup D \cup C$, then we can simply set $G_1 = G[P \cup P' \cup C]$, $P_1 = G[\{z, z_1\}]$ and $H = G[M_1]$ as parts of the wanted PCP. In practice, we need to be a little more careful to ensure the condition on the length of $P_1$ and the non-adjacency between $z$ and $H$.

Let us now find those two paths $P$ and $P'$. By definition of $Z$, $z$ also has a neighbor in $D$ or in $C$.

*(i)* If $z$ has a neighbor $x \in C$, let $y \in C$ be a vertex adjacent to $x$ on the hole. Let $x'$ and $y'$ be respectively a neighbor of $x$ and a neighbor of $y$ in $D$. Observe that $z$ is connected neither to $x'$ nor to $y$, otherwise it creates a triangle. We grow $P$ by starting from an induced path of length $k-1$ from $v$ to $x'$ and then add the path $x'xz$. Similarly, we grow $P'$ by starting from an induced path of length $k-1$ from $v$ to $y'$, and then add the edge $y'z$ if it exists, or else the path $y'yxz$. Observe that $P'$ is indeed an induced path since there is no triangle.

*(ii)* If $z$ has no neighbor in $C$, then it has at least one neighbor $x'$ in $D$. Apply Lemma 1.26 to get a vertex $y' \in D$ such that there exists a $x'y'$-path of length 3 or 5, and another one of length 4, both with interior in $C$. Observe that $x'$ and $y'$ cannot have a common neighbor $u$ in $N_{k-2} \cup \{z\}$, otherwise there would be either a triangle $x', u, y'$ (if $x'y' \in E$), or a $C_6$ using the $x'y'$-path of length 4 with interior in $C$. Now we grow $P$ by starting from an induced path of length $k-1$ from $v$ to $x'$, and add the edge $x'z$. We grow $P'$ by starting from an induced path

of length $k-1$ from $v$ to $y'$, and then add the edge $x'y'$ if it exists, otherwise add the $x'y'$-path of length 3 or 5 with interior in $C$, and then finish with the edge $x'z$.

Now come the fine tuning. Choose in fact $z_1 \in M_1 \cap N(z)$ so that $z_1$ is connected to a major connected component $M_2$ of $M_1 \setminus N(z)$. Choose $z_2$ a neighbor of $z_1$ in $M_2$ such that $z_2$ is connected to a major connected component $M_3$ of $M_2 \setminus N(z_1)$. We redefine $H = G[\{z_2 \cup M_3\}]$ and $P_1 = G[\{z, z_1, z_2\}]$. Then $P_1$ is a path of length 2, $H$ is connected and $G_1$ is colorable with 4 colors (it is easily 7 colorable as the union of a 5-hole and two paths; a careful case analysis shows that it is 4 colorable). Moreover $H$ has chromatic number at least $\chi(N'_k) - \chi(Z') - \chi(N(z)) - \chi(N(z_1))$. Since the neighborhood of any vertex is a stable set, $\chi(Z') \leqslant |D| + |C| + \chi(C) \leqslant 13$ and $\chi(N(z)), \chi(N(z_1)) \leqslant 1$. Thus $\chi(H) \geqslant \delta/2 - 15$.

□

We go on with step (ii): find a strong rooted PCP. The following lemma is proved in the same way as Lemma 1.21 by replacing the use of Lemma 1.20 by Lemma 1.27, so we omit the proof here.

**LEMMA 1.28**

*Let $G \in \mathcal{C}_{3,2k\geqslant 6}$ be a connected graph, $f$ be the function defined in Lemma 1.27, $v$ be a vertex of $G$ and $(N_0, N_1, \ldots)$ be the $v$-levelling. For every $k, \delta$ such that $\chi(N_k) \geqslant \delta + 1 \geqslant 2c' + 1$, there exists a strong rooted PCP of order 1 in $N_k$ with leftovers at least $f(\delta)$.*

Step (iii) is proved by Lemma 1.22 from the previous subsection, and was valid not only for $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ but also for $G \in \mathcal{C}_{3,2k\geqslant 6}$. So we continue with step (iv):

**LEMMA 1.29**

*Let $v$ be a vertex of a graph $G \in \mathcal{C}_{3,2k\geqslant 6}$, and let $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $S$ be a stable set inside $N_{k-1}$. Then $\chi(N(S) \cap N_k) \leqslant 2c'$.*

*Proof.* Suppose by contradiction that $\chi(N(S) \cap N_k) \geqslant 2c' + 1$. By Lemma 1.28, we can grow in $N(S) \cap N_k$ a rooted PCP of order 1, and in particular $S$ dominates $G_1$. By definition of a strong PCP, there is a 5-hole $C$ in $G_1$. Since $S$ is a dominating set of $C$, we can apply Lemma 1.26 to get two vertices $t, t' \in S$ such that one can find both an even and an odd $tt'$-path with interior in $C$ and length at least 3. Then any upper $tt'$-path close a hole of even length at least 6. □

In fact, as in the previous section, we can directly deduce from Lemmas 1.24 and 1.29 that one can lift the PCP up into $N_{k-1}$ to get a subset of vertices with high chromatic number:

**LEMMA 1.30**

*Let $G \in \mathcal{C}_{3,2k\geqslant 6}$, $v \in V(G)$ and let $(N_0, N_1, \ldots)$ be the $v$-levelling. Let $P$ be a strong rooted PCP of order 1 in a level $N_k$ (for some $k \geqslant 2$) with leftovers $\delta \geqslant 2c'$. Let $A = N(G_1) \cap N_{k-1}$ be the active lift of the PCP, then $\chi(A) \geqslant \varphi(\delta) = \frac{\delta}{2c'}$.*

We are now ready to finish the proof, this is step (v). Recall that a sketch was given and may be useful to have a less technical overview of the proof.

*Proof of Theorem 1.16.* Let $c$ be a constant such that

$$\varphi\left(f\left(\varphi\left(f\left(\frac{c}{2}-1\right)\right)-1\right)\right) \geqslant 4c' .$$

Suppose that $G \in \mathcal{C}_{3,2k\geqslant 6}$ has chromatic number $\chi(G) \geqslant c$. Then pick a vertex $v$, let $(N_0, N_1, \ldots)$ be the $v$-levelling and $N_k$ be a colorful level, then $\chi(N_k) \geqslant c_1 + 1 = c/2$. Apply Lemma 1.28 and grow inside $N_k$ a strong rooted PCP $P = (G_1, P_1, H)$ of order 1 with leftovers at least $c_2 = f(c_1)$. Then apply Lemma 1.30 and get an active lift $A = N(G_1)$ of $P$ inside $N_{k-1}$ with chromatic number at least $c_3 = \varphi(c_2)$. By Lemma 1.28, we can obtain a strong rooted PCP $P' = (G_1', P_1', H')$ inside $A$ with leftovers at least $c_4 = f(c_3 - 1)$, and by Lemma 1.30 we obtain an active lift $A'$ of $P'$ inside $N_{k-2}$ with chromatic number at least $c_5 = \varphi(c_4)$. Because of the chromatic restriction in the definition of the PCP, one can color $G_1'$ with 4 colors. Moreover, $G_1'$ dominates $A'$ by definition. Thus there exists a stable set $S \subseteq P'$ such that $\chi(N(S) \cap A') \geqslant c_6 = c_5/4$. Now $c_6 \geqslant c'$ thus Lemma 1.18 proves the existence of a 5-hole $C$ inside $N(S) \cap A'$. Let us give an overview of the situation: we have a 5-hole $C$ inside $N_{k-2}$, dominated by a stable set $S$ inside $N_{k-1}$, and every pair of vertices $t, t'$ of $S$ can be linked by a $tt'$-path $P_{down}$ with interior in $G_1 \subseteq N_k$. Lemma 1.26 gives the existence of two vertices $t, t' \in S$ linked by both an odd path and an even path of length at least 3 with interior in $C$. Closing one of these paths with $P_{down}$ gives an induced even hole of length at least 6, a contradiction. □

Observe that no optimization was made on the constants $c'$ and $c$ from Lemma 1.18 and Theorem 1.16. The proof gives the following upper bounds:

- $\chi(G) \leqslant 435\,122$ for every $G \in \mathcal{C}_{3,5,2k\geqslant 6}$, and
- $\chi(G) \leqslant 12 \cdot 10^{18}$ for every $G \in \mathcal{C}_{3,2k\geqslant 6}$.

# Chapter 2

# The Erdős-Hajnal property

HOW does the chromatic number $\chi(G)$ behave with respect to the clique number $\omega(G)$? The previous chapter gave a formal definition of *behave*, based on the impression that, if there is no huge clique, then the chromatic number *should morally* not be huge (this is of course false). We can rephrase this feeling by the following hypothesis: either $\omega(G)$ is large, or $\chi(G)$ is small. The second outcome implies that one can partition the vertices of $G$ into few stable sets: one of them has to be very large. Consequently, we came to the following conclusion: *one of $\omega(G)$ or $\alpha(G)$ has to be large*. This is the unformal statement of the Erdős-Hajnal property. Unfortunately, this does not hold in general, but forbidding a fixed graph $H$ may be enough to enforce such a property. Let us now survey results surrounding this conjecture in Section 2.1 (the interested reader may refer to the more exhaustive survey by Chudnovsky [35]), and then see a powerful tool called the Strong Erdős-Hajnal property, together with its application to $(P_k, \overline{P_k})$-free graphs in Section 2.2. We conclude in Section 2.3 with new useful tools towards the Erdős-Hajnal Conjecture by unbalancing the Strong Erdős-Hajnal property and then focusing on Easy Neighborhoods. Quite surprisingly, most tools used in this chapter can be successfully adapted in the study of the Clique-Stable Set separation problem, as described in Chapter 3 (see in particular Section 3.4.1).

Note that the main content of this chapter (Section 2.2) is covered in:

[II] **The Erdős-Hajnal Conjecture for Paths and Antipaths**, with N. BOUSQUET and S. THOMASSÉ, *Journal of Combinatorial Theory, Series B*, 113:261–264, 2015.

## 2.1 Context and Motivations

Erdős proved with a probabilistic argument that there exist graphs with no clique or stable set of size larger than $\mathcal{O}(\log n)$. However, the situation may be much different in $H$-free graphs for any fixed graph $H$. Erdős and Hajnal even conjectured that one can go from a logarithmic order to a polynomial one in such classes:

**Conjecture 2.1** [69] (*Erdős-Hajnal Conjecture*)

*For every graph $H$, there exists a constant $\beta(H) > 0$ such that every $H$-free graph $G$ has either a clique or a stable set of size at least $|V(G)|^{\beta(H)}$.*

In the same paper, they proved that $H$-free graphs definitely have a large clique or stable set, but with a weaker definition of *large*.

**Theorem 2.2** [69]

*For every graph $H$, there exists a constant $\beta(H) > 0$ such that every $H$-free graph $G$ has either a clique or a stable set of size $e^{\beta(H)\sqrt{\log|V(G)|}}$.*

Let us now introduce some classical notation. Let $\mathcal{C}$ be a hereditary class of graph, to be thought as $H$-free graphs for some fixed graph $H$, or $\mathcal{H}$-free graphs for a family of graphs $\mathcal{H}$. The class $\mathcal{C}$ is said to have the *Erdős-Hajnal property* if there exists a constant $\beta(\mathcal{C}) > 0$ such that every graph $G \in \mathcal{C}$ has either a clique or a stable set of size at least $|V(G)|^{\beta(\mathcal{C})}$.

In the light of Chapter 1, the first question that comes to mind is: what about perfect graphs?

**Observation 2.3**

*Perfect graphs have the Erdős-Hajnal property.*

*Proof.* The proof is extremely short: observe that a proper coloring is just a partition of the vertices into $\chi(G)$ stable sets of size at most $\alpha(G)$, so $|V(G)| \leqslant \chi(G)\alpha(G)$ for any graph $G$. Now, if $G$ is perfect $\chi(G) = \omega(G)$, so $|V(G)| \leqslant \omega(G)\alpha(G)$ and finally we have $\alpha(G) \geqslant \sqrt{|V(G)|}$ or $\omega(G) \geqslant \sqrt{|V(G)|}$. □

Using this observation, one can restate Conjecture 2.1 to the following equivalent statement:

**Conjecture 2.4**

*For every graph $H$, there exists a constant $\beta(H) > 0$ such that every $H$-free graph $G$ has an induced perfect subgraph of size at least $|V(G)|^{\beta(H)}$.*

Conjecture 2.4 can be easier to work with, as it only has one outcome instead of two (a clique *or* a stable set of large size). Let us now follow the classical adage: *if something works for perfect graphs, ask the same for $\chi$-bounded classes*. Here is a partial answer:

**Observation 2.5**

*Let $\mathcal{C}$ be a class of graphs having a polynomial $\chi$-bounding function $f(x) = x^c$ for some constant $c > 0$. Then $\mathcal{C}$ has the Erdős-Hajnal property.*

*Proof.* Let $G \in \mathcal{C}$. As in the proof of Observation 2.3, $|V(G)| \leqslant \chi(G)\alpha(G)$, and by property of $\mathcal{C}$, we have $\chi(G) \leqslant \omega(G)^c$. Hence $|V(G)| \leqslant \omega(G)^c\alpha(G)$. Consequently, one of $\alpha(G)$ or $\omega(G)$ is greater than $|V(G)|^{\frac{1}{c+1}}$. □

Let us keep looking for the easy cases: if $H$ is a complete graph or its complement, then $H$-free graphs have the Erdős-Hajnal property by Ramsey theorem [168]. Secondly, what happens with small graphs $H$? The answer is obviously positive if $|V(H)| \leqslant 2$. For $H = P_3$, notice that every $H$-free graph $G$ is the disjoint union of cliques, and consequently has either a clique or a stable set of size at least $\sqrt{|V(G)|}$. To shorten the list of graphs we have to check, it is useful to observe that the property is self-complementary: for any graph $H$, if $H$-free graphs have the Erdős-Hajnal property, then $\overline{H}$-free graphs also have the Erdős-Hajnal property. According to the previous observations, this implies that for every 3-vertex graph $H$, $H$-free graphs have the Erdős-Hajnal property.

Another easy case, which will be interesting with respect to the Strong Erdős-Hajnal property (see Section 2.2 for a definition and further details), is the one of *cographs*. Cographs are recursively defined as follows: a graph is a cograph if and only if

*(i)* it has only one vertex, or

*(ii)* it is the disjoint union of two cographs, or

*(iii)* it is the join of two cographs (the *join* of two graphs $G_1$ and $G_2$ is obtained by taking the vertex-disjoint union of $G_1$ and $G_2$, and adding every possible edges between $G_1$ and $G_2$)

**OBSERVATION 2.6** [186]

- *A graph is a cograph if and only if it is $P_4$-free.*
- *A cograph is perfect.*

This implies in particular that $P_4$-free graphs have the Erdős-Hajnal property. In fact, that is all we need to prove the Erdős-Hajnal property in $H$-free graphs for any 4-vertex $H$. The reason for this lies in the *substitution* operation, whose definition already appeared in Section 1.1 and which we recall here. Given two graphs $H_1$ and $H_2$ on disjoint vertex sets $V_1$ and $V_2$ respectively, each with at least two vertices, and $v \in V_1$, we say that $H$ is obtained from $H_1$ by substituting $H_2$ for $v$, or obtained from $H_1$ and $H_2$ by substitution (in a less detailed manner) if

*(i)* $V(H) = (V_1 \cup V_2) \setminus \{v\}$

*(ii)* $H[V_2] = H_2$

*(iii)* $H[V_1 \setminus \{v\}] = H_1[V_1 \setminus \{v\}]$

*(iv)* $v_1 \in V_1$ is adjacent in $H$ to $v_2 \in V_2$ if and only if $v_1$ is adjacent to $v$ in $H_1$.

A graph is *prime* if it cannot be obtained from smaller graphs by substitution. Alon, Pach and Solymosi proved that substitution behaves well with respect to the Erdős-Hajnal property:

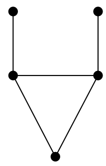


FIGURE 2.1: The 5-vertex graph called the bull.

**THEOREM 2.7** [6]

*If $H_1$-free graphs and $H_2$-free graphs have the Erdős-Hajnal property, and $H$ is obtained from $H_1$ and $H_2$ by substitution, then $H$-free graphs have the Erdős-Hajnal property.*

Such a tool is very useful to build bigger graphs $H$ inducing the Erdős-Hajnal property. In particular for 4-vertex graphs, $P_4$ is the only prime graph. Thus we can deduce the following corollary:

**COROLLARY 2.8**

*Let $H$ be a graph on at most 4 vertices. The class of $H$-free graphs has the Erdős-Hajnal property.*

Let us go one step further and examine prime 5-vertex graphs. There are only four of them:

- $C_5$, the cycle of length five,
- $P_5$, the path on five vertices,
- $\overline{P_5}$, its complement,
- the bull, a triangle with two additional disjoint pendant edges, depicted on Figure 2.1.

With a strong structural analysis of prime bull-free graphs, Chudnovsky and Safra solved the last item:

**THEOREM 2.9** [41]

*Bull-free graphs have the Erdős-Hajnal property.*

The constant $\beta(H)$ they obtained (as in the definition of the Erdős-Hajnal Conjecture) is $\frac{1}{4}$, which is best possible (as explained in [35]). Unfortunately, Conjecture 2.1 is still open for $H = C_5$, $P_5$, or $\overline{P_5}$. One could think of adding extra hypotheses to make the problem easier. Observing that the bull is self-complementary, and that the problem is still open both for $P_5$ and $\overline{P_5}$, one may successfully think of forbidding both of them:

**THEOREM 2.10** [86, 35]

*The class of $(P_5, \overline{P_5})$-free graphs have the Erdős-Hajnal property.*

In fact, this class has been extensively studied in the last few years [38, 37] and even structural results could be proved. Having a self-complementary class seems ergo helpful, although the question is still wide open for the self-complementary $C_5$. In this respect, Chudnovsky made the following conjecture, hopefully slightly easier than Conjecture 2.1:

**CONJECTURE 2.11** [35]

*For every graph $H$, there exists a positive constant $\beta(H)$ such that every $(H, \overline{H})$-free graph has either a clique or a stable set of size at least $|V(G)|^{\beta(H)}$.*

Unfortunately, the list of tractable $H$ for the Erdős-Hajnal Conjecture ends up here, since no answer is known for $H$-free graphs with $H$ a 6-vertex prime graph. However, Theorem 2.10 gave rise to a new series of partial results trying to forbid a path and an antipath: Chudnovsky and Zwols proved the Erdős-Hajnal property for $(P_5, \overline{P_6})$-free graphs [47], and this was further extended to $(P_5, \overline{P_7})$-free graphs by Chudnovsky and Seymour [45].

Besides adding extra hypotheses, another way of attacking a hard conjecture is to provide similar weaker results. On the one hand, *similar* can mean considering *almost* all $H$-free graphs instead of *all* $H$-free graphs, as proved by Loebl *et al.*:

**THEOREM 2.12** [148]

*Let $H$ be a graph and $\mathcal{P}_H$ be the class of $H$-free graphs. There exists a subclass $Q_H \subseteq \mathcal{P}_H$ such that $Q_H$ has the Erdős-Hajnal property, and*

$$\frac{|\{G \in Q_H \mid |V(G)| = n\}|}{|\{ G \in \mathcal{P}_H \mid |V(G)| = n\}|} \underset{n\to\infty}{\longrightarrow} 1\,,$$

i.e. *the proportion of graphs of $\mathcal{P}_H$ on $n$ vertices that belong to $Q_H$ tends to 1.*

On the other hand, *similar* can mean that instead of a large clique or a large stable set, one can ask for something large that looks like a clique or a stable set: a *biclique* of size $t$ is a (not necessarily induced) complete bipartite subgraph $(X, Y)$ such that both $|X|, |Y| \geqslant t$ (observe that it does not require any condition inside $X$ or inside $Y$). Erdős, Hajnal and Pach proved in [72] that for any graph $H$, there exists some $c(H) > 0$ such that for every $H$-free graph $G$, $G$ or its complement $\overline{G}$ contains a biclique of size $|V(G)|^{c(H)}$. Neither of those two outcomes gives directly a large clique or a large stable set, but this was improved upon by Fox and Sudakov:

**THEOREM 2.13** [91]

*For any graph $H$, there exists a constant $c(H) > 0$ such that every $H$-free graph $G$ has either a biclique or a stable set of size $|V(G)|^{c(H)}$.*

Following both approaches of Conjecture 2.11 and Fox and Sudakov's idea of finding a large "clique-like" or "stable set-like" subgraph, we proved the Erdős-Hajnal property for $(P_k, \overline{P_k})$-free graphs, result to which Section 2.2 is dedicated. Note that using the same tools, Bonamy, Bousquet, and Thomassé proved the following:

**THEOREM 2.14** [16]

*Let $k \geqslant 4$ be an integer and $\mathcal{C}$ be the class of graphs with no holes nor antiholes of length at least $k$. Then $\mathcal{C}$ has the Erdős-Hajnal property.*

## 2.2 The Strong Erdős-Hajnal property in the class of $(P_k, \overline{P_k})$-free graphs

In the same flavour as Theorem 2.13, Fox and Pach found an alternative definition [88] of a *large subgraph similar to a clique or a stable set*. They came up with the following definition: a class $\mathcal{C}$ of graphs has the *Strong Erdős-Hajnal property* if there exists a constant $c(\mathcal{C}) > 0$ such that for every $G \in \mathcal{C}$, at least one of $G$ or $\overline{G}$ contains a biclique of size $c(\mathcal{C})|V(G)|$. This attempt was very successful in the following sense:

**THEOREM 2.15** [5, 88]

*If $\mathcal{C}$ is a class of graphs having the Strong Erdős-Hajnal property, then $\mathcal{C}$ has the Erdős-Hajnal property.*

*Proof.* The proof uses the aforementioned properties of cographs. Let $c$ be the constant of the Strong Erdős-Hajnal property, meaning that for every $G \in \mathcal{C}$, $G$ or $\overline{G}$ contains a biclique of size $c|V(G)|$. Let $c' > 0$ be such that $c^{c'} \geqslant 1/2$. We prove by induction on $|V(G)|$ that every $G \in \mathcal{C}$ has an induced cograph of size $|V(G)|^{c'}$. By our hypothesis on $\mathcal{C}$, there exists a biclique $(X, Y)$ of size $c|V(G)|$ in $G$ or in $\overline{G}$. Applying the induction hypothesis inside both $X$ and $Y$, we get two cographs on vertex sets $V_X \subseteq X$ and $V_Y \subseteq Y$, each of size at least $(c|V(G)|)^{c'}$. Since cographs are closed under taking disjoint union and join, $G[V_X \cup V_Y]$ is a cograph, of size at least $2c^{c'}|V(G)|^{c'} \geqslant |V(G)|^{c'}$, using the fact that $c^{c'} \geqslant 1/2$.

Since cographs are perfect, they admit a clique or a stable set of square root size, which implies that every $G \in \mathcal{C}$ has a clique or a stable set of size $|V(G)|^{c'/2}$. □

However, Fox and Pach also proved the following:

**THEOREM 2.16** (Restatement of [89])

*For every constant $c > 0$ and any sufficiently large integer $n$, there exists a comparability graph on $n$ vertices that does not contain any biclique of size $cn$.*

Remember that comparability graphs are perfect, consequently they obviously have the Erdős-Hajnal property. Therefore, the Strong Erdős-Hajnal property can be a convenient tool for proving the Erdős-Hajnal property in some classes of graphs, but there is no hope to prove the Strong Erdős-Hajnal property for $H$-free graphs for every graph $H$, and not even for every perfect graphs.

Let us know state the main result of this Section:

**THEOREM 2.17**

*For every $k \geqslant 2$, the class of $(P_k, \overline{P_k})$-free graphs has the Strong Erdős-Hajnal property.*

**COROLLARY 2.18**

*The class of $(P_k, \overline{P_k})$-free graphs has the Erdős-Hajnal property.*

*Proof.* This comes directly from Theorems 2.15 and 2.17. □

To prove Theorem 2.17, we need a few definitions: given $\varepsilon > 0$, an *$\varepsilon$-stable set* $S$ in a graph $G$ is an induced subgraph of $G$ containing at most $\varepsilon \binom{|V(S)|}{2}$ edges. An *$\varepsilon$-clique* is an $\varepsilon$-stable set in $\overline{G}$, that is to say an induced subgraph $S$ of $G$ containing at least $(1-\varepsilon)\binom{|V(S)|}{2}$ edges. We also need the following result by Fox and Sudakov:

**THEOREM 2.19** [90]

*For every positive integer $k$ and every $\varepsilon \in (0, 1/2)$, there exists $\gamma > 0$ such that every graph $G$ satisfies one of the following:*

- *$G$ induces all graphs on $k$ vertices.*
- *$G$ contains an $\varepsilon$-stable set of size at least $\gamma|V(G)|$.*
- *$G$ contains an $\varepsilon$-clique of size at least $\gamma|V(G)|$.*

Note that a stronger result was previously showed by Rödl [175] with the help of Szemerédi's regularity lemma [194], but Fox and Sudakov's proof provides a much better quantitative estimate ($\gamma = 2^{-ck(\log 1/\varepsilon)^2}$ for some constant $c$ instead of a power tower function). They further conjecture that a polynomial estimate should hold, which would imply the Erdős-Hajnal conjecture.

We now come to the key lemma of our proof, which is an adaptation of Gyárfás' proof of the $\chi$-boundedness of $P_k$-free graphs (Theorem 1.10). A sketch of the proof for the triangle-free case was provided in Chapter 1 and is useful for having an intuition for the proof of Lemma 2.20. The interesting parallel lies in the replacement of *small chromatic number* and *big chromatic number* by *small number of vertices* and *big number of vertices*.

**LEMMA 2.20**

*For every $k \geqslant 2$, there exists $\varepsilon_k > 0$ and $c_k$ (with $0 < c_k \leqslant 1/2$) such that every connected graph $G$ on $n \geqslant 2$ vertices satisfies one of the following:*

- *There exists a vertex of degree more than $\varepsilon_k n$.*
- *For every vertex $v$, $G$ contains an induced $P_k$ starting at $v$.*
- *The complement $\overline{G}$ of $G$ contains a biclique of size $c_k n$.*

*Proof.* We proceed by induction on $k$. For $k = 2$, since $G$ is connected, every vertex is the endpoint of an edge (that is, a $P_2$). Thus we can arbitrarily define $\varepsilon_2 = c_2 = 1/2$.

If $k > 2$, let $\varepsilon_k = \frac{\varepsilon_{k-1}}{(2+\varepsilon_{k-1})}$ and $c_k = \frac{c_{k-1}(1-\varepsilon_k)}{2}$. Let us assume that the first item is false. We will show that the second or the third item is true. Let $v_1$ be any vertex and $S = V(G) \setminus N[v_1]$. The size $s$ of $S$ is at least $(1-\varepsilon_k)n - 1$. If $S$ has only *small* connected components, meaning of size at most $s/2$, then one can divide the connected components into two parts with at least $(s+1)/4$ vertices each, and no edges between both parts. This gives in $\overline{G}$ a biclique of size $\frac{s+1}{4} \geqslant \frac{(1-\varepsilon_k)n}{4}$, thus of size at least $c_k n$ since $c_k \leqslant \frac{1-\varepsilon_k}{4}$. Otherwise, $S$ has a *giant* connected component $S'$, meaning of size $s'$ more than $s/2$. Let $v_2$ be a vertex adjacent both to $v_1$ and to some vertex in $S'$. Observe that $v_2$ exists since $G$ is connected. Consider now the graph $G_2$ induced by $S' \cup \{v_2\}$. The maximum degree in $G_2$ is still at most $\varepsilon_k n = \varepsilon_{k-1}(1-\varepsilon_k)n/2 \leqslant \varepsilon_{k-1}(s'+1)$. By the induction hypothesis, either the second or the third item is true for $G_2$ with parameter $k-1$. The second item gives an induced $P_{k-1}$ in $G_2$ starting at $v_2$, thus an induced $P_k$ in $G$ starting at $v_1$. The third item gives a biclique of size $c_{k-1}|V(G_2)|$ in $\overline{G_2}$. Since $|V(G_2)| = s' + 1 \geqslant ((1-\varepsilon_k)/2)n$, this gives a biclique of size at least $(c_{k-1}(1-\varepsilon_k)/2)n = c_k n$ and concludes the proof. □

We can now prove our main theorem:

*Proof of Theorem 2.17.* Let $\varepsilon_k$ be as defined in Lemma 2.20 and $\varepsilon = \varepsilon_k/8 > 0$. By Theorem 2.19, there exists $\gamma > 0$ such that every graph $G$ not inducing $P_k$ or $\overline{P_k}$ does contain an $\varepsilon$-stable set or an $\varepsilon$-clique of size at least $\gamma n$. Free to consider the complement of $G$, we can assume that $G$ contains an $\varepsilon$-stable set $S_0$ of size $\gamma n$. We start by deleting in $S_0$ all the vertices having at least $2\varepsilon|S_0|$ neighbors in $S_0$. Since the average degree in $S_0$ is at most $\varepsilon|S_0|$, we do not delete more than half of the vertices. We call $S$ the remaining subgraph which is a $4\varepsilon$-stable set of size $s \geqslant \gamma n/2$ with maximum degree $\leqslant 4\varepsilon s$.

Let $G_S$ be the graph induced by $S$. Our goal is to find a constant $c$ such that $\overline{G_S}$ have a biclique of size $cs$, which gives a biclique in $\overline{G}$ of size at least $c\gamma n/2$ and concludes the proof. Assume first that $G_S$ only has *small* connected components, meaning of size less than $s/2$. Then one can partition the connected components of $G_S$ in order to get a biclique in $\overline{G_S}$ of size $s/4$. Otherwise, $G_S$ has a connected component $S'$ of size $s' \geqslant s/2$. The degree of every vertex in $S'$ is at most $8\varepsilon s' = \varepsilon_k s'$, and $S'$ does not contain any induced $P_k$ since $G$ does not. By Lemma 2.20, there exists a biclique of size $c_k s' \geqslant c_k s/2$ in the complement of the graph induced by $S'$, thus in $\overline{G_S}$. □

# 2.3 Useful tools

This section is dedicated to the description of new tools that may help to prove the Erdős-Hajnal Conjecture, hopefully at least in some specific cases.

### 2.3.1 Unbalancing the Strong Erdős-Hajnal property

As seen in the previous section, the Strong Erdős-Hajnal property defined by Fox and Pach is a very good definition of a large clique-like or stable set-like subgraph: the key is to find a large biclique in the graph or its complement. In this subsection, we would like to introduce a modified version where the two sides of the biclique can be very unbalanced: a class $\mathcal{C}$ of graphs is said to have the *Unbalanced Strong Erdős-Hajnal property* if there is two positive functions $f_1, f_2$ such that

*(i)* for every graph $G \in \mathcal{C}$ on $n$ vertices, there exists a biclique $(X, Y)$ in $G$ or in $\overline{G}$ such that $|X| \geqslant f_1(n)$ and $|Y| \geqslant f_2(n)$.

*(ii)* there exists $\beta > 0$ such that $f_1(n)^\beta + f_2(n)^\beta \geqslant n^\beta$ for every $n$.

**THEOREM 2.21**

*Let $\mathcal{C}$ be a hereditary class of graphs having the Unbalanced Strong Erdős-Hajnal property. Then $\mathcal{C}$ has the Erdős-Hajnal property.*

*Proof.* We will prove by induction on $|V(G)|$ that every graph $G \in \mathcal{C}$ has an induced cograph of size $|V(G)|^\beta$ and conclude according to the properties of cographs. The result is trivial for $|V(G)| = 1$. Now let $G \in \mathcal{C}$ be a graph on $n$ vertices, by assumption on $\mathcal{C}$ there exists a biclique $(X, Y)$ in $G$ or in $\overline{G}$ with $|X| \geqslant f_1(n)$ and $|Y| \geqslant f_2(n)$. Apply the induction hypothesis to $G[X]$ and $G[Y]$ to get two cographs on vertex sets $V_X \subseteq X$ and $V_Y \subseteq Y$ of respective sizes $|V_X| \geqslant |X|^\beta$ and $|V_Y| \geqslant |Y|^\beta$. By definition of a cograph, $G[V_X \cup V_Y]$ is also a cograph (either the disjoint union or the join of $G[V_X]$ and $G[V_Y]$), and has size

$$\begin{aligned} |V_X \cup V_Y| = |V_X| + |V_Y| &\geqslant |X|^\beta + |Y|^\beta \\ &\geqslant f_1(n)^\beta + f_2(n)^\beta \\ &\geqslant n^\beta && \text{by assumption } (ii). \end{aligned}$$

□

Observe in particular that, if $f_1(n) = f_2(n) = c \cdot n$, we can choose $\beta = -1/\log c$ and this corresponds exactly to the Strong Erdős-Hajnal property. Moreover, we can apply Theorem 2.21 to another interesting particular case, namely graphs having a $n^{1-\varepsilon}$ degeneracy:

**COROLLARY 2.22**

*Let $\mathcal{C}$ be a class of graphs and $\varepsilon > 0$ such that for every $G \in \mathcal{C}$, there exists $v \in V(G)$ such that $|N(v)| < |V(G)|^{1-\varepsilon}$. Then $\mathcal{C}$ has the Erdős-Hajnal property.*

*Proof.* Let $f_1(n) = 1$ and $f_2(n) = n - n^{1-\varepsilon}$. Let $G \in \mathcal{C}$, $n = |V(G)|$ and $v \in V(G)$ such that $|N(v)| < n^{1-\varepsilon}$. Then $(\{v\}, V(G) \setminus N[v])$ is a biclique in $\overline{G}$ with

$$|\{v\}| = f_1(n) = 1 \quad \text{and} \quad |V(G) \setminus N[v]| \geqslant f_2(n)\,.$$

Moreover, with $\beta = \varepsilon$ we get:

$$\begin{aligned} f_1(n)^\beta + f_2(n)^\beta &= 1 + (n - n^{1-\varepsilon})^\varepsilon \\ &\geqslant 1 + n^\varepsilon \left(1 - n^{-\varepsilon}\right)^\varepsilon \\ &\geqslant 1 + n^\varepsilon (1 - n^{-\varepsilon}) && \text{since } n^{-\varepsilon} \leqslant 1 \\ &\geqslant n^\varepsilon \,. \end{aligned}$$

Consequently, $\mathcal{C}$ has the Unbalanced Strong Erdős-Hajnal property. Apply Theorem 2.21 to conclude. □

It would be interesting to apply the Unbalanced Strong Erdős-Hajnal property with other values of $f_1(n)$ and $f_2(n)$ (for example, $f_1(n) = n/\log^c(n)$ and $f_2(n) = n - n/\log n$ with $\beta = \min(1, 1/c)$), but unfortunately we cannot think of a good candidate class of graphs having such a biclique (in every graph or its complement).

### 2.3.2 Easy neighborhood for the Erdős-Hajnal property

As described in Chapter 1, a fruitful branch of research provides structural results for some classes of graphs. Since the celebrated decomposition theorem for perfect graphs made it possible to prove the Strong Perfect Graph theorem [40], one can hope that structural results help a lot to understand the class of graphs under study and to prove interesting properties. For example, Addario-Berry *et al.* [2] proved that every even-hole-free graphs has a bisimplicial vertex, *i.e.* a vertex whose neighborhood is the disjoint union of two cliques. Such a result enables them to prove that $\chi(G) \leqslant 2\omega(G) - 1$. Aboulker *et al.* [1] obtained a somehow resembling result: every diamond-wheel-free graph (a *wheel* consists in a hole together with an extra center vertex having at least 3 neighbors on the hole; a *diamond-wheel* is a wheel whose center has at least three consecutive neighbors on the hole) has a vertex whose neighborhood is a disjoint union of cliques. In fact, they get on 8 different classes of graphs (each of them defined as forbidding some Truemper configurations) a result of type: *every graph in $\mathcal{C}$ has a vertex whose neighborhood is $\mathcal{H}_\mathcal{C}$-free*, where $\mathcal{H}_\mathcal{C}$ is composed of one or two 3-vertex graphs.

We will be interested in this subsection in exploiting such special cases of structural statement: suppose we would like to prove that every graph $G$ in a class of graph $\mathcal{C}$ satisfies some property $\mathcal{P}$. Suppose moreover that structural properties of $\mathcal{C}$ prove that for every graph $G \in \mathcal{C}$, there exists a vertex $v$ whose neighborhood is *easier* than $\mathcal{C}$, meaning that it lies in a strict subclass $\mathcal{C}'$ of $\mathcal{C}$. A magic tool to have in our pocket would be a statement such as *if $\mathcal{P}$ holds for every graph in $\mathcal{C}'$, then $\mathcal{P}$ holds in every graph of $\mathcal{C}$*. We proved that such a statement is true if $\mathcal{P}$ is the Erdős-Hajnal property:

**THEOREM 2.23**

*Let $\mathcal{C}'$ be a class of graphs having the Erdős-Hajnal property. Let $\mathcal{C}$ be a hereditary class of graphs such that for every $G \in \mathcal{C}$, there exists $v \in V(G)$ such that $G[N(v)] \in \mathcal{C}'$. Then $\mathcal{C}$ has the Erdős-Hajnal property.*

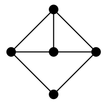


FIGURE 2.2: A minimal non-universal diamond-wheel.

To this end, we will use the aforementioned relation $n \leqslant \chi(G)\alpha(G)$ (thus one of $\chi(G)$ or $\alpha(G)$ has to be large) together with the following observation: either all the subgraphs of $G$ have a vertex of small degree, but then we can iteratively color $G$ with few colors and hence $\chi(G)$ is small; or there exists a subgraph of $G$ all vertices of which have large degree, then pick a vertex with an easy neighborhood and find in it a large enough clique or stable set. Let us make this more formal: a graph $G$ is *d-degenerate* if every induced subgraph of $G$ has a vertex of degree at most $d$. The *degeneracy* of a graph $G$ is the smallest $d$ such that $G$ is $d$-degenerate. This notion has been introduced by Lick and White [146] and can be computed in linear time [157]. It is closely related with the *coloring number* of $G$[1], introduced by Erdős and Hajnal [68]: it is the least $k$ for which $G$ has a vertex ordering such that each vertex has fewer than $k$ neighbors earlier in the ordering. Given such an ordering, one can easily color $G$ with $k$ colors by choosing the vertices in the reverse order. Thus the coloring number is an upper bound on the chromatic number (generally not tight). We can observe that a graph has degeneracy $d$ if and only if it has coloring number $d+1$. We can now prove the above theorem:

*Proof of Theorem 2.23.* Let $G \in \mathcal{C}$, $n = |V(G)|$ and $d$ be the degeneracy of $G$. Let $\varepsilon > 0$ be such that every graph $G'$ of $\mathcal{C}'$ has a clique or a stable set of size at least $|V(G')|^{\varepsilon}$. Let us prove that $G$ has a clique or a stable set of size at least $n^{\varepsilon/3}$ (in fact $n^{\varepsilon/2-\varepsilon'}$ for every $\varepsilon' > 0$). We have $n \leqslant \chi(G)\alpha(G)$, thus either $\alpha(G) \geqslant \sqrt{n}$ and we are done, or $\chi(G) \geqslant \sqrt{n}$. However as seen before, $\chi(G) \leqslant d+1$ thus $d \geqslant \sqrt{n}-1$. Moreover, $d$ is the minimum number $k$ such that $G$ is $k$-degenerate so there exists an induced subgraph $G'$ of $G$ such that every vertex of $G'$ has degree at least $d$. Now $\mathcal{C}$ is hereditary so $G' \in \mathcal{C}$ and thus there exists a vertex $v \in V(G')$ with an easy neighborhood, *i.e.* such that $G'[N(v)] \in \mathcal{C}'$. By property of $\mathcal{C}'$, we find a clique or a stable set in $G'[N(v)]$ of size $|N(v)|^{\varepsilon} \geqslant d^{\varepsilon}$. This is a clique or a stable set in $G$ of size at least $(\sqrt{n}-1)^{\varepsilon} \geqslant n^{\varepsilon/3}$. □

As discussed before, one can apply this result to even-hole-free graphs and to diamond-wheel-free graphs, but this does not give any new result since even-hole-free graphs do not contain $C_4$, and diamond-wheel-free graphs do not contain the non-prime 5-vertex graph described on Figure 2.2. However, based on Theorem 2.14 by Bonamy *et al.*, we can obtain the following corollary, where a *universal wheel* (resp. *universal antiwheel*) of length $k$ is a hole (resp. antihole) of length $k$ with an additional vertex adjacent to every vertex on the hole.

[1]Not to be mistaken for the chromatic number of $G$

**COROLLARY 2.24**

*Let $k \geqslant 4$ be an integer and $\mathcal{C}$ be the class of graphs with no universal wheel or antiwheel of length at least $k$. Then $\mathcal{C}$ has the Erdős-Hajnal property.*

*Proof.* Let $\mathcal{C}'$ be the class of graphs with no holes nor antiholes of length at least $k$. Let $G \in \mathcal{C}$ and $v \in V(G)$. Then $G[N(v)]$ belong to $\mathcal{C}'$, otherwise any hole or antihole of length at least $k$ together with $v$ would form a universal wheel or antiwheel of length at least $k$. By Theorem 2.14, $\mathcal{C}'$ has the Erdős-Hajnal property. By Theorem 2.23, $\mathcal{C}$ has the Erdős-Hajnal property. $\square$

We can observe that obtaining an analog of Theorem 2.23 for the Strong Erdős-Hajnal property seems much harder (in particular the previous proof cannot easily be adapted), and may be wrong.

# Chapter 3

# Clique-Stable Set Separation

THE third kind of interaction under study is maybe a less standard problem in graph theory: the Clique-Stable Set Separation. The deep motivation for it comes from combinatorial optimization and Yannakakis' seminal paper on Extended Formulations and will be detailed in Chapter 4. For now, let us state the question as a communication complexity problem: let Alice, Bob and the Prover be three characters and $G$ be a graph that they all know. At the beginning, they can discuss and agree on a protocol. Then Alice is given a clique $K$, Bob is given a stable set $S$, and only the Prover knows both of them. The goal is for Alice and Bob to decide whether $K$ and $S$ have a non-empty intersection. The Prover can help them but instead of just asserting *Yes* or *No*, he has to provide a *Yes-certificate* or *No-certificate* to convince Alice and Bob of the right answer. The goal is to find a certificate as short as possible. As we will see in this Chapter, the *No-certificates* are of particular interest.

We first define properly the problem, then briefly survey the upper and lower bounds known so far, as well as the trivial cases. We then continue with positive answers in more sophisticated classes of graphs: we start with random graphs, although one could think of those graphs having a lot of intertwined cliques and stable sets, and then we study $H$-free graphs for any split graph $H$. We also highlight interesting similarities in proof techniques between the Erdős-Hajnal property and the Clique-Stable Set Separation and apply this on $(P_k, \overline{P_k})$-free graphs. Finally, we focus on perfect graphs with no balanced skew-partition.

Note that the content of this chapter is covered in the following two papers:

[III] **Clique versus Independent Set**, with N. BOUSQUET and S. THOMASSÉ, *European Journal of Combinatorics*, 40:73–92, 2014.

[IV] **Clique-Stable set Separation in perfect graphs with no balanced skew-partition**, with T. TRUNCK, *ArXiv:1312.2730 (Computer Science>Discrete Mathematics)*, Submitted for publication.

## 3.1 Definition and Context

Let us first come back to the definitions: we are interested in a *non-deterministic* protocol for the *Clique versus Independent Set problem*[1], but let us first start with the *deterministic* setting, which is simpler: at the beginning, the graph $G$ is public and Alice and Bob can discuss for free, they can share some information about the graph (for example numbering the vertices) and agree on a protocol. Then Alice is given a clique $K$ (but $K$ is not shown to Bob), Bob is given a stable set $S$ (but $S$ is not shown to Alice), and the goal is for Alice and Bob to decide whether $K$ and $S$ have a non-empty intersection. They can communicate but they have to pay for every bit that they send to each other. When they have found the answer, Alice or Bob outputs *Yes* (if $K$ and $S$ intersect) or *No* (if they do not). Of course, Bob could send his whole input to Alice, but this is too costly in the worst case.

Let us now come to *non-deterministic* protocols which have been introduced by Lipton and Sedgewick in 1981 [147]. In the non-deterministic protocols, Alice can *guess a certificate* for the right answer, that is to say she can guess some information that, once checked by her and Bob, proves that the clique and the stable set intersect (resp. do not intersect). Symmetry is lost between answering *Yes* or *No*, as in the computational complexity setting (problems in NP are not necessarily in co-NP and vice-versa). It is never easy to understand how non-determinism works, so let us tell the story differently and introduce a third character called the Prover. Instead of *guessing* the answer, the Prover *knows* the answer, transmits it to Alice and Bob and tries to convince them that he is not lying.

So, at the beginning, $G$ is public and the three characters can discuss for free, as in the deterministic setting, and they can agree on a protocol. Then Alice is given a clique $K$, Bob is given a stable set $S$, and only the Prover knows both of them. The goal is for Alice and Bob to decide whether $K$ and $S$ have a non-empty intersection, with the help of the Prover that they do not really trust. As already mentioned, the symmetry between *Yes* or *No* is lost, so let us carry this asymmetry over the Prover: we assume that it is publicly known that the Prover wants to convince Alice and Bob that the answer is $b \in \{Yes, No\}$ (no matter what the correct answer is). The Prover writes a sequence of bits, which we call a $b$-certificate, on a blackboard that both Alice and Bob can read. Alice and Bob have each two buttons in front of them : either *Accept the certificate* or *Reject the certificate*. Knowing her (respectively his) own input and what is written on the blackboard, Alice (respectively Bob) decides which button to press, according to the predetermined protocol. If both of them press *Accept*, then the certificate is accepted and the system outputs $b$. Otherwise, the certificate is rejected, meaning that the Prover was unable to convince Alice and Bob that the answer was $b$. The protocol is *$b$-correct* if the following holds: the Prover can provide an accepted

[1]Historical note: the problem was introduced by Yannakakis [211] in 1991, but it was not given a name; the oldest appearance of this name that we could find dates from 1999 [79] when its usage became standard (see e.g. Huang and Sudakov 2012 [122]). Although we use here *stable set* to designate an *independent set*, we keep the historical name for the communication complexity problem.

$b$-certificate if and only if the correct answer is $b$. Since chalk is expensive, the Prover wants to minimize the length of the $b$-certificate in the worst-case. More formally, the *cost* of a protocol is the maximum number of bits needed to write the $b$-certificate over all possible entries. The *non-deterministic communication complexity for b* is the minimum cost of a $b$-correct protocol[2].

It is easy to find a *Yes*-correct protocol with low cost. Suppose that the Prover wants to convince Alice and Bob that the answer is *Yes*. If $K$ and $S$ indeed intersect[3] in $\{v_i\}$, the Prover can write $i$ as a *Yes*-certificate on the blackboard; Alice (resp. Bob) presses *Accept* if and only if $v_i \in K$ (resp. $v_i \in S$). If both Alice and Bob accept, then it is clear that $K$ and $S$ indeed intersect. Hence, it is easy to see that there exists an accepted *Yes*-certificate if and only if the correct answer is *Yes*, which means that the protocol is *Yes*-correct. The cost of this protocol is $\lceil \log |V(G)| \rceil$.

Constructing a non-trivial *No*-correct protocol is much less obvious: what could be a *No*-certificate? Suppose the three characters agree at the beginning on $k$ different ways $(B_1, W_1), \dots, (B_k, W_k)$ of cutting the graph into two parts: the *black* vertices $B_i$ and the *white* vertices $W_i$; we assume $W_i = V(G) \setminus B_i$ and hence $(B_i, W_i)$ is called a *cut*. At first, just assume that they enumerate all the $2^n$ possible cuts. Now if $K \cap S = \emptyset$, the Prover can find $i$ such that all the vertices of $K$ are white and all the vertices of $S$ are black in the $i$-th cut $(B_i, W_i)$. Then he writes $i$ on the blackboard as a *No*-certificate, and Alice (resp. Bob) presses *Accept* if and only if $K \subseteq B_i$ (resp. $S \subseteq W_i$). If they both accept, it is clear that $K$ and $S$ indeed do not intersect. Conversely, it is also clear that the Prover can find an accepted *No*-certificate if $K \cap S = \emptyset$. Consequently the protocol is *No*-correct and its cost is $\lceil \log k \rceil = \mathcal{O}(n)$, which is much worse than the $\mathcal{O}(\log n)$ upper bound on the non-deterministic communication complexity for *Yes*. The question we study here is whether *it is possible to ensure both the* No*-correctness and a $\mathcal{O}(\log n)$ cost* (i.e. *is k polynomial in n)?*

The *No*-correctness is thus defined by the following condition on the initially chosen set $\mathcal{F}$ of cuts: for every clique $K$ and stable set $S$, either $K \cap S \neq \emptyset$ or there exists a cut $(B, W) \in \mathcal{F}$ such that $K \subseteq B$ and $S \subseteq W$ (the cut *separates* $K$ and $S$). A set $\mathcal{F}$ of cuts that satisfies this condition is called a *Clique-Stable Set Separator*[4] (*CS-Separator* for short). Its *size* is just the number of cuts it contains.

One can ask how important is the role played by CS-separators in this communication complexity problem. Indeed, this is *one* solution to provide a *No-certificate*, but maybe we can think of something better? Yannakakis proved that we cannot:

[2]We take some liberties here with respect to the classical terminology. Indeed, the terms *b-correct* and *non-deterministic communication complexity for b* are not standard, but we use them to emphasize the distinction between non-deterministic protocols for the answer *Yes* and for the answer *No*. For a more standard introduction on communication complexity, see Subsection 4.8.

[3]Observe that a clique and a stable set cannot intersect in more than one vertex.

[4]Historical note: it was previously called *splitting family* by Yannakakis [211] and *separating family* by Lovász [153].

**THEOREM 3.1** [211]

*For every graph $G$ and integer $k$, the following are equivalent:*

- *There exists a CS-Separator for $G$ of size at most $2^k$.*
- *There exists a* No*-correct non-deterministic protocol for the* Clique versus Independent Set problem *on $G$ of communication complexity $k$.*

*Proof.* Finding a *No*-correct non-deterministic protocol given a CS-Separator of size $2^k$ has been described in the above discussion. Let us now assume the existence of such a *No*-correct protocol of cost $k$. Then the Prover uses at most $2^k$ different *No*-certificates over all possible inputs, let us call theses certificates $c_1, \ldots, c_{2^k}$. Now, for $1 \leqslant i \leqslant 2^k$, let us call $\mathcal{C}_i$ the set of inputs for which the Prover sends $c_i$, that is to say:

$$\mathcal{C}_i = \left\{ (K,S) \;\middle|\; \begin{array}{l} K \subseteq V(G) \text{ is a clique,} \\ S \subseteq V(G) \text{ is a stable set,} \\ \text{and } c_i \text{ is an accepted } \textit{No-certificate} \text{ for the input } (K,S). \end{array} \right\}$$

Let us now emphasize that, when Alice and Bob choose to accept or reject the certificate, the only piece of information they have access to is their own part of the input ($K$ for Alice, $S$ for Bob) and the message $c_i$ written on the blackboard.

This is a deterministic choice for Alice, given $K$ and $c_i$ (resp. for Bob, given $S$ and $c_i$). By correctness assumption, if $(K,S) \in \mathcal{C}_i$ then $c_i$ is an accepted *No*-certificate so $K \cap S = \emptyset$. Moreover, consider $(K,S),(K',S') \in \mathcal{C}_i$ and prove that $c_i$ is also an accepted *No*-certificate for input $(K,S')$ (and $(K',S)$ by similar arguments): given $K$ and $c_i$, Alice presses the button *Accept* (because this is what she does when the input is $(K,S)$); given $S'$ and $c_i$, Bob presses the button *Accept* (because this is what he does if the input is $(K',S')$). Thus $c_i$ is an accepted *No*-certificate for input $(K,S')$, so $K \cap S' = \emptyset$ by correctness of the protocol. Consequently, by setting

$$B_i = \bigcup_{(K,S)\in\mathcal{C}_i} K, \quad W_i = V(G) \setminus B_i, \quad \text{for } 1 \leqslant i \leqslant 2^k$$

we ensure that $(B_i, W_i)$ separates every $(K,S) \in \mathcal{C}_i$: we indeed have a CS-Separator of size $2^k$. □

As a consequence, the above question concerning communication complexity can be equivalently restated as follows: *given a graph $G$, does there exist a CS-Separator of size polynomial in $|V(G)|$?* Or, in a slightly different point of view, we can wonder *for which graphs can we do so?* Finding upper or lower bounds for CS-Separators in the general case or in some specific classes of graphs is what we informally call the *CS-Separation*. As already mentioned, it is easy to find a CS-Separator of size $2^n$ by taking all possible cuts, but this is too costly (this is asymptotically as bad as writing Bob's whole stable set). Can we achieve a better upper bound? The answer is yes. To prove this result, Yannakakis designed a deterministic protocol, as described at the beginning of this section. The cost of a deterministic protocol is the total number of bits exchanged in the worst case. The deterministic communication complexity of a problem is the minimum cost over all correct protocols, where *correct* means here that Alice and Bob must always output the right answer. Observe that any correct deterministic protocol can be turned into a $b$-correct non-deterministic protocol, for $b \in \{Yes, No\}$: the Prover writes on the blackboard the whole conversation and Alice and Bob check whether it matches what they would have said in this case. The cost is the same as the initial deterministic protocol, and the $b$-correctness is ensured by the correctness of the initial protocol. Thus designing a deterministic protocol with cost $k$ gives also a $k$ upper bound for the non-deterministic communication complexity for *Yes* and for *No*.

**THEOREM 3.2** [211]

*For every graph G on n vertices, there exists a deterministic protocol for the* Clique versus Independent Set problem *with communication complexity* $\frac{1}{2}\log^2 n + o(\log^2 n)$*. As a consequence, there exists a CS-Separator for G of size* $n^{(\log n)/2+o(\log n)}$.

*Proof.* Let $K$ be Alice's clique, $S$ be Bob's stable set and let us set $G_0 = G$. Let us design a deterministic protocol with several rounds, each of them dividing by two the size of the remaining graph $G_i$ with the property that $K \cap S \subseteq G_i$. At each round $i$, Alice checks whether she has a vertex $v \in K$ of degree less than $|V(G_i)|/2$ in $G_i$. If yes, she sends $v$ and Bob knows that the intersection vertex between $K$ and $S$, if any, is either $v$ itself or one of its neighbors. If $v \in S$, Bob outputs *Yes*. Otherwise, they both set $G_{i+1} = G_i[N(v)]$ and continue with the next round. If Alice does not have such a vertex, she notifies Bob and then he checks whether he has a vertex $v \in S$ of degree at least $|V(G_i)|/2$. If yes, he sends $v$, and similarly to the previous case, Alice outputs *Yes* if $v \in K$, otherwise they set $G_{i+1} = G_i \setminus N[v]$. If Bob does not have a suitable vertex to send, he outputs *No* since every vertex of $K$ has small degree and every vertex of $S$ has large degree. For each round $i$, we have $|V(G_{i+1})| < |V(G_i)|/2 \leqslant n/2^{i+1}$, and the round costs $\lceil \log |V(G_i)| \rceil$, which is at most $\lceil \log n \rceil - i$. Moreover, the protocol finishes in at most $k = \lceil \log n \rceil$ rounds and thus costs at most

$$\sum_{i=1}^{k}(k-i) = k^2 - \frac{k(k+1)}{2} = \frac{1}{2}\log^2 n + o(\log^2 n)\,,$$

which concludes the proof. □

As for lower bounds on the size of CS-Separators, it has been a long-standing open problem to find a superlinear one in the general case, and it was finally found by Huang and Sudakov in 2012 as a counterexample to the Alon-Saks-Seymour conjecture (the statement of the conjecture and the links with the CS-separation can be found in Chapter 5).

**THEOREM 3.3** [122]

*There exists an infinite family of graphs $(G_n)_{n\in\mathbb{N}}$ such that any CS-Separator of $G_n$ has size $\Omega(|V(G_n)|^{6/5})$.*

The lower bound was improved to $\Omega(|V(G)|^{\frac{3}{2}})$ and then to $\Omega(|V(G)|^{2-o(1)})$ by Amano and Shigeta in 2013 [8, 190]. It is nice to observe that all three proofs are constructive. The gap was still huge between $\Omega(n^{2-o(1)})$ and $\mathcal{O}(n^{\log n/2})$, and in particular the question of finding a polynomial upper bound was widely opened until Göös' very recent result [108]: he found a family of graphs for which the size of a CS-Separator is $|V(G)|^{\Omega(\log^{0.128}|V(G)|)}$, ruining any hope of a general polynomial upper bound.

The initial question that arose to Yannakakis was in fact only concerned with perfect graphs, but no better upper bound is known even for them. However, a polynomial upper bound was provided for some subclasses of perfect graphs in [211]: chordal graphs and their complements, and comparability graphs and their complements. Let us see how Yannakakis achieved these results. To begin with, observe that finding a CS-Separator is a self-complementary problem since the family $\mathcal{F} = \{(B_1, W_1), \ldots, (B_k, W_k)\}$ is a CS-Separator for $G$ if and only if the family $\overline{\mathcal{F}} = \{(W_1, B_1), \ldots, (W_k, B_k)\}$ is a CS-Separator for $\overline{G}$. Let us continue with an easy observation which, informally, states that we only need to focus on inclusion-wise maximal cliques and stable sets :

**OBSERVATION 3.4**

*Let $G = (V, E)$ be a graph on $n$ vertices and $\mathcal{F}$ be a family of $k$ cuts such that for every inclusion-wise maximal clique $K$ and inclusion-wise maximal stable set $S$, there exists a cut in $\mathcal{F}$ that separates $K$ and $S$. Then $G$ has a CS-Separator of size $k + 2n$.*

*Proof.* For every $v \in V$, let $Cut_{1,v}$ be the cut $(N[v], V \setminus N[v])$ and $Cut_{2,v}$ be the cut $(N(v), V \setminus N(v))$. Let us prove that $\mathcal{F}' = \mathcal{F} \cup \{Cut_{1,v} \mid v \in V\} \cup \{Cut_{2,v} \mid v \in V\}$ is a CS-separator. Let $K$ be a clique and $S$ be a stable set. Extend $K$ and $S$ by adding vertices to get an inclusion-wise maximal clique $K'$ and an inclusion-wise maximal stable set $S'$. Either $K'$ and $S'$ do not intersect, and there is a cut in $\mathcal{F}$ that separates $K'$ and $S'$ (thus $K$ and $S$); or $K'$ and $S'$ intersect in some vertex $v$ (recall that a clique and a stable set intersect on at most one vertex): if $v \in K$, then $Cut_{1,v}$ separates $K$ and $S$, otherwise $Cut_{2,v}$ does. $\square$

Based on this lemma, from now on we may make a slight abuse of notation and call a *CS-Separator* any family of cuts that separates all the inclusion-wise maximal cliques and inclusion-wise maximal stable sets, regardless the case of non-maximal cliques and stable sets. Indeed, we are interested in providing polynomial upper bounds for the size of CS-Separators, and Observation 3.4 ensures that we can ignore non-maximal cliques and stable sets, up to adding a linear number of cuts. We can now easily deduce the following:

**OBSERVATION 3.5**

*Let $\mathcal{C}$ be a class of graphs and $c > 0$ be a constant such that for every $G \in \mathcal{C}$, the graph $G$ admits $\mathcal{O}(|V(G)|^c)$ different inclusion-wise maximal cliques (or stable sets). Then every $G \in \mathcal{C}$ admits a $\mathcal{O}(|V(G)|^c)$ CS-Separator.*

*Proof.* Let $G \in \mathcal{C}$ and, up to changing $G$ into $\overline{G}$, assume $G$ has at most $\mathcal{O}(|V(G)|^c)$ maximal cliques. For every maximal clique $K$, we consider the cut $(K, V(G) \setminus K)$. The set of all these cuts gives a CS-Separator of size $\mathcal{O}(|V(G)|^c)$. □

We can apply Observation 3.5 to the following particular cases:

**OBSERVATION 3.6** (partially rephrased from [211])

- *Let $k$ be a positive integer. If $\omega(G) \leqslant k$ or $\alpha(G) \leqslant k$ then $G$ admits a $\mathcal{O}(|V(G)|^k)$ CS-Separator.*
- *Every chordal graph $G$ admits a $\mathcal{O}(|V(G)|)$ CS-Separator.*
- *More generally, every $C_4$-free graph $G$ admits CS-Separator of size $\mathcal{O}(|V(G)|^2)$.*

*Proof.* The first item is trivial by Observation 3.4. The second one relies on the well-known property of chordal graphs having a linear number of inclusion-wise maximal cliques [95]. The last one is a consequence of a result by Alekseev [3] stating that any $kK_2$-free graph (*i.e.* graphs with no induced matching of size $k$) has at most $\mathcal{O}(|V(G)|^{2k-2})$ maximal stable sets. A graph is $C_4$-free if and only if its complement is $2K_2$-free, so we conclude with Observation 3.5. □

The proof for comparability graphs does not rely on the same argument. It uses more precisely the properties of those graphs:

**THEOREM 3.7** [211]

*For every comparability graph $G$, there exists a* No*-correct non-deterministic protocol for the* Clique versus Independent Set problem *on $G$ of complexity* $2\lceil \log n \rceil$. *This implies the existence of a CS-Separator of size $\mathcal{O}(n^2)$ for $G$.*

*Proof.* By definition of comparability graph, there is an order $\prec$ on $V(G)$ consistent with an orientation of the edges. Observe that a clique $K$ is a chain $v_1 \prec \ldots \prec v_k$ in the order and that a stable set $S$ is an antichain. Suppose that $K \cap S = \emptyset$, then three disjoint cases can occur (intuitively, these are the three possible answers when looking for the greatest vertex of $K$ among those having a greater neighbor in $S$):

*(i)* *$S$ is somehow greater than $K$*: there exists $s \in S$ such that $v_k \prec s$ (and thus $v_i \prec s$ for every $v_i \in K$).

*(ii)* *$S$ is intermediate compared to $K$*: there exists $v_i \in K$ and $s \in S$ such that $v_i \prec s$ and $v_{i+1} \not\prec s'$ for every $s' \in S$.

*(iii) S is smaller or incomparable to K*: for every $s \in S$, we have $v_1 \not\prec s$ (but maybe $s \prec v_1$). Thus $v_i \not\prec s$ for every $v_i \in K$ and $s \in S$.

Here is the protocol: in case (i), the Prover writes $v_k$ (and, say, one bit to differentiate from case (iii)) on the blackboard. Alice checks that it is indeed her greatest node, and Bob checks that there exists $s \in S$ such that $v_k \prec s$. If they both accept the certificate, then they can be sure that $K$ and $S$ do not intersect, for otherwise there would be $s' \in K \cap S$ such that $s' \prec v_k \prec s$ (or $s' = v_k \prec s$), a contradiction with $S$ being a stable set.

In case (ii), the Prover writes $v_i$ and $v_{i+1}$: Alice checks that these are indeed two consecutive vertices in her clique, and Bob checks that $v_{i+1} \notin S$, and that for every $s' \in S$, $v_{i+1} \not\prec s'$ and finally that there exists $s \in S$ such that $v_i \prec s$. Then they can be sure that $S$ do not intersect $K$: otherwise, there exists $s' \in K \cap S$. If $s' \prec v_i$, then $s' \prec v_i \prec s$, a contradiction; if $v_i \prec s'$, then either $s' = v_{i+1}$ or $v_{i+1} \prec s'$, both yielding a contradiction.

Finally in case (iii), the Prover writes $v_1$ on the blackboard (and one bit to distinguish from case (i)): Alice checks that $v_1$ is her smallest vertex, and Bob checks that $v_1 \not\prec s$ for every $s \in S$. Then it is clear that no vertex of $S$ can be in the clique.

Thus the protocol is *No*-correct, and the cost is $2\lceil \log n \rceil$ (since, in the worst case, the Prover has to write the name of two vertices). □

To find out more on communication complexity and related matters, the reader can consult Lovász' survey [153] and the book devoted to the subject by Kushilevitz and Nisan [144].

For links between the CS-separation and other existing problems, the reader can go directly to the next chapters: Chapter 4 explains Yannakakis's combinatorial optimization motivation for introducing this problem, and provides an introduction to the world of Extended Formulations. Chapter 5 explores the links with the Alon-Saks-Seymour conjecture whereas Chapter 6 compares the CS-Separation with two Constraint Satisfaction Problems called THE STUBBORN PROBLEM and 3-COMPATIBLE-COLORING-PROBLEM.

Our goal in this chapter is to provide tools to find classes of graphs having a polynomial CS-Separator, in order to get closer to an answer for the perfect graph case or the general case.

# 3.2 Random graphs

The first very natural thing to start with is random graphs. Indeed, cliques and stable sets in random graphs do not have a well-structured behavior, they are quite intertwined. Because of this, one could think of random graphs as good candidates to provide a superpolynomial lower bound on the size of a CS-Separator. However, when the edges are drawn with probability $1/2$, the maximum cliques and stable sets have size roughly $2\log(|V(G)|)$ which, as we will see, is *not too large* in the sense that we can find a wise way to separate all the pairs of subsets of vertices of this size. Now when the edge probability changes, say it becomes smaller, then maximum stable sets can grow but maximum cliques decrease. This gives us enough control to find a polynomial-size CS-Separator in random graphs whatever the edge probability. Let us now go for a few formal definitions.

Let $n$ be a positive integer and $p \in [0,1]$ (observe that $p$ can depend on $n$). We will work on the *Erdős-Rényi model* [73]: the random graph $G(n,p)$ is a probability space over the set of graphs on the vertex set $\{1,\dots,n\}$ determined by $\mathbb{P}[ij \in E] = p$, with these events mutually independent. A family $\mathcal{F}$ of cuts on a graph $G$ with $n$ vertices is a *complete $(a_1,a_2)$-separator* if for every pair $(A_1,A_2)$ of disjoint subsets of vertices with $|A_1| \leqslant a_1$, $|A_2| \leqslant a_2$, there exists a cut $(B,W) \in \mathcal{F}$ separating $A_1$ and $A_2$, namely $A_1 \subseteq B$ and $A_2 \subseteq W$.

**THEOREM 3.8**

*Let $n$ be a positive integer and $p \in [0,1]$ be the probability of an edge (possibly depending on $n$). Then there exists a family $\mathcal{F}_{n,p}$ of cuts of size $\mathcal{O}(n^7)$ such that for every graph $G \in G(n,p)$, the probability that $\mathcal{F}_{n,p}$ is a CS-separator for $G$ tends to 1 when $n$ goes to infinity.*

*Proof.* We distinguish two cases: assume first that $p \leqslant 1/\sqrt{n}$ and consider the probability to have a clique of size 6:

$$\mathbb{P}(\exists \text{ a clique of size } 6) \leqslant \binom{n}{6} p^{\binom{6}{2}} \leqslant \frac{n^6}{(\sqrt{n})^{15}} \leqslant n^{-3/2} \underset{n\to+\infty}{\longrightarrow} 0\,.$$

Hence every potential clique have size at most five. Define the family $\mathcal{F}_{n,p}$ of size $\mathcal{O}(n^5)$:

$$\mathcal{F}_{n,p} = \{(B,W) |\ B \subseteq V(G), |B| \leqslant 5, W = V(G) \setminus B\}\,,$$

then the statement holds with $\mathcal{F}_{n,p}$. If $1-p \leqslant 1/\sqrt{n}$, the proof is the same as above by exchanging the roles of cliques and stable sets and by taking

$$\mathcal{F}_{n,p} = \{(B,W) |\ B \subseteq V(G), |W| \leqslant 5, W = V(G) \setminus B\}\,.$$

For the second case, we can now suppose that $p > 1/\sqrt{n}$ and $1-p > 1/\sqrt{n}$. Following classical results [15] for the case where $p$ is fixed and independent from $n$, let

$$r_\omega = \frac{3\log n}{-\log p} \qquad \text{and} \qquad r_\alpha = \frac{3\log n}{-\log(1-p)}\,.$$

The first goal is to construct a complete $(r_\omega, r_\alpha)$-separator. Draw a random partition $(V_1,V_2)$ where each vertex is put in $V_1$ independently from the others with probability $p$, and in $V_2$ otherwise. Let $A_1$ and $A_2$ be two disjoint subsets of vertices of respective size $r_\omega$ and $r_\alpha$. There are at most $4^n$ such pairs. The probability that $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ is at least $p^{r_\omega}(1-p)^{r_\alpha}$. Observe that $p^{r_\omega}(1-p)^{r_\alpha} = 1/n^6$. Then on average $(A_1,A_2)$ is separated by at least $1/n^6$ of all the partitions. By double-counting on

$$T = \sum_{(A_1,A_2)\in\mathcal{P}} \sum_{(B,W)\in\mathcal{C}} \mathbb{1}(A_1 \subseteq B,\ A_2 \subseteq W)$$

where $\mathcal{C}$ is the set of all possible cuts and $\mathcal{P}$ is the set of all pairs of subsets $(A_1, A_2) \subseteq V(G)^2$ such that $|A_1| \leqslant r_\omega$ and $|A_2| \leqslant r_\alpha$, we obtain the existence of a cut $(B, W) \in \mathcal{C}$ that separates at least $|\mathcal{P}|/n^6$ pairs of $\mathcal{P}$.

We delete these newly separated pairs and add the partition $(B, W)$ to $\mathcal{F}_{n,p}$. There remain at most $(1 - 1/n^6) \cdot 4^n$ pairs. The same probability for a pair $(A_1, A_2) \in \mathcal{P}$ to be cut by a random partition still holds, hence we can iterate the process $i$ times until $(1 - 1/n^6)^i \cdot 4^n \leqslant 1$. This is satisfied for $i = 2n^7$. Thus starting from $\mathcal{F}_{n,p} = \emptyset$ and adding one by one the selected cuts, we achieve a complete $(r_\omega, r_\alpha)$-separator of size $\mathcal{O}(n^7)$.

The second goal is to prove that the probability that $\mathcal{F}_{n,p}$ is a CS-separator for $G$ tends to 1 when $n$ goes to infinity. It is enough to prove that the probability that there exists a clique (resp. stable set) of size $r_\omega$ (resp. $r_\alpha$ ) tends to 0 when $n$ goes to infinity. Both are similar by exchanging $p$ (resp. clique) and $1 - p$ (resp. stable set). Observe that

$$\mathbb{P}(\exists K, |K| = r_\omega, K \text{ is a clique}) \leqslant \binom{n}{r_\omega} p^{\binom{r_\omega}{2}}$$

Standard calculation using the Stirling approximation shows that this expression is equivalent to $(2\pi)^{-1/2} f(n)$ where

$$f(n) = \left(1 - \frac{r_\omega}{n}\right)^{-n-1/2} \left(\frac{n}{r_\omega} - 1\right)^{r_\omega} r_\omega^{-1/2} p^{\frac{r_\omega(r_\omega-1)}{2}}$$

Observe now that, since $1 - p > 1/\sqrt{n}$, then

$$\frac{r_\omega}{n} \leqslant \frac{3\log n}{\sqrt{n} + o(\sqrt{n})} \underset{n\to+\infty}{\longrightarrow} 0 \qquad \text{thus} \qquad -\left(n + \frac{1}{2}\right) \log\left(1 - \frac{r_\omega}{n}\right) = r_\omega + o(r_\omega)$$

Then standard calculation gives

$$\log(f(n)) \leqslant r_\omega + o(r_\omega) + r_\omega \log n - r_\omega \log r_\omega - \frac{1}{2}\log r_\omega + \frac{r_\omega(r_\omega - 1)}{2} \log p$$

and

$$\frac{r_\omega(r_\omega - 1)}{2} \log p = -\frac{3}{2} r_\omega \log n + \frac{3}{2} \log n$$

Moreover, since $p > 1/\sqrt{n}$, then $r_\omega \geqslant 6$ which implies $(r_\omega + 1/2)\log r_\omega \geqslant 0$ and $r_\omega \log n \underset{n\to+\infty}{\longrightarrow} +\infty$. Thus

$$\begin{aligned}
\log(f(n)) &\leqslant r_\omega + o(r_\omega) + r_\omega \log n - \frac{3}{2} r_\omega \log n + \frac{3}{2}\log n \\
&\leqslant -\left(\frac{1}{2} + o(1)\right) r_\omega \log n + \frac{3}{2}\log n \\
&\leqslant -\left(\frac{1}{2} + o(1)\right) 6\log n + \frac{3}{2}\log n \\
&\leqslant \left(-\frac{3}{2} + o(1)\right) \log n \underset{n\to+\infty}{\longrightarrow} -\infty\,.
\end{aligned}$$

□

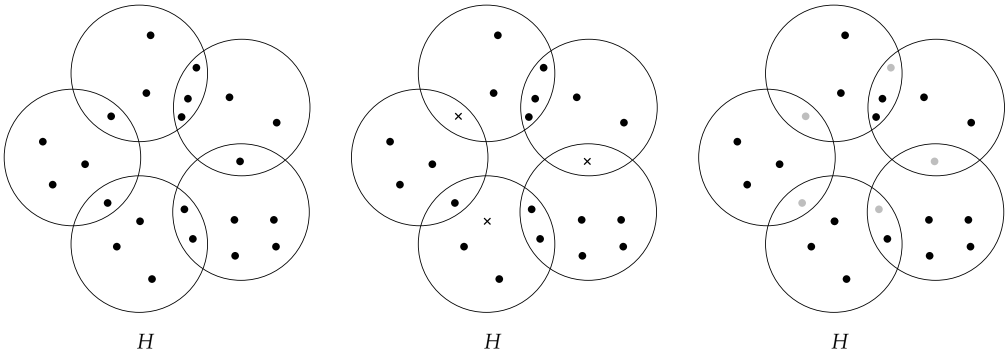
Note here that no optimization was made on the degree of the polynomial. For instance, replacing the constant 3 by $(5/2+\varepsilon)$ in the definition of $r_\omega$ and $r_\alpha$ leads to a CS-separator of size $\mathcal{O}(n^{6+2\varepsilon})$.

We provide here an upper bound for CS-Separators in random graphs, but there is still an interesting remaining question: *can we find a lower bound on the degree of the polynomial for the size of a Clique-Stable Set Separator in random graphs, in particular for the special case $p = 1/2$?*

# 3.3 Forbidding a fixed split graph

Remember that in Chapter 2, we observed that the clique and stable set behaviour is expected to be much different between random graphs and $H$-free graphs, for any graph $H$: this is basically the idea behind the Erdős-Hajnal conjecture. In this respect, proving the existence of a polynomial-size CS-Separator for $H$-free graphs (for any fixed $H$) would be a result very complementary to Theorem 3.8. Unfortunately, for now we could not achieve such a result, but we still have a positive answer for the case where $H$ is a *split* graph, meaning that $V(H)$ can be partitioned into a clique and a stable set. The starting point of this result is the following: because of Yannakakis' result on comparability graphs (Theorem 3.7), we were interested in the CS-Separation on net-free graphs (remember that comparability graphs do not contain the graph called *net*, depicted on Figure 1.2, which is itself a split graph). It turned out that the method we found for net-free graphs could be adapted for $H$-free graphs for every split graph $H$.

The proof uses tools coming from the world of hypergraphs, so let us give the necessary definitions. A *hypergraph* $H = (V, E)$ is a generalization of a graph: $V$ is still a set of *vertices*, and $E$ is a set of *hyperedges*, *i.e.* sets of vertices, with no control on their size. A graph is just a hypergraph where every hyperedge contains exactly two vertices. To avoid confusion, in the rest of this section, $H$ will be used only to denote a hypergraph, and $\Gamma$ will be used to denote the split graph we want to forbid. Let us now state some definitions concerning hypergraphs. Let $H = (V, E)$ be a hypergraph. The *transversality* $\tau(H)$ (see Figure 3.1) is the minimum cardinality of a *transversal*, *i.e.*




(a) A hypergraph

(b) The cross vertices form a transversal of min. size: $\tau(H) = 3$.

(c) $\tau^*(H) = 2.5$: as shown by $w(\bullet) = 0$ and $w(\circ) = 1/2$.

FIGURE 3.1: Illustrations for the definition of *transversality* and *fractional transversality*.

a subset of vertices intersecting each hyperedge. The transversality corresponds to an optimal solution of the following integer linear program:

$$\text{Minimize:} \quad \sum_{x \in V} w(x)$$

$$\text{Subject to:} \quad \begin{array}{l} \forall x \in V, w(x) \in \{0,1\} \\ \forall e \in E, \sum_{x \in e} w(x) \geqslant 1 \end{array}$$

The *fractional transversality* $\tau^*(H)$ (see Figure 3.1) is the optimum value of the fractional relaxation of the above linear program, which means that the first condition is replaced by: for every $x \in V$, $w(x) \geqslant 0$. It is easy to see that $\tau(H) \geqslant \tau^*(H)$ but the equality may or may not hold (see an example where the inequality is strict on Figure 3.1). The *Vapnik-Chervonenkis dimension* (or *VC-dimension* for short) [204] (see Figure 3.2) of a hypergraph $H = (V, E)$ is the maximum cardinality of a *shattered set*, *i.e.* a set of vertices $A \subseteq V$ such that for every $A' \subseteq A$ there is a hyperedge $e \in E$ so that $e \cap A = A'$. The following bound due to Haussler and Welzl links the transversality, the VC-dimension and the fractional transversality.

**THEOREM 3.9** [116]

*Every hypergraph $H$ with VC-dimension $d$ satisfies*

$$\tau(H) \leqslant 16d\tau^*(H)\log(d\tau^*(H)).$$

We can now state the main result of this Section. Please note that we use two auxiliary lemmas (Lemmas 3.11 and 3.12), which are stated in the middle of the proof and whose proofs are postponed to the end of the Section.

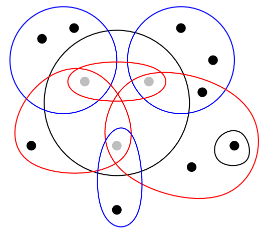


FIGURE 3.2: A hypergraph with 12 vertices and 8 hyperedges (the hyperedges are colored only in order to make the figure easier to read). The set $A$ of three grey vertices is *shattered*: for every $A' \subseteq A$, there exists a hyperedge $e$ such that $e \cap A = A'$ (a black one if $|A'| = 0$ or 3; a blue one if $|A'| = 1$; a red one if $|A'| = 2$). Moreover it is the largest shattered set, so the VC-dimension is 3.

**THEOREM 3.10**

*Let $\Gamma$ be a fixed split graph. There exists a constant $t > 0$ such that every $\Gamma$-free graph $G$ admits a CS-Separator of size $\mathcal{O}(|V(G)|^t)$.*

*Proof.* The vertices of $\Gamma$ are partitioned into $(\Gamma_1, \Gamma_2)$ where $\Gamma_1$ is a clique and $\Gamma_2$ is a stable set. Let $\gamma = \max(|\Gamma_1|, |\Gamma_2|)$ and $t = 64\gamma(\log(\gamma) + 2)$. Let $G = (V, E)$ be a $\Gamma$-free graph on $n$ vertices. The idea is to first define a family of cuts (that may seem arbitrary at first sight), and then prove that it is a CS-Separator for $G$.

Let $\mathcal{F}$ be the following family of cuts: for every stable set $\{s_1, \ldots, s_r\}$ with $r \leqslant t$, we note $B = \cup_{1 \leqslant i \leqslant r} N(s_i)$ and put $(B, V \setminus B)$ in $\mathcal{F}$. Similarly, for every clique $\{x_1, \ldots, x_r\}$ with $r \leqslant t$, we note $B = \cap_{1 \leqslant i \leqslant r} N[x_i]$ and put $(B, V \setminus B)$ in $\mathcal{F}$. Since each member of $\mathcal{F}$ is defined with a set of at most $t$ vertices, the size of $\mathcal{F}$ is at most $\mathcal{O}(n^t)$. Let us now prove that $\mathcal{F}$ is a CS-separator. Let $K$ be a clique and $S$ be a stable set disjoint from $K$. We build a hypergraph $H$ with vertex set $S$ and hyperedges constructed as follows: for every $x \in K$, build the hyperedge $S \cap N_G(x)$ (see Figure 3.3(b)). Symmetrically, build $H'$ a hypergraph with vertex set $K$. For every $s \in S$, build the hyperedge $K \setminus N_G(s)$. The goal is to prove using Theorem 3.9 that $H$ or $H'$ has bounded transversality $\tau$. This will enable us to prove that $K$ and $S$ are separated by $\mathcal{F}$.

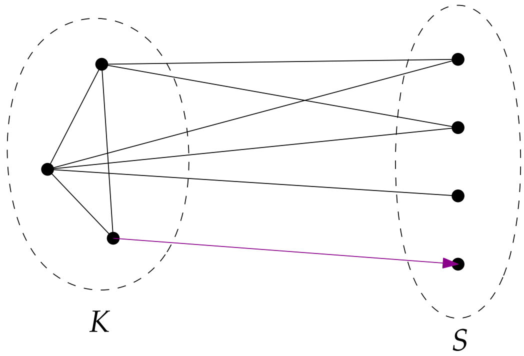
To begin with, let us introduce an auxiliary oriented graph AUX with vertex set $K \cup S$. For every $x \in K$ and $s \in S$, put the arc $(x, s)$ if $xs \in E$, and put the arc $(s, x)$ otherwise (see Figure 3.3(c)). For a weight function $w : V \to \mathbb{R}^+$ and a subset of vertices $T \subseteq V$, we define $w(T) = \sum_{x \in T} w(x)$.

**LEMMA 3.11**

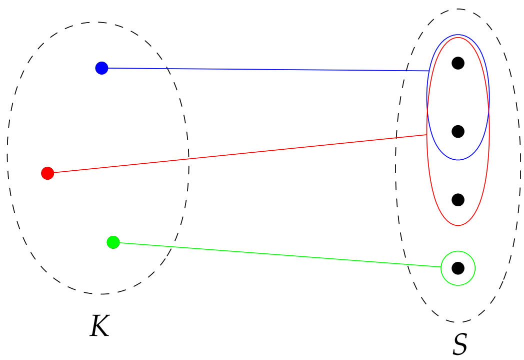
*In AUX, there exists either:*

(i) *a weight function $w : S \to \mathbb{R}^+$ such that $w(S) = 2$ and for every $x \in K$, $w(N^+(x)) \geqslant 1$, or*

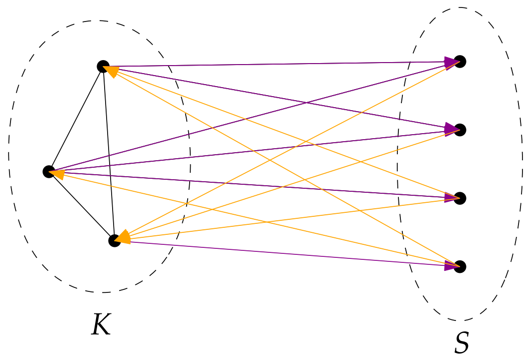
(ii) *a weight function $w : K \to \mathbb{R}^+$ such that $w(K) = 2$ and $w(N^+(s)) \geqslant 1$ for every $s \in S$.*




(a) A clique $K$ and a stable $S$ in $G$.




(b) Hypergraph $H$ where hyperedges are built from the neighborhood of vertices from $K$.




(c) Graph AUX built from $K$ and $S$.

FIGURE 3.3: Illustration of the proof of Theorem 3.10. For more visibility in (c), forward arcs are drawn in dark purple and backward arcs in light orange.

In the following, we suppose we are in case (i) and we prove that $H$ has bounded transversality. Case (ii) is handled symmetrically by switching $H$ and $H'$.

**LEMMA 3.12**

*The hypergraph $H$ has fractional transversality $\tau^*(H)$ at most 2 and VC-dimension upper bounded by $2\gamma - 1$.*

Applying Theorem 3.9 and Lemma 3.12 to $H$, we obtain

$$\tau(H) \leqslant 16d\tau^*(H)\log(d\tau^*(H)) \leqslant 64\gamma(\log(\gamma) + 2) = t.$$

Hence $\tau(H)$ is bounded by $t$ which only depends on $\Gamma$. Let $\tau = \tau(H)$, then there must be $s_1, \ldots, s_\tau \in S$ such that each hyperedge of $H$ contains at least one $s_i$.

Since hyperedges stand for neighborhoods of vertices of $K$, this implies that every $x \in K$ has a neighbor among $\{s_1, \ldots, s_\tau\}$. By setting $B = \cup_{1 \leqslant i \leqslant \tau} N_G(s_i)$, we obtain that $K \subseteq B$ and $S \subseteq V \setminus B$ since $V \setminus B = \cap_{1 \leqslant i \leqslant \tau}(V \setminus N_G(s_i))$ and all the vertices of $S$ are non-adjacent to every $s_i$. This means that the cut $(B, V \setminus B) \in \mathcal{F}$ built from the stable set $s_1, \ldots, s_\tau$ separates $K$ and $S$.

When case (ii) of Lemma 3.11 occurs, $H'$ has transversality $\tau(H') \leqslant t$, so there are $\tau$ vertices $x_1, \ldots, x_\tau \in K$ such that every $s \in S$ has a non-neighbor among $x_1, \ldots, x_\tau$ (with $\tau = \tau(H')$). Consequently, we obtain that $S \subseteq \cup_{1 \leqslant i \leqslant t}(V \setminus N_G[x_i])$. Moreover, by setting $B = \cap_{1 \leqslant i \leqslant t} N_G[x_i]$, we have $K \subseteq B$ since $x_1, \ldots, x_\tau$ are in the same clique $K$. Thus the cut $(B, V \setminus B) \in \mathcal{F}$ built from the clique $x_1, \ldots, x_\tau$ separates $K$ and $S$. □

In order to prove Lemma 3.11, we need the following two lemmas. Given a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we note $x \neq 0$ if there exists $i$ such that $x_i \neq 0$ and we note $x \geqslant 0$ if for every $i$, $x_i \geqslant 0$. We use the following variant of the Hahn-Banach separation theorem:

**LEMMA 3.13**

*Let $A$ be a $n \times m$ matrix. Then at least one of the following holds:*

(i) *There exists $w \in \mathbb{R}^m$ such that $w \geqslant 0$, $w \neq 0$ and $Aw \geqslant 0$, or*

(ii) *There exists $y \in \mathbb{R}^n$ such that $y \geqslant 0$, $y \neq 0$ and ${}^t yA \leqslant 0$.*

*Proof.* Call $\mathbb{R}^n_+ \subseteq \mathbb{R}^n$ the convex set composed of all vectors with only non-negative coordinates. We call $a_1, \ldots, a_m$ the column vectors of $A$, and we examine the set of non-negative combinations of them: $A_{vec} = \{\lambda_1 a_1 + \ldots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{R}^+\}$. If $\mathbb{R}^n_+ \cap A_{vec} \neq \{0\}$, then there exists $w \in \mathbb{R}^m$ fulfilling the requirements of the first item. Otherwise, the interior of $\mathbb{R}^n_+$ and the interior of $A_{vec}$ are disjoint and, according to the Hahn-Banach separation theorem, there is a hyperplane separating them. Call its normal vector on the positive side $y \in \mathbb{R}^n$, then $y$ fulfills the requirements of the second item. □

From this we derive Lemma 3.14:

**LEMMA 3.14**

> *Let $G$ be an oriented graph on vertex set $V$. Then there exists a weight function $w : V \to [0,1]$ such that $w(N^+(x)) \geqslant w(N^-(x))$ for every vertex $x$ and $w(V) = 1$.*

*Proof.* Let $A$ be the adjacency matrix of the oriented graph $G$, that is to say that $A_{x,y} = 1$ if $(x, y)$ is an arc, or $-1$ if $(y, x)$ is an arc, or 0 otherwise (since $G$ is an oriented graph, exactly one condition holds). Apply Lemma 3.13 to $A$. Either case (i) occurs and then $w$ is a nonnegative weight function on the columns of $A$, with at least one non zero weight. Moreover, $Aw \geqslant 0$ so we get $w(N^+(x)) \geqslant w(N^-(x))$ for every $x \in V$. We conclude by rescaling the weight function with a factor $1/w(V)$.

Otherwise, case (ii) occurs and there is $y \in \mathbb{R}^n$ with $y \neq 0$ such that ${}^t yA \leqslant 0$. We get by transposition ${}^t Ay \leqslant 0$ thus $-Ay \leqslant 0$ since $A$ is an antisymmetric matrix, and then $Ay \geqslant 0$. We conclude as in the previous case. □

We can now prove Lemma 3.11:

*Proof of Lemma 3.11.* Apply Lemma 3.14 to the oriented graph AUX to obtain a weight function $w' : V \to [0,1]$. Then $w'(V) = 1$, so either $w'(K) > 0$ or $w'(S) > 0$ (or both). Assume $w'(S) > 0$ (the other case is handled symmetrically). Consider the new weight function $w$ defined by $w(x) = 2w'(x)/w'(S)$ if $x \in S$, and 0 otherwise. Then for every $x \in K$, on the one hand $w(N^+(x)) \geqslant w(N^-(x))$ by extension of the property of $w'$, and on the other hand, $N^+(x) \cup N^-(x) = S$ by construction of AUX. Thus $w(N^+(x)) \geqslant w(S)/2 = 1$ since $w(S) = 2$. □

Finally, we prove Lemma 3.12 assuming that case (i) of Lemma 3.11 occurs:

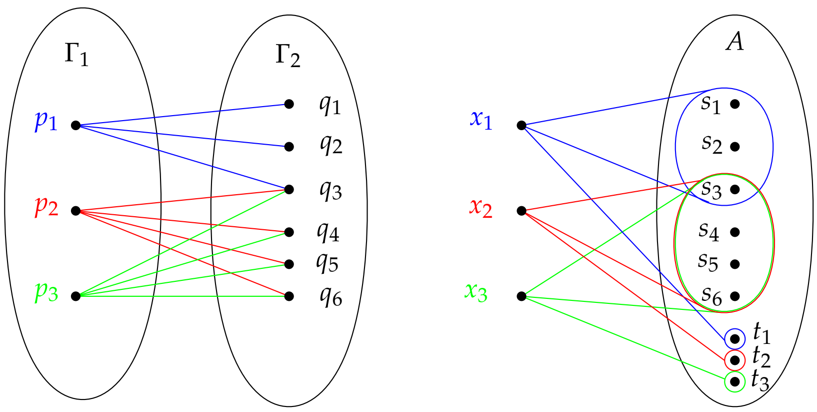
*Proof of Lemma 3.12.* We first focus on the bound for $\tau^*(H)$. Let us prove that the weight function $w$ given by Lemma 3.11 provides a solution to the fractional transversality linear program. Let $e$ be a hyperedge built from the neighborhood of $x \in K$. Recall that this neighborhood is precisely $N^+(x)$ in AUX, then we have:

$$\sum_{s \in e} w(s) = w(N^+(x)) \geqslant 1.$$

Thus $w$ satisfies the constraints of the fractional transversality, and $w(S) \leqslant 2$, which implies $\tau^* \leqslant 2$.

We now bound the VC-dimension of $H$ by $2\gamma - 1$ (see Figure 3.4). Assume by contradiction that there is a shattered set $A = \{s_1, \ldots, s_\gamma, t_1, \ldots, t_\gamma\}$ of $2\gamma$ vertices of $H$, *i.e.* for every $A' \subseteq A$ there is a hyperedge $e \in E$ so that $e \cap A = A'$. The aim is to exploit the shattering to find an induced $\Gamma$, which builds a contradiction. Recall that the forbidden split graph $\Gamma$ is the union of a clique $\Gamma_1 = \{p_1, \ldots, p_r\}$ and a stable set $\Gamma_2 = \{q_1, \ldots, q_{r'}\}$ (with $r, r' \leqslant \gamma$).

Let $p_i \in \Gamma_1$, let $\{q_{i_1}, \ldots, q_{i_k}\} = N_\Gamma(p_i) \cap \Gamma_2$ be the set of its neighbors in $\Gamma_2$. Consider $\mathcal{N}_i = \{s_{i_1}, \ldots, s_{i_k}\} \cup \{t_i\}$ (possible because $|\Gamma_1|, |\Gamma_2| \leqslant \gamma$). By assumption on




(a) A split graph $\Gamma$ (edges inside $\Gamma_1$ are not displayed).

(b) A shattered set $A$ in $H$ that mimics $\Gamma$.

FIGURE 3.4: Illustration for the proof of Lemma 3.12.

$A$, there exists $e \in E$ such that $e \cap A = \mathcal{N}_i$. Let $x_i \in K$ be the vertex whose neighborhood corresponds to the hyperedge $e$, then the neighborhood of $x_i$ in $A$ is exactly $\mathcal{N}_i$. Let $\mathcal{N} = \{s_1, \dots, s_\gamma\}$. Now, forget about the existence of $t_1, \dots, t_\gamma$, and observe that $N_G(x_i) \cap \mathcal{N} = \{s_{i_1}, \dots, s_{i_k}\}$. Then $G[\{x_1, \dots, x_r\} \cup \mathcal{N}]$ is an induced $\Gamma$, which is a contradiction. $\square$

Note that the presence of $t_1, \dots, t_\gamma$ is useful in case where two vertices of $\Gamma_1$ are twins with respect to $\Gamma_2$, meaning that their neighborhoods restricted to $\Gamma_2$ are the same subset $N$. Then, $A$ does not ensure that there exist two hyperedges intersecting $A$ in exactly $N$. So the vertices $t_1, \dots, t_\gamma$ ensure that for two distinct vertices $p_i, p_j$ of $\Gamma_1$, the sets $\mathcal{N}_i$ and $\mathcal{N}_j$ are different. In fact, only $t_1, \dots, t_{\log \gamma}$ are needed to make $\mathcal{N}_i$ and $\mathcal{N}_j$ distinct: for $p_i \in \Gamma_1$, code $i$ in binary over $\log \gamma$ bits and define $\mathcal{N}_i$ to be the union of $\{s_{i_1}, \dots, s_{i_k}\}$ with the set of $t_j$ such that the $j$-th bit is one. Thus the VC-dimension of $H$ is bounded by $\gamma + \log \gamma$.

## 3.4 Techniques in common with the study of the Erdős-Hajnal property

Interestingly, there are some common tools between the CS-Separation and the Erdős-Hajnal property (see the dedicated Chapter 2 for definitions and background about the Erdős-Hajnal property). In particular, if the Strong Erdős-Hajnal property holds in a hereditary class of graphs, not only one can deduce the Erdős-Hajnal property, but also a polynomial upper-bound for CS-Separators in this class. Moreover, the Easy neighborhood property can also be adapted, by changing the definition of *easy*: instead of asking for a large clique or a large stable set in the neighborhood, now it means that there is a polynomial-size CS-Separator in the graph induced by the neighborhood. Finally, we will study the role played by modules.

Such similarities between the CS-Separation and the Erdős-Hajnal property led us to believe that the following may be true:

**Conjecture 3.15**

*Let $H$ be a fixed graph. Then there exists $c > 0$ such that every $H$-free graph $G$ admits a $\mathcal{O}(|V(G)|^c)$ CS-Separator.*

### 3.4.1 $(P_k, \overline{P_k})$-free graphs

Recall that the graph $P_k$ is the path with $k$ vertices, and the graph $\overline{P_k}$ is its complement. We proved in Chapter 2 the following theorem:

**Theorem 2.17**

*For every $k \geqslant 2$, the class of $(P_k, \overline{P_k})$-free graphs has the Strong Erdős-Hajnal property.*

From this we deduced that the Erdős-Hajnal property holds in this class. We now prove a hidden face of the Strong Erdős-Hajnal property which concerns the CS-Separation:

**Theorem 3.16**

*Let $\mathcal{C}$ be a hereditary class of graphs having the Strong Erdős-Hajnal property. Then there exists $c > 0$ such that every $G \in \mathcal{C}$ has a CS-Separator of size $\mathcal{O}(|V(G)|^c)$.*

*Proof.* The class $\mathcal{C}$ has the Strong Erdős-Hajnal property so there exists $t > 0$ such that for every $G \in \mathcal{C}$, there is a biclique of size at least $t|V(G)|$ in $G$ or in $\overline{G}$. The goal is to prove that every graph $G \in \mathcal{C}$ admits a CS-separator of size $|V(G)|^c$ with $c = (-1/\log(1-t))$. We proceed by contradiction and assume that $G$ is a minimal counterexample.

Let $n = V(G)$ and assume that $\overline{G}$ has a biclique $(X, Y)$ of size $tn$. We define $Z = V(G) \setminus (X \cup Y)$. By minimality of $G$, $G[X \cup Z]$ admits a CS-separator $F_X$ of size $(|X| + |Z|)^c$, and $G[Y \cup Z]$ admits a CS-separator $F_Y$ of size $(|Y| + |Z|)^c$. Let us build $F$ aiming at being a CS-separator for $G$. For every cut $(B, W)$ in $F_X$, build the cut $(B, W \cup Y)$, and similarly for every cut $(B, W)$ in $F_Y$, build the cut $(B, W \cup X)$. We show that $F$ is indeed a CS-separator: let $K$ be a clique of $G$ and $S$ be a stable set disjoint from $K$, then either $K \subseteq X \cup Z$, or $K \subseteq Y \cup Z$ since there is no edge between $X$ and $Y$. By symmetry, suppose $K \subseteq X \cup Z$, then there exists a cut $(B, W)$ in $F_X$ that separates $K$ and $S \cap (X \cup Z)$ and the corresponding cut $(B, W \cup Y)$ in $F$ separates $K$ and $S$. Finally, $F$ has size at most $2 \cdot ((1-t)n)^c \leqslant n^c$.

If now $\overline{G}$ does not have a biclique, then $G$ has one, let us call it $(X, Y)$. With the same technique (by replacing every cut $(B, W) \in F_X$ by $(B \cup Y, W)$ in $F$, and every cut $(B, W) \in F_Y$ by $(B \cup X, W)$ in $F$), we obtain a CS-Separator $F$ of size at most $n^c$. □

From Theorems 2.17 and 3.16, we thus obtain the following result:

**THEOREM 3.17**

> *For every $k$, there exists a positive constant $c_k$ such that every $(P_k, \overline{P_k})$-free graph has a CS-Separator of size $\mathcal{O}(|V(G)|^{c_k})$.*

Of course, the underlying natural question is the following: *is there a polynomial-size CS-Separator for $P_k$-free graphs, for every $k > 0$?* The answer is trivially *Yes* for $k \leqslant 4$ ($P_2$-free graphs are stable sets, $P_3$-free graphs are disjoint union of cliques, and $P_4$-free graphs are cographs for which we can easily prove the result by induction[5]), and open for $k \geqslant 6$. The case $k = 5$ was solved as a consequence of a recent result by Lokshtanov, Vatshelle, and Villanger [149], where they design a polynomial-time algorithm for computing the Maximum Weighted Stable Set in $P_5$-free graphs. Despite the fact that it does not use or provide common tools with the Erdős-Hajnal property (it is widely open to know whether the Erdős-Hajnal property holds for $P_5$-free graphs), we state it in this Section.

### 3.4.2 The case of $P_5$-free graphs

As discussed above, we use a recent result due to Lokshtanov, Vatshelle, and Villanger [149]. They prove that the Maximum Weighted Stable Set can be computed in polynomial time in $P_5$-free graphs, via a deep analysis of the so-called *potential maximal cliques*[6] in such graphs. What we need is in fact their strong intermediate result. Before stating it, let us provide some definitions.

A *triangulation* of a graph $G = (V, E)$ is a graph $H = (V, E \uplus F)$ (obtained from $G$ by adding a set of edges $F$ called *fill edges*) such that every cycle of length at least four has a *chord*, that is an edge between two nonconsecutive vertices of the cycle. It is a *minimal* triangulation if $H' = (V, E \uplus F')$ is not a triangulation for every $F' \subsetneq F$.

**THEOREM 3.18** (rephrased from [149])

> *Every $P_5$-free graph $G = (V, E)$ has a family $\Pi$ of subsets of $V$ with size at most $3|V(G)|^7$, such that the following holds: for every non-singleton maximal stable set $S$ of $G$, there exists a minimal triangulation $H$ of $G$ such that every maximal clique of $H$ is in $\Pi$ and every fill edge has both extremities in $V \setminus S$.*

As announced, we can easily deduce the following as a corollary of Theorem 3.18:

**THEOREM 3.19**

> *For every $P_5$-free graph $G$, there exists a CS-separator of size $\mathcal{O}(n^8)$.*

---

[5] Moreover, $P_4$ is split and self-complementary, so both Theorems 3.10 and 3.17 also prove it.

[6] A *potential maximal clique* of $G$ is a maximal clique in a minimal triangulation of $G$. See next paragraph for the missing definitions.

*Proof.* Let $V = V(G)$ and let $\Pi$ be the family output by the algorithm of Theorem 3.18. Define $F = \Pi \uplus \Pi' \uplus F_0$ where

$$\Pi' = \{(U \setminus \{x\}, V \setminus (U \setminus \{x\}))|\ U \in \Pi, x \in V\}$$
$$F_0 = \{(V \setminus \{x\}, \{x\})|\ x \in V\}$$

Let $K$ and $S$ be respectively a clique and a stable set of $G$ which do not intersect. If $|S| = 1$, then $F_0$ separates $K$ from $S$. Otherwise, by property of $\Pi$, there exists a minimal triangulation $H$ of $G$ such that every maximal clique of $H$ is in $\Pi$ and every fill edge has both extremities in $V \setminus S$, in particular $S$ is still a stable set in $H$. Let $K'$ be a maximal clique of $H$ such that $K \subseteq K'$. Then $|K' \cap S| \leqslant 1$ and $K' \in \Pi$. In particular $(K' \setminus (K' \cap S), (V \setminus K') \cup (K' \cap S))$ separates $K$ and $S$ and belongs to $F$. □

Note that the family $\Pi$ can be efficiently constructed.

### 3.4.3 Easy neighborhood

The outline of this subsection is the following informal statement: if every graph of some hereditary class admits a vertex whose neighborhood is *easy*, then every graph of the class is *easy*. What is important is how to define *easiness* such that such a statement holds. We saw in Section 2.3.2 that satisfying the Erdős-Hajnal property is a good definition of *easiness*, and we show here that having a polynomial CS-Separator is also a good definition.

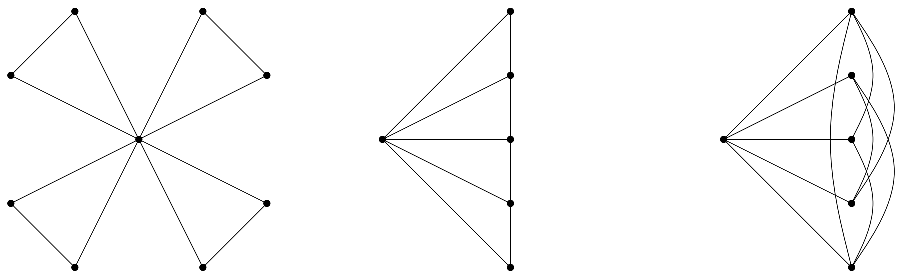
**THEOREM 3.20**

*Let $\mathcal{C}$ be a hereditary class of graphs and $p, c > 0$ such that for every $G \in \mathcal{C}$, there exists $v \in V(G)$ such that $G[N(v)]$ admits a CS-Separator of size $p \cdot |N(v)|^c$. Then every graph $G \in \mathcal{C}$ admits a CS-Separator of size $\mathcal{O}(|V(G)|^{c+1})$.*

*Proof.* We prove that every $G \in \mathcal{C}$ has a CS-Separator of size $p|V(G)|^{c+1}$. We proceed by contradiction and consider a minimum counterexample $G$ on $n$ vertices. Let $v \in V(G)$ be a vertex with an *easy neighborhood*, meaning that $G[N(v)]$ admits a CS-Separator $\mathcal{F}_1$ of size $p|N(v)|^c \leqslant p(n-1)^c$. By minimality of $G$, the graph $G \setminus \{v\}$ admits a CS-Separator $\mathcal{F}_2$ of size $p(n-1)^{c+1}$. We construct a family $\mathcal{F}$ of cuts and prove that it is a CS-Separator for $G$: for every cut $(B_1, W_1) \in \mathcal{F}_1$, build the cut $(B_1 \cup \{v\}, (V(G) \setminus N[v]) \cup W_1)$; for every cut $(B_2, W_2) \in \mathcal{F}_2$, build the cut $(B_2, W_2 \cup \{v\})$. We can easily see that $\mathcal{F}$ has size

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leqslant p(n-1)^c + p(n-1)^{c+1} \leqslant p(n-1)^c n \leqslant pn^{c+1}\,.$$

Let us finally prove that $\mathcal{F}$ is a CS-Separator for $G$: let $K$ be a clique and $S$ be a stable set disjoint from $K$. If $v \in K$, then $K \subseteq N[v]$ since $K$ is a clique. Thus there is a cut $(B_1, W_1) \in \mathcal{F}_1$ that separates the clique $K' = K \cap N(v)$ and the stable set $S' = S \cap N(v)$. Consequently, $(B_1 \cup \{v\}, (V(G) \setminus N[v]) \cup W_1) \in \mathcal{F}$ separates $K$ and $S$. Otherwise, $v \notin K$ and there is a cut $(B_2, W_2) \in \mathcal{F}_2$ that separates $K' = K$ and $S' = S \setminus \{v\}$. Consequently, $(B_2, W_2 \cup \{v\})$ separates $K$ and $S$. □



(a) The 4-windmill (b) The fan of order 5 (c) The pseudo-antifan of order 5

FIGURE 3.5: A windmill, a fan and a pseudo-antifan.

Applying this result gives a polynomial upper-bound for CS-Separators in some new classes of graphs: a *k-windmill* (see Figure 3.5.(a)) is the graph obtained from an induced matching on $k$ edges by adding a universal vertex (*i.e.* a vertex adjacent to every other vertex). A *fan* of order $k$ (see Figure 3.5.(b)) is composed of a path on $k$ vertices with an extra universal vertex. A *pseudo-antifan* of order $k$ (see Figure 3.5.(c)) is composed of the complement of a path of length $k$ with an additional universal vertex. Remember that a diamond-wheel is composed of a hole together with an additional vertex adjacent to at least three consecutive vertices on the hole. As mentioned in Section 2.3.2, for every diamond-wheel-free graph, there exists a vertex whose neighborhood is the disjoint union of cliques [1]. We can thus obtain the following:

**THEOREM 3.21**

*Let $k$ be a positive integer.*

- *For every $k$-windmill-free graph $G$, there exists a $\mathcal{O}(|V(G)|^{2k-1})$ CS-Separator.*
- *There exists $t_k > 0$ such that every graph $G$ with no fan and no pseudo-antifan of order $k$ admits a $\mathcal{O}(|V(G)|^{k+1})$ CS-Separator.*
- *For every diamond-wheel-free graph $G$, there exists a $\mathcal{O}(|V(G)|^2)$ CS-Separator.*

*Proof.* For each of these three classes, we just need to show that the assumptions of Theorem 3.20 are satisfied.

- If $G$ is a $k$-windmill-free graph, then for every vertex $v$, the graph $G[N(v)]$ contains no induced matching on $k$ edges. By Alekseev's result [3], the graph $G[N(v)]$ has at most $\mathcal{O}(|N(v)|^{2k-2})$ maximal stable sets, which implies that it has a $\mathcal{O}(|N(v)|^{2k-2})$ CS-Separator.
- Let $c_k$ be the constant of Theorem 3.17. If $G$ has no fan and no pseudo-antifan

of order $k$, then for every vertex $v$, the graph $G[N(v)]$ is $(P_k, \overline{P_k})$-free. By Theorem 3.17, $G[N(v)]$ has a $\mathcal{O}(|N(v)|^{c_k})$ CS-Separator.

- If $G$ is a diamond-wheel-free graph, let $v$ be the (possible not unique) vertex whose neighborhood is a disjoint union of cliques. Then the graph $G[N(v)]$ has a $\mathcal{O}(|N(v)|)$ CS-Separator.

□

### 3.4.4 Modules

We end this section by a short observation concerning the role played by modules in the CS-Separation. Informally, it states that we can contract a module and construct a polynomial-size CS-Separator by induction. Interestingly, the proof technique is quite close to the technique used in the previous subsection for the *Easy Neighborhood*: cut the graph into pieces and observe that we can reconstruct a CS-Separator for the whole graph out of few CS-Separators on these pieces. In fact, this has similarities with the techniques used to prove that some kind of decompositions cannot appear in a minimally imperfect graph (see Section 1.1 for more details). The same kind of technique will be used in the next section for 2-joins.

We recall that a module $M$ (also called homogeneous set) in a graph $G$ is a set $M$ of $m$ vertices such that $2 \leqslant m \leqslant |V(G)| - 2$ and every $v \in V(G) \setminus M$ is adjacent either to every vertex of $M$, or to none. In other words, all vertices of $M$ have the same set of neighbors outside of $M$.

**THEOREM 3.22**

*Let $p \geqslant 1$ be a constant. Let $\mathcal{C}$ be a hereditary class of graphs and assume $G$ is the smallest graph in $\mathcal{C}$ that does not admit a CS-Separator of size $p \cdot |V(G)|^2$. Then $G$ has no module.*

*Proof.* We proceed by contradiction and suppose that $G$ is a graph on $n \geqslant 2$ vertices with a module $M$ of size $m$. We can assume that $n \geqslant 5$ for otherwise the set of all possible cuts in $G$ is a CS-Separator of size $2^n \leqslant n^2$. By definition of a module, $G$ admits a decomposition $(M, X, Y)$ with $M$ complete to $X$ and $M$ anticomplete to $Y$. Let $v \in M$. By minimality of $G$, there exists a CS-separator $\mathcal{F}_1$ on $G[(X \cup Y \cup \{v\})]$ of size at most $p \cdot (n - m + 1)^2$. Similarly, there exists a CS-separator $\mathcal{F}_2$ on $G[M]$ of size at most $p \cdot m^2$. Let us now build $\mathcal{F}$ aiming at being a CS-separator for $G$, by extending the cuts of $\mathcal{F}_1$ and $\mathcal{F}_2$: for all cut $(B_1, W_1) \in \mathcal{F}_1$, add $M$ on the same side as $v$, that is to say build $(B_1 \cup M, W_1)$ if $v \in B$, and $(B_1, W_1 \cup M)$ otherwise. Moreover, for all $(B_2, W_2) \in \mathcal{F}_2$, let us build $(B_2 \cup X, W_2 \cup Y)$ in $\mathcal{F}$. The goal is now to prove that $\mathcal{F}$ is a CS-separator of size at most $p \cdot n^2$. Let $K$ be a clique in $G$ and $S$ be a stable set disjoint from $K$. Suppose as a first case that both $K$ and $S$ intersect $M$, then $K \subseteq M \cup X$ and $S \subseteq M \cup Y$ by definition of $X$ and $Y$. By definition of $\mathcal{F}_2$, there is a cut $(B_2, W_2)$ in $F_2$ that separates $K' = M \cap K$ and $S' = M \cap S$ and the cut of $\mathcal{F}$ induced by $(B_2, W_2)$ separates $K$ and $S$. Suppose as a second case that none of $K$ nor

$S$ intersect $M$, then there is a cut in $\mathcal{F}_1$ separating $K$ and $S$, and the corresponding cut in $\mathcal{F}$ still does. The third and last case occurs when $K$ intersects $M$ and $S$ does not (or the contrary by symmetric arguments). Then build in $G[X \cup Y \cup \{v\}]$ the clique $K' = (K \setminus M) \cup \{v\}$. By definition of $\mathcal{F}_1$, there exists a cut in $\mathcal{F}_1$ that separates $K'$ and $S$ and the corresponding cut in $\mathcal{F}$ separates $K$ and $S$. The symmetric case is handled by taking $S' = (S \setminus M) \cup \{v\}$ and a cut in $\mathcal{F}_1$ separating $K$ and $S'$. As a conclusion, $\mathcal{F}$ is a CS-separator for $G$.

Finally let us show that $|\mathcal{F}| \leqslant p \cdot n^2$. By construction, we can easily see that $|\mathcal{F}| \leqslant |\mathcal{F}_1| + |\mathcal{F}_2| \leqslant p \cdot ((n-m+1)^2 + m^2)$. If $P(x) = (n-x+1)^2 + x^2 - n^2$, we only need to prove that $P(x) \leqslant 0$ for every $2 \leqslant x \leqslant n-1$. But $P$ is a polynomial of degree 2 with leading coefficient $2 > 0$, and $P(2) = P(n-1) = -2(n-5) \leqslant 0$. The convexity of $P$ proves the statement for $2 \leqslant x \leqslant n-1$. □

## 3.5 Perfect graphs with no balanced skew-partition

Yannakakis' initial motivation for introducing the CS-Separation was concerned only with *perfect graphs*, so we obviously tried to focus on this case. As described in Chapter 1, decomposition techniques play a great role in works on perfect graphs[7], and they even led Chudnovsky, Robertson, Seymour and Thomas to the proof of the Strong Perfect Graph theorem: a graph is perfect if and only if it is Berge, meaning that it contains no odd holes nor odd antiholes. Moreover, Section 3.4 showed that decomposition techniques can also be useful for the CS-Separation, that is why we try here to get a CS-Separation result on perfect graphs *via* decompositions.

Remember that the decomposition theorem for Berge graphs (Theorem 1.6) states that any Berge graph $G$ is either *basic*, or has a *2-join* (in $G$ or in $\overline{G}$), or has a *balanced skew-partition*. The 2-join has a good algorithmic behavior for coloring purposes, but the balanced skew-partition is much less friendly. We will see that the 2-join also has a good behavior with respect to the CS-Separation, whereas we could not obtain such a result for the balanced skew-partition. Consequently, instead of proving the CS-Separation in *all* perfect graphs, we would like to reach a weaker goal and prove the CS-Separation for perfect graphs that can be recursively decomposed with 2-join until reaching a basic class. *Is there a natural subclass of perfect graphs admitting such a recursive decomposition?* We are not the first to ask this question: Chudnovsky, Trotignon, Trunck and Vušković were facing the same problem when trying to combinatorially and efficiently color perfect graphs. They came out with the following remark: if a Berge graph has no balanced skew-partition and is not basic, then by the Decomposition Theorem it has a 2-join. Let us decompose this graph with this 2-join, now can we recursively continue? Could we assert that this decomposed graph still

[7]If not familiar with historical results and concepts on perfect graphs, the reader is advised to read Section 1.1 to better understand the motivations and techniques of this section.

has no balanced skew-partition? They proved that the answer is *yes*, and thus they suceeded in finding a combinatorial polynomial-time algorithm to compute the Maximum Weighted Stable Set in Berge graphs with no balanced skew-partition, from which they deduce a coloring algorithm.

They used a powerful tool, so-called *trigraphs*, which is a generalization of graphs. It was introduced by Chudnovsky in her PhD thesis [32] to simplify the statement and the proof of the Strong Perfect Graph Theorem. Indeed, the original statement provided four different outcomes, but she proved that one of them (the homogeneous pair) is not necessary. Trigraphs are also very useful in the study of bull-free graphs [33, 34, 196] and claw-free graphs [43]. Using the previous study of Berge trigraphs with no balanced skew-partition, we prove that Berge graphs with no balanced skew-partition have a polynomial CS-Separator.

A natural question that comes to mind, once a result is proved, is *how can we generalize it?* In this case, two different directions were open for generalizations: first, do we need all the properties of 2-joins? Can we do the same with a more general kind of decomposition? Yes, we were able to handle a new kind of decomposition called the *generalized k-join*. The second direction was: *can we prove more than the CS-Separation?* Indeed, according to the results of the previous section, it is tempting to ask for the Strong Erdős-Hajnal property instead of the CS-Separation[8]. Although perfect graphs trivially have the Erdős-Hajnal property, the case of the Strong Erdős-Hajnal property is made particularly interesting by Fox and Pach's result stating that some comparability graphs do not have the Strong Erdős-Hajnal property (Theorem 2.16). Consequently, the Strong Erdős-Hajnal property does not hold for every perfect graphs, but *does it hold for the subclass of perfect graphs under study?* We prove that the answer is yes, and we moreover combine both generalizations and prove that the answer is also yes for trigraphs that can be recursively decompose *via* generalized $k$-join.

The fact that the Strong Erdős-Hajnal property holds in Berge graphs with no balanced skew partition shows that this subclass is much less general than the whole class of perfect graphs. This observation is confirmed by another recent work [166] by Penev who also studied the class of Berge graphs with no balanced skew-partition and proved that they admit a 2-clique-coloring (*i.e.* there exists a non-proper coloring with two colors such that every inclusion-wise maximal clique is not monochromatic). This is known to be false for the whole class of perfect graphs (see a counterexample on Figure 3.6), but it is still conjectured that all perfect graphs admit a 3-clique-coloring.

It should be noticed that the class of Berge graphs with no balanced skew-partition is not hereditary (because removing a vertex may create a balanced skew-partition), so the CS-Separation is not a consequence of the Strong Erdős-Hajnal property and needs a full proof. Moreover, this class is in some sense transverse to classical hereditary subclasses of perfect graphs: for instance $P_4$, which is a bipartite, chordal and comparability graph has a balanced skew-partition (takes the extremities as the non-

[8] Remember that the Strong Erdős-Hajnal property implies the CS-Separation in hereditary classes of graphs.

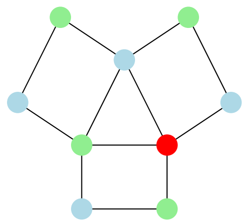


FIGURE 3.6: A perfect graph which cannot be 2-clique-colored.

connected part $A$, and the two middle vertices as the non-anticonnected part $B$). However, $P_4$ is an induced subgraph of $C_6$, which has no balanced skew-partition. So sometimes one can *kill* all the balanced skew-partitions by adding some vertices. Trotignon and Maffray proved that given a basic graph $G$ on $n$ vertices having a balanced skew-partition, there exists a basic graph $G'$ on $\mathcal{O}(n^2)$ vertices which has no balanced skew-partition and contains $G$ as an induced subgraph [155]. Some degenerated cases are to be considered: graphs with at most 3 vertices as well as cliques and stable sets do not have a balanced skew-partition. Moreover, Trotignon showed [198] that every double-split graph does not have a balanced skew-partition. In addition to this, observe that any clique-cutset of size at least 2 yields to a balanced skew-partition: as a consequence, paths, chordal graphs and cographs have always a balanced skew-partition, up to a few degenerated cases. Table 3.1 compares the class of Berge graphs with no balanced skew-partition with some examples of well-known subclasses of perfect graphs. In particular, there exist two non-trivial perfect graphs lying in none of the above mentioned classes (basic, chordal graphs, comparability graphs, cographs), one of them having a balanced skew-partition, the other not having any.

We start in Subsection 3.5.1 by introducing trigraphs and all related definitions. In Subsection 3.5.2, we state the decomposition theorem from [46] for Berge trigraphs with no balanced skew-partition. The results come in the last two subsections: Subsection 3.5.3 is concerned with finding polynomial-size CS-Separators in Berge trigraphs with no balanced skew-partition. In the same subsection, we extend this result to classes of trigraphs closed by generalized $k$-join, provided that the *basic* class admits polynomial-size CS-Separators. As for Subsection 3.5.4, it is dedicated to proving that the Strong Erdős-Hajnal property holds in perfect graphs with no balanced skew-partition, and in the generalized case (with a similar assumption on the *basic* class).

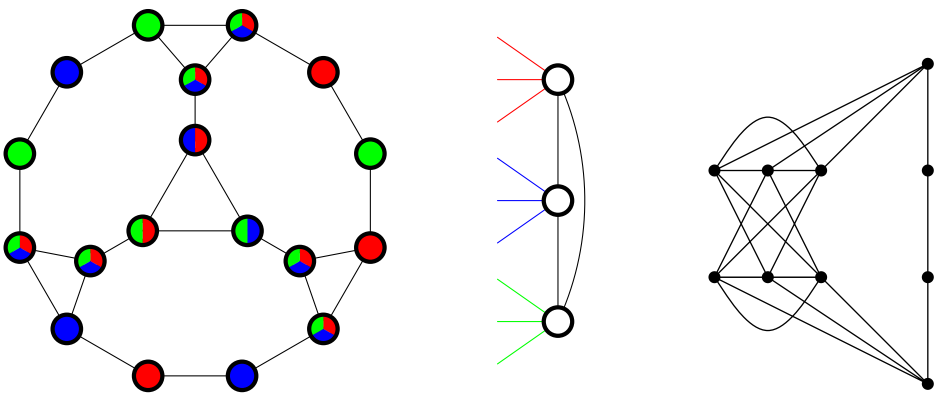
### 3.5.1 Definitions

#### Intuition

We first need to introduce trigraphs: this is a generalization of graphs where a new kind of adjacency between vertices is defined: the *semi-adjacency*. The intuition between two semi-adjacent vertices, also called a *switchable pair*, is that in some situation,

| | With a BSP | With no BSP |
|---|---|---|
| Bipartite graph | $P_4$ | $C_4$ |
| Compl. of a bipartite graph | $P_4$ | $C_4$ |
| Line graph of a bip. graph | $P_4$ | $C_4$ |
| Complement of a line graph of a bip. graph | $P_4$ | $C_4$ |
| Double-split | None | $C_4$ |
| Comparability graph | $P_4$ | $C_4$ |
| Path | $P_k$ for $k \geqslant 4$ | None |
| Chordal | All (except deg. cases) | $K_t, S_t, \overline{C_4}$ , $t \geqslant 4$ |
| Cograph | All (except deg. cases) | $K_t, S_t, C_4, \overline{C_4}, t \geqslant 4$ |
| None of the above classes | Worst Berge Graph Known so Far | Zambelli's graph |

TABLE 3.1: Classical subclasses of perfect graphs compared with perfect graphs with no balanced skew-partition (BSP for short). Graphs with less than 4 vertices are not considered. See Figure 3.7 for a description of the Worst Berge Graph Known So Far and Zambelli's graph.



(a) The Worst Berge Graph Known So Far. (discovered by Chudnovsky and Seymour; displayed in [199]). Red edges (resp. blue edges, green edges) go to red (resp. blue, green) vertices.

(b) Zambelli's graph.

FIGURE 3.7: The two non-trivial perfect graphs dealt with in Table 3.1: the first does have a BSP, the second one does not.

they are considered as adjacent, and in some other situations, they are considered as non-adjacent. This implies to be very careful about terminology, for example in a trigraph two vertices are said *adjacent* if there is a *real* edge between them but also if they are semi-adjacent. What if we want to speak about *really adjacent* vertices, in the old-fashioned way? The dedicated terminology is *strongly adjacent*, adapted to *strong neighborhood*, *strong clique* and so on.

Because of this, we need to redefine all the usual notions on graphs to adapt them on trigraphs. Reading the following definitions is quite tedious, however giving such definitions seems mandatory for the accuracy of the results. That is why we provide below some intuition and unformal definitions for the most important concepts used in the rest of the section. Thus the reader may skip the formal definitions and refer to them later, when some details are needed.

The set of switchable pairs of $T$ is denoted by $\sigma(T)$. The *complement* of a trigraph is naturally obtained by exchanging strong adjacency with strong antiadjacency, and the semi-adjacency stays the same. A trigraph has a *hole* (resp. an *antihole*, a *path*) if the switchable pairs can be turned into edges or non-edges (each independently) so that the resulting graph has a hole (resp. antihole, path). A trigraph is *Berge* if and only if it has no odd hole and no odd antihole. This operation of turning the switchable pairs into strong edges or strong antiedges led to graphs that are called *realizations* of the trigraph, and the same operation that switch only *some* switchable pairs led to trigraphs that are called *semi-realizations* of the initial trigraph. The *full realization* of $T$ is the graph obtained from $T$ by turning every switchable pair into a strong edge.

The trigraphs we are interested in come from decomposing Berge graphs along 2-joins. As we will see in the next section, this yields only to few switchable pairs, or at least distant switchable pairs. This property is useful both for decomposing trigraphs and for proving the CS-Separation in basic classes. In a nutshell, the class $\mathcal{F}$ of trigraphs studied in the following is Berge trigraphs for which any switchable pair cannot share a common endpoint with more than one other switchable pair. In other words, the *switchable components* (connected components of the graph obtained from $T$ by deleting every strong edge and strong antiedge, and turning the switchable pairs into edges) are paths of length at most 2. The correct definition is much more precise, but basically, this is all we need.

Let us now go for formal definitions.

### Trigraphs

For a set $X$, we denote by $\binom{X}{2}$ the set of all subsets of $X$ of size 2. For brevity of notation an element $\{u, v\}$ of $\binom{X}{2}$ is also denoted by $uv$ or $vu$. A *trigraph* $T$ consists of a finite set $V(T)$, called the *vertex set* of $T$, and a map $\theta : \binom{V(T)}{2} \longrightarrow \{-1, 0, 1\}$, called the *adjacency function*.

Two distinct vertices of $T$ are said to be *strongly adjacent* if $\theta(uv) = 1$, *strongly antiadjacent* if $\theta(uv) = -1$, and *semiadjacent* if $\theta(uv) = 0$. We say that $u$ and $v$ are *adjacent* if they are either strongly adjacent, or semiadjacent; and *antiadjacent* if they are either strongly antiadjacent, or semiadjacent. An *edge* (*antiedge*) is a pair of adjacent (antiadjacent) vertices. If $u$ and $v$ are adjacent (antiadjacent), we also say that $u$ is *adjacent (antiadjacent) to* $v$, or that $u$ is a *neighbor (antineighbor)* of $v$. The *open neighborhood* $N(u)$ of $u$ is the set of neighbors of $u$, and the *closed neighborhood* $N[u]$ of $u$ is $N(u) \cup \{u\}$. If $u$ and $v$ are strongly adjacent (strongly antiadjacent), then $u$ is a *strong neighbor (strong antineighbor)* of $v$. Let $\sigma(T)$ the set of all semiadjacent pairs of $T$. Thus, a trigraph $T$ is a graph if $\sigma(T)$ is empty. A pair $\{u,v\} \subseteq V(T)$ of distinct vertices is a *switchable pair* if $\theta(uv) = 0$, a *strong edge* if $\theta(uv) = 1$ and a *strong antiedge* if $\theta(uv) = -1$. An edge $uv$ (antiedge, strong edge, strong antiedge, switchable pair) is *between* two sets $A \subseteq V(T)$ and $B \subseteq V(T)$ if $u \in A$ and $v \in B$ or if $u \in B$ and $v \in A$.

Let $T$ be a trigraph. The *complement* $\overline{T}$ of $T$ is a trigraph with the same vertex set as $T$, and adjacency function $\overline{\theta} = -\theta$. Let $A \subset V(T)$ and $b \in V(T) \setminus A$. We say that $b$ is *strongly complete* to $A$ if $b$ is strongly adjacent to every vertex of $A$; $b$ is *strongly anticomplete* to $A$ if $b$ is strongly antiadjacent to every vertex of $A$; $b$ is *complete* to $A$ if $b$ is adjacent to every vertex of $A$; and $b$ is *anticomplete* to $A$ if $b$ is antiadjacent to every vertex of $A$. For two disjoint subsets $A, B$ of $V(T)$, $B$ is *strongly complete (strongly anticomplete, complete, anticomplete)* to $A$ if every vertex of $B$ is strongly complete (strongly anticomplete, complete, anticomplete) to $A$.

A *clique* in $T$ is a set of vertices all pairwise adjacent, and a *strong clique* is a set of vertices all pairwise strongly adjacent. A *stable set* is a set of vertices all pairwise antiadjacent, and a *strong stable set* is a set of vertices all pairwise strongly antiadjacent. For $X \subset V(T)$ the trigraph *induced by $T$ on $X$* (denoted by $T[X]$) has vertex set $X$, and adjacency function that is the restriction of $\theta$ to $\binom{X}{2}$. Isomorphism between trigraphs is defined in the natural way, and for two trigraphs $T$ and $H$ we say that $H$ is an *induced subtrigraph* of $T$ (or *$T$ contains $H$ as an induced subtrigraph*) if $H$ is isomorphic to $T[X]$ for some $X \subseteq V(T)$. Since in this paper we are only concerned with the induced subtrigraph containment relation, we say that *$T$ contains $H$* if $T$ contains $H$ as an induced subtrigraph. We denote by $T \setminus X$ the trigraph $T[V(T) \setminus X]$.

Let $T$ be a trigraph. A *path* $P$ of $T$ is a sequence of distinct vertices $p_1, \dots, p_k$ such that either $k = 1$, or for $i, j \in \{1, \dots, k\}$, $p_i$ is adjacent to $p_j$ if $|i - j| = 1$ and $p_i$ is antiadjacent to $p_j$ if $|i - j| > 1$. We say that $P$ is a path *from $p_1$ to $p_k$*, its *interior* is the set $\{p_2, \dots, p_{k-1}\}$, and the *length* of $P$ is $k - 1$. Observe that, since a graph is also a trigraph, it follows that a path in a graph, the way we have defined it, is what is sometimes in literature called a chordless path.

A *hole* in a trigraph $T$ is an induced subtrigraph $H$ of $T$ with vertices $h_1, \dots, h_k$ such that $k \geqslant 4$, and for $i, j \in \{1, \dots, k\}$, $h_i$ is adjacent to $h_j$ if $|i - j| = 1$ or $|i - j| = k - 1$; and $h_i$ is antiadjacent to $h_j$ if $1 < |i - j| < k - 1$. The *length* of a hole is the number of vertices in it. An *antipath* (*antihole*) in $T$ is an induced subtrigraph of $T$ whose complement is a path (hole) in $\overline{T}$.

A *semirealization* of a trigraph $T$ is any trigraph $T'$ with vertex set $V(T)$ that satisfies the following: for all $uv \in \binom{V(T)}{2}$, if $uv$ is a strong edge in $T$, then it is also a strong edge in $T'$, and if $uv$ is a strong antiedge in $T$, then it is also a strong antiedge in $T'$. Sometimes we will describe a semirealization of $T$ as an *assignment of values* to switchable pairs of $T$, with three possible values: "strong edge", "strong antiedge" and "switchable pair". A *realization* of $T$ is any graph that is semirealization of $T$ (so, any semirealization where all switchable pairs are assigned the value "strong edge" or "strong antiedge"). The realization where all switchable pairs are assigned the value "strong edge" is called the *full realization* of $T$.

Let $T$ be a trigraph. For $X \subseteq V(T)$, we say that $X$ and $T[X]$ are *connected* (*anticonnected*) if the full realization of $T[X]$ ($\overline{T[X]}$) is connected. A *connected component* (or simply *component*) of $X$ is a maximal connected subset of $X$, and an *anticonnected component* (or simply *anticomponent*) of $X$ is a maximal anticonnected subset of $X$.

A trigraph $T$ is *Berge* if it contains no odd hole and no odd antihole. Therefore, a trigraph is Berge if and only if its complement is. We observe that $T$ is Berge if and only if every realization (semirealization) of $T$ is Berge.

Finally let us define the class of trigraphs we are working on. Let $T$ be a trigraph, denote by $\Sigma(T)$ the graph with vertex set $V(T)$ and edge set $\sigma(T)$ (the switchable pairs of $T$). The connected components of $\Sigma(T)$ are called the *switchable components* of $T$. Let $\mathcal{F}$ be the class of Berge trigraphs such that the following hold:

- Every switchable component of $T$ has at most two edges (and therefore no vertex has more than two neighbors in $\Sigma(T)$).

- Let $v \in V(T)$ have degree two in $\Sigma(T)$, denote its neighbors by $x$ and $y$. Then either $v$ is strongly complete to $V(T) \setminus \{v, x, y\}$ in $T$, and $x$ is strongly adjacent to $y$ in $T$, or $v$ is strongly anticomplete to $V(T) \setminus \{v, x, y\}$ in $T$, and $x$ is strongly antiadjacent to $y$ in $T$.

Observe that $T \in \mathcal{F}$ if and only if $\overline{T} \in \mathcal{F}$.

### 3.5.2 Decomposing trigraphs of $\mathcal{F}$

This section recalls definitions and results from [46] that we use in the next subsection. Our goal is to state the decomposition theorem for trigraphs of $\mathcal{F}$ and to define the blocks of decomposition. First we need some definitions.

#### Basic trigraphs

We need the counterparts of *bipartite graphs* (and their complements), *line graphs of bipartite graphs* (and their complements), and *double-split graphs* which are the basic classes for decomposing Berge graphs. For the trigraph case, the basic classes are *bipartite trigraphs* and their complements, *line trigraphs* and their complements, and *doubled trigraphs*.

A trigraph $T$ is *bipartite* if its vertex set can be partitioned into two strong stable sets. A trigraph $T$ is a *line trigraph* if the full realization of $T$ is the line graph of a bipartite graph and every clique of size at least 3 in $T$ is a strong clique. Let us now define the trigraph analogue of the double split graph, namely the doubled trigraph.

A *good partition* of a trigraph $T$ is a partition $(X, Y)$ of $V(T)$ (possibly, $X = \emptyset$ or $Y = \emptyset$) such that:

- Every component of $T[X]$ has at most two vertices, and every anticomponent of $T[Y]$ has at most two vertices.
- No switchable pair of $T$ meets both $X$ and $Y$.
- For every component $C_X$ of $T[X]$, every anticomponent $C_Y$ of $T[Y]$, and every vertex $v$ in $C_X \cup C_Y$, there exists at most one strong edge and at most one strong antiedge between $C_X$ and $C_Y$ that is incident to $v$.

A trigraph is *doubled* if it has a good partition. A trigraph is *basic* if it is either a bipartite trigraph, the complement of a bipartite trigraph, a line trigraph, the complement of a line trigraph or a doubled trigraph. Basic trigraphs behave well with respect to induced subtrigraphs and complementation as stated by the following lemma.

**LEMMA 3.23** [46]

*Basic trigraphs are Berge and are closed under taking induced subtrigraphs, semirealizations, realizations and complementation.*

**Decompositions**

We now describe the decompositions that we need for the decomposition theorem. They generalize the decompositions used in the Strong Perfect Graph Theorem [40], and in addition all the *important crossing* edges and non-edges in those graph decompositions are required to be strong edges and strong antiedges of the trigraph, respectively.

First, a *2-join* in a trigraph $T$ (see Figure 3.8.(a) for an illustration) is a partition $(X_1, X_2)$ of $V(T)$ such that there exist disjoint sets $A_1, B_1, C_1, A_2, B_2, C_2 \subseteq V(T)$ satisfying:

- $X_1 = A_1 \cup B_1 \cup C_1$ and $X_2 = A_2 \cup B_2 \cup C_2$;
- $A_1, A_2, B_1$ and $B_2$ are non-empty;
- no switchable pair meets both $X_1$ and $X_2$;
- every vertex of $A_1$ is strongly adjacent to every vertex of $A_2$, and every vertex of $B_1$ is strongly adjacent to every vertex of $B_2$;
- there are no other strong edges between $X_1$ and $X_2$;
- for $i = 1, 2$ $|X_i| \geqslant 3$;
- for $i = 1, 2$, if $|A_i| = |B_i| = 1$, then the full realization of $T[X_i]$ is not a path of length two joining the members of $A_i$ and $B_i$;

- for $i = 1, 2$, every component of $T[X_i]$ meets both $A_i$ and $B_i$ (this condition is usually required only for a *proper* 2-join, but we will only deal with proper 2-join in the following).

The partition $(A_1, B_1, C_1, A_2, B_2, C_2)$ is called a *split of the 2-join* $(X_1, X_2)$. A *complement 2-join* of a trigraph $T$ is a 2-join in $\overline{T}$. When proceeding by induction on the number of vertices, we sometimes want to contract one side of a 2-join into three vertices and assert that the resulting trigraph is smaller. This does not come directly from the definition (we assume only $|X_i| \geqslant 3$), but can be deduced from the following technical lemma:

**LEMMA 3.24** [46]

*Let $T$ be a trigraph from $\mathcal{F}$ with no balanced skew-partition, and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of a 2-join $(X_1, X_2)$ in $T$. Then $|X_i| \geqslant 4$, for $i = 1, 2$.*

Moreover, when decomposing a 2-join, we need to be careful about the parity of the lengths of paths from $A_i$ and $B_i$ in order not to create an odd hole. In this respect, the following lemma is useful:

**LEMMA 3.25** [46]

*Let $T$ be a Berge trigraph and $(A_1, B_1, C_1, A_2, B_2, C_2)$ a split of a 2-join of $T$. Then all paths with one end in $A_i$, one end in $B_i$ and interior in $C_i$, for $i = 1, 2$, have lengths of the same parity.*

*Proof.* Otherwise, for $i = 1, 2$, let $P_i$ be a path with one end in $A_i$, one end in $B_i$ and interior in $C_i$, such that $P_1$ and $P_2$ have lengths of different parity. They form an odd hole, a contradiction. □

Consequently, a 2-join in a Berge trigraph is said *odd* or *even* according to the parity of the lengths of the paths between $A_i$ and $B_i$. The above lemma ensures the correctness of the definition.

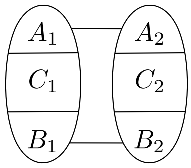
Our second decomposition is the balanced skew-partition. A *skew-partition* is a partition $(A, B)$ of $V(T)$ so that $A$ is not connected and $B$ is not anticonnected. It is moreover *balanced* if there is no odd path of length greater than 1 with ends in $B$ and interior in $A$, and there is no odd antipath of length greater than 1 with ends in $A$ and interior in $B$.

We are now ready to state the decomposition theorem.

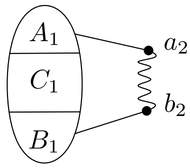
**THEOREM 3.26** ([46], adapted from [32])

*Every trigraph in $\mathcal{F}$ is either basic, or admits a balanced skew-partition, a 2-join, or a complement 2-join.*

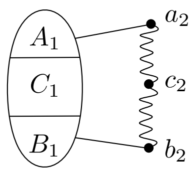
We now define the *blocks of decomposition* $T_{X_1}$ and $T_{X_2}$ of a 2-join $(X_1, X_2)$ in a trigraph $T$ (an illustration of blocks of decomposition can be found in Figure 3.8). Let




(a) A 2-join.




(b) Block of decomposition $T_{X_1}$ for an odd 2-join.




(c) Block of decomposition $T_{X_1}$ for an even 2-join.

FIGURE 3.8: Diagram for a 2-join and its blocks of decomposition. Straight lines stand for strongly complete sets, and wiggly edges stand for switchable pairs. No other edge can cross between left and right.

$(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of $(X_1, X_2)$, then intuitively the block, say, $T_{X_1}$ is obtained from $T$ by keeping $X_1$ as it is and contracting $X_2$ into few vertices, depending on the parity of the 2-join: 2 vertices for odd 2-joins (one for $A_2$, one for $B_2$), and 3 vertices for even 2-joins (one extra vertex for $C_2$). What we want is a good behavior with respect to induction: assume that we want to prove some property $\mathcal{P}$ (here, a poynomial-size CS-Separator) in every trigraph $T$ of $\mathcal{F}$ with no balanced skew-partition, then we handle the 2-join case by contracting the trigraph into these two blocks, apply the induction hypothesis on them, and prove the property $\mathcal{P}$ on $T$ knowing that it holds on the blocks. The important part here is *apply the induction hypothesis*, which we can do only if the blocks of decomposition stay in the class. This is what trigraphs are useful for: replacing the switchable pairs by strong edges or strong antiedges in the blocks of decompositions described below may create a balanced skew-partition: hence we could not apply induction hypothesis.

If the 2-join is odd, we build the block of decomposition $T_{X_1}$ as follows: we start with $T[A_1 \cup B_1 \cup C_1]$. We then add two new marker vertices $a_2$ and $b_2$ such that $a_2$ is strongly complete to $A_1$, $b_2$ is strongly complete to $B_1$, $a_2b_2$ is a switchable pair, and there are no other edges between $\{a_2, b_2\}$ and $X_1$. Note that $\{a_2, b_2\}$ is a switchable component of $T_{X_1}$. The block of decomposition $T_{X_2}$ is defined similarly with marker vertices $a_1$ and $b_1$.

If the 2-join is even, we build the block of decomposition $T_{X_1}$ as follows: once again, we start with $T[A_1 \cup B_1 \cup C_1]$. We then add three new marker vertices $a_2$, $b_2$ and $c_2$ such that $a_2$ is strongly complete to $A_1$, $b_2$ is strongly complete to $B_1$, $a_2c_2$ and $c_2b_2$ are switchable pairs, and there are no other edges between $\{a_2, b_2, c_2\}$ and $X_1$. The block of decomposition $T_{X_2}$ is defined similarly with marker vertices $a_1$, $b_1$ and $c_1$.

We define the blocks of decomposition of a complement 2-join $(X_1, X_2)$ in $T$ as the complement of the blocks of decomposition of the 2-join $(X_1, X_2)$ in $\overline{T}$.

**THEOREM 3.27** [46]

*If $(X_1, X_2)$ is a 2-join or a complement 2-join of a trigraph $T$ from $\mathcal{F}$ with no balanced skew-partition, then $T_{X_1}$ and $T_{X_2}$ are trigraphs from $\mathcal{F}$ with no balanced skew-partition.*

### 3.5.3 Clique-Stable Set separation

**Preliminaries**

Before reaching our goal, we need to define what is a CS-Separator in a trigraph, and discuss which trivial properties still hold or not. Observe that there are not many changes with the corresponding definitions for graphs.

Let $T$ be a trigraph. A *cut* is a partition of $V(T)$ into two parts $B, W \subseteq V(T)$ (hence $W = V(T) \setminus B$). It *separates* a clique $K$ and a stable set $S$ if $K \subseteq B$ and $S \subseteq W$. Sometimes we will call $B$ the *clique side* of the cut and $W$ the *stable set side* of the cut. What we really need to decide is what kind of cliques and stable sets we want to separate in trigraphs: only strong cliques and strong stable sets or all cliques and all stable sets (that can contain switchable pairs)? We choose the latter, to have a stronger assumption when applying induction hypothesis later on in the proofs: we say that a family $F$ of cuts is a *CS-separator* if for every (not necessarily strong) clique $K$ and every (not necessarily strong) stable set $S$ which do not intersect, there exists a cut in $F$ that separates $K$ and $S$. Just as for graphs, finding a CS-Separator is a self-complementary problem: suppose there exists a CS-separator of size $k$ on $T$, then we build a CS-separator of size $k$ on $\overline{T}$ by building for every cut $(B, W)$ the cut $(W, B)$.

In a graph, a clique and a stable set can intersect on at most one vertex. This property was useful to prove that we only need to focus on inclusion-wise maximal cliques and inclusion-wise maximal stable sets. This is no longer the case for trigraphs, for which a clique and a stable set can intersect on a switchable component $V$, provided this component contains only switchable pairs, i.e. such that for every $u, v \in V$, $u = v$ or $uv \in \sigma(T)$. However, when restricted to trigraphs of $\mathcal{F}$, a clique and a stable set can intersect on at most one vertex or one switchable pair, so we can derive the following counterpart of Observation 3.4:

**OBSERVATION 3.28**

*If a trigraph $T$ of $\mathcal{F}$ admits a family $F$ of cuts separating all the inclusion-wise maximal cliques and the inclusion-wise maximal stable sets, then it admits a CS-separator of size at most $|F| + O(n^2)$.*

*Proof.* For every $x \in V(T)$, let $Cut_{1,x}$ be the cut $(N[x], V(T) \setminus N[x])$ and $Cut_{2,x}$ be the cut $(N(x), V(T) \setminus N(x))$. For every switchable pair $xy$, let

$$\begin{aligned}
Cut_{1,xy} &= (U, V(T) \setminus U) \quad \text{with } U = N[x] \cap N[y] \\
Cut_{2,xy} &= (U, V(T) \setminus U) \quad \text{with } U = N[x] \cap N(y) \\
Cut_{3,xy} &= (U, V(T) \setminus U) \quad \text{with } U = N(x) \cap N[y] \\
Cut_{4,xy} &= (U, V(T) \setminus U) \quad \text{with } U = N(x) \cap N(y)
\end{aligned}$$

Let $F'$ be the union of $F$ with all these cuts for every $x \in V(T)$, $xy \in \sigma(T)$, and let us prove that $F'$ is a CS-separator. Let $K$ be a clique and $S$ be a stable set disjoint from $K$. Extend $K$ and $S$ by adding vertices to get a maximal clique $K'$ and a maximal stable set $S'$. Three cases are to be considered. Either $K'$ and $S'$ do not intersect, and there is a cut in $F$ that separates $K'$ from $S'$ (thus $K$ from $S$). Or $K'$ and $S'$ intersect on a vertex $x$: if $x \in K$, then $Cut_{1,x}$ separates $K$ from $S$, otherwise $Cut_{2,x}$ does. Or else $K'$ and $S'$ intersect on a switchable pair $xy$ (recall that a clique and a stable set can intersect on at most one vertex or one switchable pair): in this case, the same argument can be applied with $Cut_{i,xy}$ for some $i \in \{1,2,3,4\}$ depending on the intersection between $\{x,y\}$ and $K$. □

In particular, as for the graph case, if $T \in \mathcal{F}$ has at most $\mathcal{O}(|V(T)|^c)$ maximal cliques (or stable sets) for some constant $c \geqslant 2$, then there is a CS-separator of size $\mathcal{O}(|V(T)|^c)$.

**In Berge graphs with no balanced skew-partition**

We now focus on proving that the trigraphs of $\mathcal{F}$ with no balanced skew-partition admit a quadratic CS-separator. The outline is to prove this by induction on the trigraph: either the trigraph is basic, and we handle this case in Lemma 3.29; or the trigraph (or its complement) can be decomposed by 2-join, and we handle this case in Lemma 3.30 by contracting each side of the 2-join into blocks of decompositions. We put the pieces together in Theorem 3.31.

We begin with the case of basic trigraphs:

**LEMMA 3.29**

*There exists a constant c such that every basic trigraph T admits a CS-separator of size $c|V(T)|^2$.*

*Proof.* Since the problem is self-complementary, we consider only the cases of bipartite graphs, line trigraphs and doubled trigraphs. Cliques in a bipartite trigraph have size at most 2, thus there is at most a quadratic number of them. If $T$ is a line trigraph, then its full realization is the line graph of a bipartite graph $G$ thus $T$ has a linear number of maximal cliques because each of them corresponds to a vertex of $G$. By Observation 3.28, this implies the existence of a CS-separator of quadratic size.

If $T$ is a doubled trigraph, let $(X,Y)$ be a good partition of $T$ and consider the following family of cuts: first, build the cut $(Y,X)$, and in the second place, for every $Z = \{x\}$ with $x \in X$ or $Z = \emptyset$, and for every $Z' = \{y\}$ with $y \in Y$ or $Z' = \emptyset$, build the cut $((Y \cup Z) \setminus Z', (X \cup Z') \setminus Z)$. Finally, for every pair $x, y \in V$, build the cut $(\{x,y\}, V(T) \setminus \{x,y\})$, and $(V(T) \setminus \{x,y\}, \{x,y\})$. These cuts form a CS-separator: let $K$ be a clique in $T$ and $S$ be a stable set disjoint from $K$, then $|K \cap X| \leqslant 2$ and $|S \cap Y| \leqslant 2$. If $|K \cap X| = 2$, then $K$ has size exactly 2 since no vertex of $Y$ has two adjacent neighbors in $X$. So the cut $(K, V \setminus K)$ separates $K$ and $S$. By similar arguments, if $|S \cap Y| = 2$ then $S$ has size 2 and $(V \setminus S, S)$ separates $K$ and $S$. Otherwise, $|K \cap X| \leqslant 1$ and $|S \cap Y| \leqslant 1$ and then $(Y \cup (K \cap X) \setminus (S \cap Y), X \cup (S \cap Y) \setminus (K \cap X))$ separates $K$ and $S$. □

Next, we handle the case of a 2-join in the trigraph and show how to reconstruct a CS-separator from the CS-separators of the blocks of decompositions.

**LEMMA 3.30**

*Let $T$ be a trigraph admitting a 2-join $(X_1, X_2)$. If the blocks of decomposition $T_{X_1}$ and $T_{X_2}$ admit a CS-separator of size respectively $k_1$ and $k_2$, then $T$ admits a CS-separator of size $k_1 + k_2$.*

*Proof.* Let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of $(X_1, X_2)$, $T_{X_1}$ (resp. $T_{X_2}$) be the block of decomposition with marker vertices $a_2, b_2$, and possibly $c_2$ (depending on the parity of the 2-join) (resp. $a_1, b_1$, and possibly $c_1$). Observe that there is no need to distinguish between an odd or an even 2-join, because $c_1$ and $c_2$ play no role. Let $F_1$ be a CS-separator of $T_{X_1}$ of size $k_1$ and $F_2$ be a CS-separator of $T_{X_2}$ of size $k_2$.

Let us build $F$ aiming at being a CS-separator for $T$. For each cut $(U, W) \in F_1$, build a cut as follows: start with $U' = U \cap X_1$ and $W' = W \cap X_1$. If $a_2 \in U$, add $A_2$ to $U'$, otherwise add $A_2$ to $W'$. Moreover if $b_2 \in U$, add $B_2$ to $U'$, otherwise add $B_2$ to $W'$. Now build the cut $(U', W' \cup C_2)$ with the resulting sets $U'$ and $W'$. In other words, we put $A_2$ on the same side as $a_2$, $B_2$ on the same side as $b_2$, and $C_2$ on the stable set side. For each cut $(U, W)$ in $F_2$, we do the similar construction: start from $(U \cap X_2, W \cap X_2)$, then put $A_1$ on the same side as $a_1$, $B_1$ on the same side as $b_1$, and finally put $C_1$ on the stable set side.

The family $F$ of cuts is indeed a CS-separator: let $K$ be a clique and $S$ be a stable set disjoint from $K$. As a first case, suppose that $K \subseteq X_1$. We define $S' = (S \cap X_1) \cup S_{a_2,b_2}$ where $S_{a_2,b_2} \subseteq \{a_2, b_2\}$ contains $a_2$ (resp. $b_2$) if and only if $S$ intersects $A_2$ (resp. $B_2$). $S'$ is a stable set of $T_{X_1}$, so there is a cut in $F_1$ separating the pair $K$ and $S'$. The corresponding cut in $F$ separates $K$ and $S$. The case $K \subseteq X_2$ is handled symmetrically.

As a second case, suppose that $K$ intersects both $X_1$ and $X_2$. Then $K \cap C_1 = \emptyset$ and $K \subseteq A_1 \cup A_2$ or $K \subseteq B_1 \cup B_2$. Assume by symmetry that $K \subseteq A_1 \cup A_2$. Observe that $S$ cannot intersect both $A_1$ and $A_2$ which are strongly complete, so without loss of generality we assume that it does not intersect $A_2$. Let $K' = (K \cap A_1) \cup \{a_2\}$ and $S' = (S \cap X_1) \cup S_{b_2}$ where $S_{b_2} = \{b_2\}$ if $S$ intersects $B_2$, and $S_{b_2} = \emptyset$ otherwise. $K'$ is a clique and $S'$ is a stable set of $T_{X_1}$ so there exists a cut in $F_1$ separating them, and the corresponding cut in $F$ separates $K$ and $S$. Then $F$ is a CS-separator. □

This leads us to the main theorem of this section:

**THEOREM 3.31**

*Every trigraph $T$ of $\mathcal{F}$ with no balanced skew-partition admits a CS-separator of size $\mathcal{O}(|V(T)|^2)$.*

*Proof.* Let $c'$ be the constant of Lemma 3.29 and $c = \max(c', 2^{24})$. Let us prove by induction that every trigraph of $T$ on $n$ vertices admits a CS-separator of size $cn^2$. The initialization is concerned with basic trigraphs, for which Lemma 3.29 shows that a CS-separator of size $c'n^2$ exists, and with trigraphs of size less than 24. For them, one can consider every subset $U$ of vertices and take the cut $(U, V \setminus U)$ which form a trivial CS-separator of size at most $2^{24}n^2$.

Consequently, we can now assume that the trigraph $T$ is not basic and has at least 25 vertices. By applying Theorem 3.26, we know that $T$ has a 2-join $(X_1, X_2)$

(or a complement 2-join, in which case we switch to $\overline{T}$ since the problem is self-complementary). We define $n_1 = |X_1|$, then by Lemma 3.24 we can assume that $4 \leqslant n_1 \leqslant n-4$. Applying Theorem 3.27, we can apply the induction hypothesis on the blocks of decomposition $T_{X_1}$ and $T_{X_2}$ to get a CS-separator of size respectively at most $k_1 = c(n_1+3)^2$ and $k_2 = c(n-n_1+3)^2$. By Lemma 3.30, $T$ admits a CS-separator of size $k_1+k_2$. The goal is to prove that $k_1+k_2 \leqslant cn^2$.

Let $P(n_1) = c(n_1+3)^2 + c(n-n_1+3)^2 - cn^2$. Observe that $P$ is a degree 2 polynomial with leading coefficient $2c > 0$. Moreover, $P(4) = P(n-4) = -2c(n-25) \leqslant 0$ so by convexity of $P$, $P(n_1) \leqslant 0$ for every $4 \leqslant n_1 \leqslant n-4$, which achieves the proof. □

### Closure by generalized $k$-join

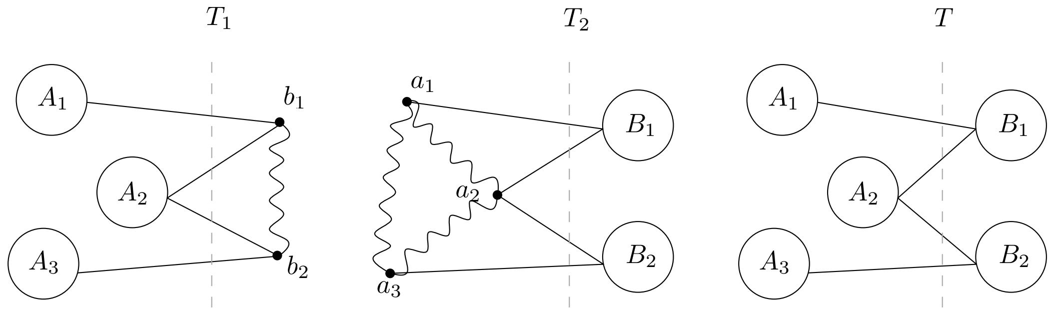
We present here a way to extend the result of the Clique-Stable separation on Berge graphs with no balanced skew-partition to larger classes of graphs, based on a generalization of the 2-join. Let $\mathcal{C}$ be a class of graphs, which should be seen as *basic* graphs. For any integer $k \geqslant 1$, we construct the class $\mathcal{C}^{\leqslant k}$ of trigraphs in the following way: a trigraph $T$ belongs to $\mathcal{C}^{\leqslant k}$ if and only if there exists a partition $X_1, \ldots, X_r$ of $V(T)$ such that

- for every $1 \leqslant i \leqslant r$, $1 \leqslant |X_i| \leqslant k$,
- for every $1 \leqslant i \leqslant r$, $\binom{X_i}{2} \subseteq \sigma(T)$,
- for every $1 \leqslant i \neq j \leqslant r$, $(X_i \times X_j) \cap \sigma(T) = \emptyset$, and
- there exists a graph $G$ in $\mathcal{C}$ such that $G$ is a realization of $T$.

In other words, starting from a graph $G$ of $\mathcal{C}$, we partition its vertices into small parts (of size at most $k$), and change all adjacencies inside the parts into switchable pairs.

We now define the *generalized $k$-join* between two trigraphs $T_1$ and $T_2$ (see Figure 3.9 for an illustration), which generalize the 2-join and is quite similar to the $H$-join defined in [24]. Let $T_1$ and $T_2$ be two trigraphs having the following properties with $1 \leqslant r, s \leqslant k$:

- $V(T_1)$ is partitioned into $(A_1, \ldots, A_r, B = \{b_1, \ldots, b_s\})$ and $A_j \neq \emptyset$ for every $1 \leqslant j \leqslant r$.
- $V(T_2)$ is partitioned into $(B_1, \ldots, B_s, A = \{a_1, \ldots, a_r\})$ and $B_i \neq \emptyset$ for every $1 \leqslant i \leqslant s$.
- $\binom{B}{2} \subseteq \sigma(T_1)$ and $\binom{A}{2} \subseteq \sigma(T_2)$, meaning that $A$ and $B$ contain only switchable pairs.




(a) In $T_1$, $b_1b_2$ is a switchable pair, $b_1$ is strongly complete to $A_1$ and $A_2$ and strongly anticomplete to $A_3$; $b_2$ is strongly complete to $A_2$ and $A_3$ and strongly anticomplete to $A_1$. There can be any adjacency in the left part.

(b) In $T_2$, $\{a_1, a_2, a_3\}$ contains only switchable pairs, $B_1$ is strongly complete to $\{a_1, a_2\}$ and strongly anticomplete to $a_3$; $B_2$ is strongly complete to $\{a_2, a_3\}$ and strongly anticomplete to $a_1$. There can be any adjacency in the right part.

(c) In $T$, $B_1$ is strongly complete to $A_1$ and $A_2$ and strongly anticomplete to $A_3$; $B_2$ is strongly complete to $A_2$ and $A_3$ and strongly anticomplete to $A_1$. The adjacencies inside the left part and the right part are preserved.

FIGURE 3.9: Example of a generalized 3-join $T$ of $T_1$ and $T_2$ with $r = 3$ and $s = 2$.

- For every $1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r$, $b_i$ and $a_j$ are either both strongly complete or both strongly anticomplete to respectively $A_j$ and $B_i$. In other words, there exists a bipartite graph describing the adjacency between $B$ and $(A_1, \dots, A_r)$, and the same bipartite graph describes the adjacency between $(B_1, \dots, B_s)$ and $A$.

Then the generalized $k$-join of $T_1$ and $T_2$ is defined as the trigraph $T$ with vertex set $V(T) = A_1 \cup \ldots \cup A_r \cup B_1 \cup \ldots \cup B_s$. Let $\theta_1$ and $\theta_2$ be the adjacency function of $T_1$ and $T_2$, respectively. As much as possible, the adjacency function $\theta$ of $T$ follows $\theta_1$ and $\theta_2$ (meaning $\theta(uv) = \theta_1(uv)$ for $uv \in \binom{V(T_1) \cap V(T)}{2}$ and $\theta(uv) = \theta_2(uv)$ for $uv \in \binom{V(T_2) \cap V(T)}{2}$), and for $a \in A_j, b \in B_i, \theta(ab) = 1$ if $b_i$ and $A_j$ are strongly complete in $T_1$ (or, equivalently, if $a_j$ and $B_i$ are strongly complete in $T_2$), and $-1$ otherwise.

We finally define $\overline{\mathcal{C}^{\leqslant k}}$ to be the smallest class containing $\mathcal{C}^{\leqslant k}$ and closed under generalized $k$-join.

**LEMMA 3.32**

*If every graph $G$ of $\mathcal{C}$ admits a CS-separator of size $m$, then every trigraph $T$ of $\overline{\mathcal{C}^{\leqslant k}}$ admits a CS-separator of size $m^{k^2}$.*

*Proof.* First we claim that if there exists a CS-separator $F$ of size $m$ then the family of cuts $F' = \{(\cap_{i=1}^k U_i, \cup_{i=1}^k W_i) | (U_1, W_1), \dots, (U_k, W_k) \in F\}$ has size $m^k$ and separates every clique from every union of at most $k$ stable sets. Indeed if $K$ is a clique and $S_1, \dots, S_k$ are $k$ stable sets such that they do not intersect $K$ then there exists in $F$ $k$ partitions $(U_1, W_1), \dots, (U_k, W_k)$ such that $(U_i, W_i)$ separates $K$ and $S_i$. Now

$(\cap_{i=1}^k U_i, \cup_{i=1}^k W_i)$ is a partition that separates $K$ from $\cup_{i=1}^k S_i$. Using the same argument we can build a family $F''$ of cuts of size $m^{k^2}$ that separates every union of at most $k$ cliques from every union of at most $k$ stable sets. Now let $T$ be a trigraph of $\mathcal{C}^{\leqslant k}$ and let $G \in \mathcal{C}$ such that $G$ is a realization of $T$. Let $X_1, \ldots, X_r$ be the partition of $V(T)$ as in the definition of $\mathcal{C}^{\leqslant k}$. Notice that a clique $K$ (resp. stable set $S$) in $T$ is a union of at most $k$ cliques (resp. stable sets) in $G$: indeed, by taking one vertex in $K \cap X_i$ (if not empty) for each $1 \leqslant i \leqslant r$, we build a clique of $G$; repeating this operation at most $k$ times covers $K$ with $k$ cliques of $G$. It follows that there exists a CS-separator of $T$ of size $m^{k^2}$. □

**LEMMA 3.33**

*If $T_1, T_2 \in \overline{\mathcal{C}^{\leqslant k}}$ admit CS-separators of size respectively $m_1$ and $m_2$, then the generalized k-join $T$ of $T_1$ and $T_2$ admits a CS-separator of size $m_1 + m_2$.*

*Proof.* The proof is very similar to the one of Lemma 3.30. We follow the notations introduced in the definition of the generalized $k$-join. Let $F_1$ (resp. $F_2$) be a CS-separator of size $m_1$ (resp. $m_2$) on $T_1$ (resp. $T_2$). Let us build $F$ aiming at being a CS-separator on $T$. For every cut $(U, W)$ in $F_1$, build the cut $(U', W')$ with the following process: start with $U' = U \cap \cup_{j=0}^r A_j$ and $W' = W \cap \cup_{j=0}^r A_j$; now for every $1 \leqslant i \leqslant s$, if $b_i \in U$, then add $B_i$ to $U'$, otherwise add $B_i$ to $W'$. In other words, we take a cut similar to $(U, W)$ by putting $B_i$ in the same side as $b_i$. We do the symmetric operation for every cut $(U, W)$ in $F_2$ by putting $A_j$ in the same side as $a_j$.

$F$ is indeed a CS-separator: let $K$ be a clique and $S$ be a stable set disjoint from $K$. Suppose as a first case that one part of the partition $(A_1, \ldots, A_r, B_1, \ldots, B_s)$ intersects both $K$ and $S$. Without loss of generality, we assume that $A_1 \cap K \neq \emptyset$ and $A_1 \cap S \neq \emptyset$. Since for every $i$, $A_1$ is either strongly complete or strongly anticomplete to $B_i$, $B_i$ cannot intersect both $K$ and $S$. Consider the following sets in $T_1$: $K' = (K \cap V(T)) \cup K_b$ and $S' = (S \cap V(T)) \cup S_b$ where $K_b = \{b_i | K \cap B_i \neq \emptyset\}$ and $S_b = \{b_i | S \cap B_i \neq \emptyset\}$. $K'$ is a clique in $T_1$, $S'$ is a stable set in $T_1$, and there is a cut separating them in $F_1$. The corresponding cut in $F$ separates $K$ and $S$.

In the case when no part of the partition intersects both $K$ and $S$, an analogous argument applies. □

**THEOREM 3.34**

*If every graph $G$ of $\mathcal{C}$ admits a CS-separator of size $\mathcal{O}(|V(G)|^c)$, then every trigraph $T$ of $\overline{\mathcal{C}^{\leqslant k}}$ admits a CS-separator of size $\mathcal{O}(|V(T)|^{k^2c})$. In particular, every realization $G'$ of a trigraph of $\overline{\mathcal{C}^{\leqslant k}}$ admits a CS-separator of size $\mathcal{O}(|V(G')|^{k^2c})$.*

*Proof.* Let $p'$ be the constant such that every $G \in \mathcal{C}$ admits a CS-separator of size $p'|V(G)|^c$, and let $p_0$ be a large constant to be defined later. We prove by induction that there exists a CS-separator of size $pn^{k^2c}$ with $p = \max(p', 2^{p_0})$. The base case is divided into two cases: the trigraphs of $\mathcal{C}^{\leqslant k}$, for which the property is verified according to Lemma 3.32, and the trigraphs of size at most $p_0$ (for which one can consider every subset $U$ of vertices and take the cut $(U, V \setminus U)$ which form a trivial CS-separator of size at most $2^{p_0} n^{k^2 c}$).

Consequently, we can now assume that $T$ is the generalized $k$-join of $T_1$ and $T_2$ with at least $p_0$ vertices. Let $n_1 = |T_1|$ and $n_2 = |T_2|$ with $n_1 + n_2 = n + r + s$ and $r + s + 1 \leqslant n_1, n_2, \leqslant n - 1$. By induction, there exists a CS-separator of size $pn_1^{k^2 c}$ on $T_1$ and one of size $pn_2^{k^2 c}$ on $T_2$. By Lemma 3.33, there exists a CS-separator on $T$ of size $pn_1^{k^2 c} + pn_2^{k^2 c}$. The goal is to prove $pn_1^{k^2 c} + pn_2^{k^2 c} \leqslant pn^{k^2 c}$.

Notice that $n_1 + n_2 = n - 1 + r + s + 1$ so by convexity of $x \mapsto x^c$ on $\mathbb{R}^+$, we know that $n_1^{k^2 c} + n_2^{k^2 c} \leqslant (n-1)^{k^2 c} + (r+s+1)^{k^2 c}$. Moreover, $r + s + 1 \leqslant 2k + 1$. Now we can define $p_0$ large enough such that for every $n \geqslant p_0$, $n^{k^2 c} - (n-1)^{k^2 c} \geqslant (2k+1)^{k^2 c}$. Then $n_1^{k^2 c} + n_2^{k^2 c} \leqslant n^{k^2 c}$, which concludes the proof. □

### 3.5.4 Strong Erdős-Hajnal property

**In Berge trigraphs with no balanced skew-partition**

As before for the CS-Separation, we need to adapt the definition of the Strong Erdős-Hajnal property to trigraphs. In fact we need a weighted version to make the proof work. When one faces a 2-join with split $(A_1, B_1, C_1, A_2, B_2, C_2)$, the idea is to contract $A_i$, $B_i$, and $C_i$ for $i = 1$ or 2, *via* the blocks of decomposition, until either we find an obvious large biclique or complement biclique in the trigraph $T$, or we end up on a basic trigraph. We handle the case of basic trigraphs, which are easier because they are well-structured, and we then prove that we can go backward and transform a biclique (or a complement biclique) in the contraction of $T$ into a biclique (or a complement biclique) in $T$, which is possible because the weight on every vertex $v$ stands for all the vertices that have been contracted into $v$.

However, this sketch of proof is too good to be true: in the case of an odd 2-join or odd complement 2-join $(X_1, X_2)$ with split $(A_1, B_1, C_1, A_2, B_2, C_2)$, the block of decomposition $T_{X_1}$ does not contain any vertex that *stands for* $C_2$. Thus we have to put the weight of $C_2$ on the switchable pair $a_2 b_2$, and remember whether $C_2$ was strongly anticomplete (in case of a 2-join) or strongly complete (in case of a complement 2-join) to $X_1$. This may propagate if we further contract $a_2 b_2$.

Let us now introduce some formal notation. A *weighted trigraph* is a pair $(T, w)$ where $T$ is a trigraph and $w$ is a weight function which assigns:

- to every vertex $v \in V(T)$, a triple $w(v) = (w_r(v), w_c(v), w_{\overline{c}}(v))$. Each coordinate has to be a non-negative integer, and $w_r(v)$ is called the *real weight* of $v$, whereas $w_c(v)$ (resp. $w_{\overline{c}}(v)$) is called the *extra-complete* (resp. *extra-anticomplete*) weight of $v$.

- to every switchable pair $uv \in \sigma(T)$, a pair $w(uv) = (w_c(uv), w_{\overline{c}}(uv))$. Similarly, each coordinate has to be a non-negative integer and $w_c(v)$ (resp. $w_{\overline{c}}(v)$) is called the *extra-complete* (resp. *extra-anticomplete*) weight of $uv$.

The extra-anticomplete (resp. extra-complete) weight will stand for vertices that have been deleted during the decomposition of an odd 2-join (resp. odd complement 2-join) - the $C_2$ in the above discussion - and thus which were strongly anticomplete (resp. strongly complete) to the other side of the 2-join.

Let us briefly mention the natural notations related to weight that will be used below: given a set of vertices $U \subseteq V(T)$, the weight of $U$ is

$$w(U) = (w_r(U), w_c(U), w_{\bar{c}}(U)) \quad \text{where} \quad \begin{aligned} w_r(U) &= \sum_{v \in U} w_r(v) \\ w_c(U) &= \sum_{\substack{u,v \in U \\ uv \in \sigma(T)}} w_c(uv) + \sum_{v \in U} w_c(v) \\ w_{\bar{c}}(U) &= \sum_{\substack{u,v \in U \\ uv \in \sigma(T)}} w_{\bar{c}}(uv) + \sum_{v \in U} w_{\bar{c}}(v) \end{aligned}$$

The *total weight* of $U$ is $w_t(U) = w_r(U) + w_c(U) + w_{\bar{c}}(U)$. By abuse of notation, we will write $w(T)$ instead of $w(V(T))$, and in particular the total weight of $T$ will be denoted $w_t(T)$. Given two disjoint sets of vertices $A$ and $B$, the *crossing weight* $w(A, B)$ is defined as the weight of the switchable pairs with one endpoint in $A$ and the other in $B$, namely:

$$w(A, B) = (w_c(A, B), w_{\bar{c}}(A, B)) = \left( \sum_{\substack{a \in A, b \in B \\ ab \in \sigma(T)}} w_c(ab), \sum_{\substack{a \in A, b \in B \\ ab \in \sigma(T)}} w_{\bar{c}}(ab) \right).$$

A bad behavior for a weight function is to concentrate all the weight at the same place, or to have a too large extra-complete and extra-anticomplete weight. This is why we introduce the notion of balanced weight function: a weight function $w$ is *balanced* if:

- for every $v \in V(T)$, $w_r(v) \leqslant \frac{1}{55} \cdot w_t(T)$,
- for every $x \in V(T)$ or $x \in \sigma(T)$, $\max(w_c(x), w_{\bar{c}}(x)) \leqslant \frac{1}{55} \cdot w_t(T)$, and
- $w_c(T) + w_{\bar{c}}(T) \leqslant \frac{7}{55} \cdot w_t(T)$.

A *virgin weight* on $T$ is a weight $w$ such that $w_c(T) = w_{\bar{c}}(T) = 0$, *i.e.* every extra-complete or extra-anticomplete weight is null. In such a case, we will drop the subscript and simply denote $w(v)$ for $w_r(v)$.

A *biclique* is a pair $(X, Y)$ of disjoint subsets of vertices such that $X$ is strongly complete to $Y$, and its *weight* is $\min(w_r(X), w_r(Y))$. A *complement biclique* in $T$ is a biclique in $\overline{T}$. Instead of looking for a biclique or a complement biclique of large *size*, from now on the goal is to find a biclique or a complement biclique of large *weight*, that is to say a constant fraction of $w_t(T)$.

We need a few more definitions, in particular the contraction of a trigraph which explains how to contract subsets of vertices in case of a 2-join or complement 2-join: let $(T, w)$ be a weighted trigraph such that $T$ admits a 2-join or complement 2-join $(X_1, X_2)$. Without loss of generality, we can assume that $X_1$ is the *heavier* part, *i.e.* $w_t(X_1) \geqslant w_t(T)/2$ (possible because $w_t(T) = w_t(X_1) + w_t(X_2)$, since no switchable pair has one endpoint in $X_1$ and the other in $X_2$), and let $(A_1, B_1, C_1, A_2, B_2, C_2)$ be a split of $(X_1, X_2)$. Let us define the weighted trigraph $(T', w')$ to be the *contraction of $(T, w)$* if $T'$ is the block of decomposition $T_{X_1}$ and its weight function $w'$ is defined as follows:

- For every vertex $v \in X_1$, we define $w'(v) = w(v)$.
- For marker vertices $a_2$ and $b_2$, we set $w'(a_2) = w(A_2)$ and $w'(b_2) = w(B_2)$.
- In case of an even (complement or not) 2-join, the marker vertex $c_2$ exists and we define $w'(c_2) = w(C_2)$. Moreover, we define $w'(a_2c_2) = w(A_2, C_2)$ and $w'(b_2c_2) = w(B_2, C_2)$.
- In case of an odd 2-join, the marker vertex $c_2$ does not exist so things become slightly more complicated. Since we want to preserve the total weight, the switchable pair $a_2b_2$ has to take a lot of weight, including the real weight of $C_2$; $w_r(C_2)$ is thus given as an extra-anticomplete weight to $a_2b_2$ because $C_2$ is strongly anticomplete to every other vertex outside of $A_2 \cup B_2$. For this reason, we define

$$\begin{aligned}
&w'(a_2b_2) = (w'_c(a_2b_2), w'_{\bar{c}}(a_2b_2)), \text{ where} \\
&w'_c(a_2b_2) = w_c(C_2) + w_c(A_2, B_2) + w_c(A_2, C_2) + w_c(B_2, C_2) \text{ and} \\
&w'_{\bar{c}}(a_2b_2) = w_{\bar{c}}(C_2) + w_{\bar{c}}(A_2, B_2) + w_{\bar{c}}(A_2, C_2) + w_{\bar{c}}(B_2, C_2) + w_r(C_2)\,.
\end{aligned}$$

- In case of an odd complement 2-join, the same problem occurs and we give the real weight $w_r(C_2)$ as an extra-complete weight to $a_2b_2$. We thus define

$$\begin{aligned}
&w'(a_2b_2) = (w'_c(a_2b_2), w'_{\bar{c}}(a_2b_2)), \text{ where} \\
&w'_c(a_2b_2) = w_c(C_2) + w_c(A_2, B_2) + w_c(A_2, C_2) + w_c(B_2, C_2) + w_r(C_2) \text{ and} \\
&w'_{\bar{c}}(a_2b_2) = w_{\bar{c}}(C_2) + w_{\bar{c}}(A_2, B_2) + w_{\bar{c}}(A_2, C_2) + w_{\bar{c}}(B_2, C_2)\,.
\end{aligned}$$

To recover information about the original trigraph after several steps of contraction, we need to introduce the notion of *model*. Intuitively, imagine that a weighted trigraph $(T, w)$ is obtain from an initial weighted trigraph $(T_0, w_0)$ by several contractions, then we can partition the vertices of the original trigraph $T_0$ into subsets of vertices that have been contracted to the same vertex or the same switchable pair of $T$. Moreover, we want to make sure that the real weight of a vertex $v$ in $T$ is the weight of the set of vertices that have been contracted to $v$. We also want that the strong adjacency and strong antiadjacency in $T$ reflects the strong adjacency and strong antiadjacency in $T_0$. Finally, we want the extra-complete (resp. extra-anticomplete) weight in

$T$ to stand for subsets of vertices of $T_0$ that have been deleted, but which were strongly complete (resp. strongly anticomplete) to (almost) all the rest of $T_0$. Formally, given a trigraph $T_0$ equipped with a virgin weight $w_0$, a weighted trigraph $(T, w)$ is a *model* of $(T_0, w_0)$ if the following conditions are fulfilled (see Figure 3.10 for an example):

- *the partition condition:* there exists a *partition mapping* $\beta$ which:
  - to every vertex $v \in V(T)$, assigns a triple
  $$\beta(v) = (\beta_r(v), \beta_c(v), \beta_{\overline{c}}(v))$$
  of disjoint subsets of vertices of $T_0$ (possibly empty). We define the *team* of $v$ as $\beta_t(v) = \beta_r(v) \cup \beta_c(v) \cup \beta_{\overline{c}}(v)$. For convenience, the set $\beta_r(v)$ (resp. the set $\beta_c(v)$, the set $\beta_{\overline{c}}(v)$) is called the *real team* (resp. *extra-complete team*, *extra-anticomplete team*) of $v$.
  - to every switchable pair $uv \in \sigma(T)$, assigns a pair
  $$\beta(uv) = (\beta_c(uv), \beta_{\overline{c}}(uv))$$
  of disjoint subsets of vertices of $T_0$ (possible empty). We also define the *team* of $uv$ as $\beta_t(uv) = \beta_c(uv) \cup \beta_{\overline{c}}(uv)$, and we call $\beta_c(uv)$ (resp. $\beta_{\overline{c}}(uv)$) the extra-complete team (resp. the extra-anticomplete team) of $uv$, similarly to the previous case.

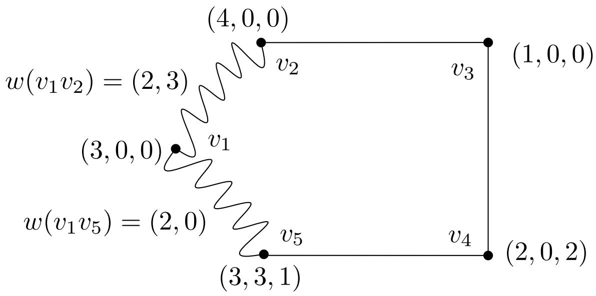
  Moreover, any two teams must be disjoint and the union of all teams is $V(T_0)$. In other words, $V(T_0)$ is partitioned into *teams*, each team being assigned to a vertex of $T$ or to a switchable pair of $T$, and each team is divided into three (or two, for a switchable pair) disjoint parts.

  Similarly to the weight function, for a subset of vertices $U \subseteq V(T)$ we define

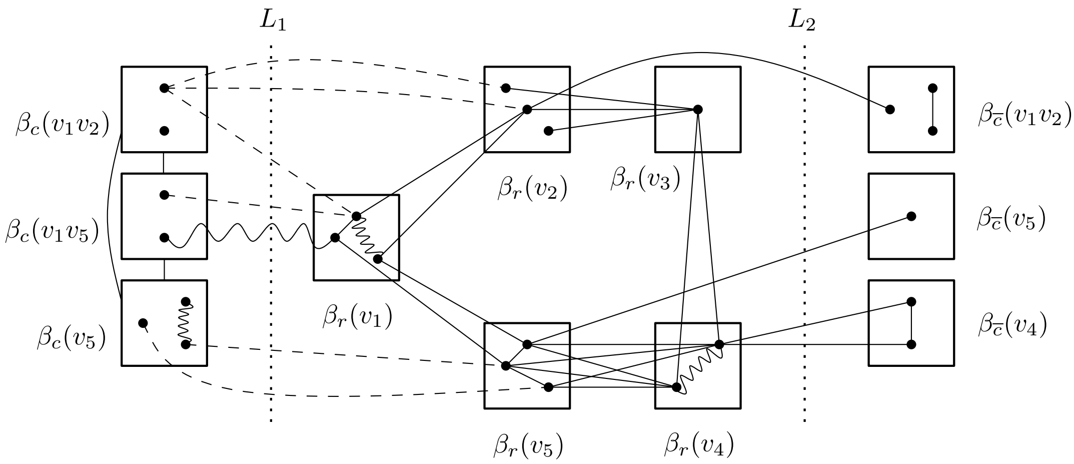
$$\beta(U) = (\beta_r(U), \beta_c(U), \beta_{\overline{c}}(U)) \text{ where } \beta_r(U) = \bigcup_{v \in U} \beta_r(v)$$
$$\beta_c(U) = \bigcup_{v \in U} \beta_c(v) \cup \bigcup_{\substack{u,v \in U \\ uv \in \sigma(T)}} \beta_c(uv)$$
$$\beta_{\overline{c}}(U) = \bigcup_{v \in U} \beta_{\overline{c}}(v) \cup \bigcup_{\substack{u,v \in U \\ uv \in \sigma(T)}} \beta_{\overline{c}}(uv)$$

  and for two disjoint subsets of vertices $A, B \subseteq V(T)$,

$$\beta(A, B) = (\beta_c(A, B), \beta_{\overline{c}}(A, B)) \quad \text{where} \quad \beta_c(A, B) = \bigcup_{\substack{a \in A, b \in B \\ ab \in \sigma(T)}} \beta_c(ab)$$
$$\beta_{\overline{c}}(A, B) = \bigcup_{\substack{a \in A, b \in B \\ ab \in \sigma(T)}} \beta_{\overline{c}}(ab)$$




(a) A weighted trigraph $(T, w)$




(b) A weighted trigraph $(T_0, w_0)$ with virgin weight $w_0$ defined by $w_0(v) = (1, 0, 0)$ for every vertex $v \in V(T_0)$ and $w_0(uv) = (0, 0)$ for every $uv \in \sigma(T_0)$.

FIGURE 3.10: Illustration for the definition of a model: the weighted trigraph $(T, w)$ depicted in (a) is a model of the weighted trigraph $(T_0, w_0)$ depicted in (b), as witnessed by the partition mapping $\beta$ (empty teams are not depicted). Each vertex at the left-hand side of dotted line $L_1$ is assumed to be strongly adjacent to every other vertex except if a non-edge is explicitly drawn (with a dashed edge for strong antiedge and with a wiggly edge for a switchable pair). Similarly, each vertex at the right-hand side of dotted line $L_2$ is assumed to be strongly antiadjacent to every other vertex except if an edge is explicitly drawn.

- *the weight condition:* $w_t(T) = w_0(T_0)$ and for every vertex $v$,

$$w_r(v) = w_0(\beta_r(v))\ , \quad w_c(v) = w_0(\beta_c(v)) \quad \text{and} \quad w_{\overline{c}}(v) = w_0(\beta_{\overline{c}}(v))$$

  In other words, the real weight $w_r(v)$ of $v$ is the weight of its real team $\beta_r(v)$ in $(T_0, w_0)$, its extra-complete (resp extra-anticomplete) weight is the weight of extra-complete team $\beta_c(v)$ (resp. extra-anticomplete team $\beta_{\overline{c}}(v)$) in $(T_0, w_0)$. In the same fashion, for every switchable pair $uv \in \sigma(T)$,

$$w_c(uv) = w_0(\beta_c(uv)) \quad \text{and} \quad w_{\overline{c}}(uv) = w_0(\beta_{\overline{c}}(uv))\ .$$

- *the strong adjacency condition:* if two vertices $u$ and $v$ are strongly adjacent in $T$, then $\beta_r(u)$ and $\beta_r(v)$ are strongly complete in $T_0$. If $u$ and $v$ are strongly antiadjacent in $T$, then $\beta_r(u)$ and $\beta_r(v)$ are strongly anticomplete in $T_0$.

- *the extra-condition:* Informally, the extra-complete team of $v$ (resp. $uv$) is strongly complete to every other extra-complete team, and is also strongly complete to every real team, except maybe the real team of $v$ (resp. of $u$ and $v$). In the same way, the extra-anticomplete team of $v$ (resp. $uv$) is strongly anticomplete to every other extra-anticomplete team, and is also strongly anticomplete to every real team, except maybe the real team of $v$ (resp. $u$ and $v$). Formally: for every vertex $v$, $\beta_c(v)$ is strongly complete to every $\beta_c(x)$ for $x \in V(T)$, $x \neq v$ or $x \in \sigma(T)$, and to every $\beta_r(y)$ for $y \neq v$. For every switchable pair $uv \in \sigma(T)$, $\beta_c(uv)$ is strongly complete to every $\beta_c(x)$ for $x \in V(T)$ or $x \in \sigma(T)$, $x \neq uv$, and to every $\beta_r(y)$ for $y \neq u, v$. For every vertex $v$, $\beta_{\overline{c}}(v)$ is strongly anticomplete to every $\beta_{\overline{c}}(x)$ for $x \in V(T)$, $x \neq v$ or $x \in \sigma(T)$, and to every $\beta_r(y)$ for $y \neq v$. For every switchable pair $uv \in \sigma(T)$, $\beta_{\overline{c}}(uv)$ is strongly anticomplete to every $\beta_{\overline{c}}(x)$ for $x \in V(T)$ or $x \in \sigma(T)$, $x \neq uv$, and to every $\beta_r(y)$ for $y \neq u, v$.

We are now ready for the proof, let us first provide a sketch: start from a trigraph $T_0$ with a balanced virgin weight $w_0$, in which we want to find a biclique or complement biclique fo large weight. Keep contracting it, and prove that at each step, we still have a model of $(T_0, w_0)$ with balanced weight. Stop either when the teams provide a biclique or complement biclique of large weight in $(T_0, w_0)$, or when we end up with a basic trigraph. In the latter case, delete the extra-complete and extra-anticomplete weight, find a biclique or complement biclique of large weight in the basic trigraph, and show that it gives a biclique or complement biclique of large weight in $T_0$.

**LEMMA 3.35**

*Let $(T_0, w_0)$ be a trigraph of $\mathcal{F}$ with no balanced skew-partition, equipped with a balanced virgin weight $w_0$. Assume that $(T, w)$ is a model of $(T_0, w_0)$ such that $w$ is balanced and $T$ is a non-basic trigraph of $\mathcal{F}$ with no balanced skew-partition. Then at least one of the following holds:*

- *There exists a biclique or complement biclique in $(T_0, w_0)$ of weight at least $\frac{1}{55} \cdot w_0(T_0)$.*
- *The contraction $(T', w')$ of $(T, w)$ is a model of $(T_0, w_0)$ and $w'$ is balanced. Moreover, $T'$ is a trigraph of $\mathcal{F}$ with no balanced skew-partition.*

*Proof.* First of all, let us check that the second item is well-defined: by assumption, $T$ is a trigraph of $\mathcal{F}$ with no balanced skew-partition and is not basic, thus $T$ has a 2-join or a complement 2-join $(X_1, X_2)$. Consequently, the contraction of $(T, w)$ is well-defined and $T'$ is the block of decomposition $T_{X_1}$ or $T_{X_2}$. By Theorem 3.27, $T'$ is a trigraph of $\mathcal{F}$ with no balanced skew-partition. Up to exchanging $T$ and $\overline{T}$, assume that $(X_1, X_2)$ is a 2-join with split $(A_1, B_1, C_1, A_2, B_2, C_2)$ and, without loss of generality, we assume that $w_t(X_1) \geqslant w_t(T)/2$ (consequently $T' = T_{X_1}$).

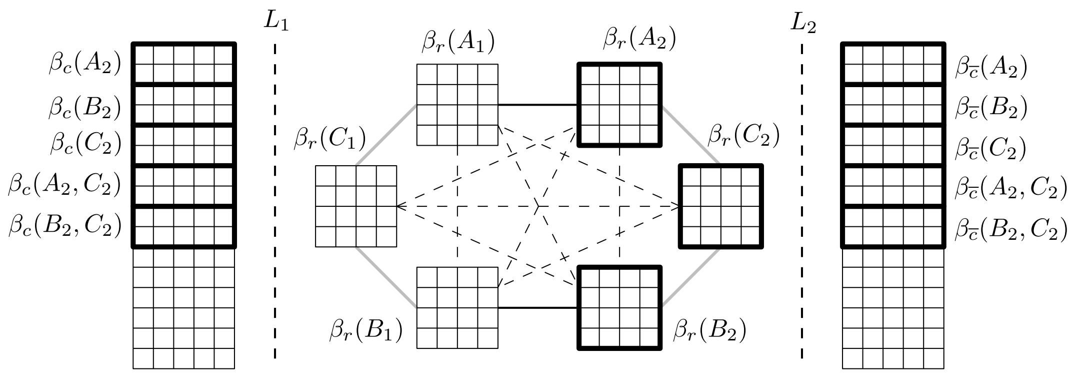
**Case 1:** $(X_1, X_2)$ **is an even 2-join.**

We first prove that $(T', w')$ is a model of $(T_0, w_0)$. Since $(T, w)$ is a model of $(T_0, w_0)$, there exists a partition mapping $\beta$ that witnesses it. Let us build $\beta'$ a partition mapping for $(T', w')$ in the following natural way (see Figure 3.11 (a)):

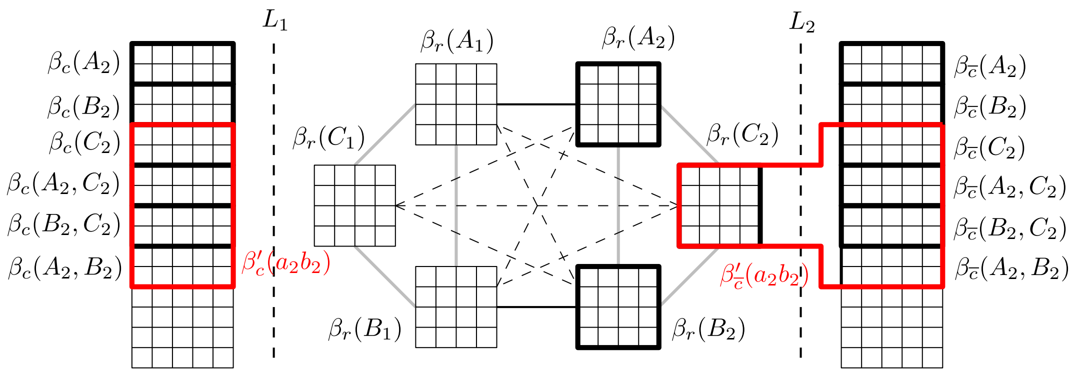
- for every $v \in X_1$, we set $\beta'(v) = \beta(v)$ and for every $u, v \in X_1$, $uv \in \sigma(T')$, we set $\beta'(uv) = \beta(uv)$.
- $\beta'(a_2) = \beta(A_2)$, $\beta'(b_2) = \beta(B_2)$, $\beta'(c_2) = \beta(C_2)$.
- $\beta'(a_2c_2) = \beta(A_2, C_2)$ and $\beta'(b_2c_2) = \beta(B_2, C_2)$

We easily see that the *weight condition* is ensured. Let us check the *strong adjacency condition*. Let $u, v \in V(T')$ be two strongly adjacent (or strongly antiadjacent) vertices, we need to prove that $\beta'_r(u)$ and $\beta'_r(v)$ are strongly complete (or strongly anticomplete) in $T_0$:

- if $\{u, v\} \cap \{a_2, b_2, c_2\} = \emptyset$, then $u$ and $v$ were already strongly adjacent (or strongly antiadjacent) in $T$ so the condition holds since $(T, w)$ is a model of $(T_0, w_0)$.
- if $u = a_2$ and $u$ and $v$ are strongly adjacent: then $v \in A_1$ and thus $v$ was strongly complete to $A_2$ in $T$. Consequently, $\beta_r(v)$ is strongly complete to $\beta_r(a)$ in $T_0$ for every $a \in A_2$, and thus $\beta'_r(v) = \beta_r(v)$ is strongly complete to $\beta'_r(a_2) = \beta_r(A_2)$.
- if $u = a_2$ and $u$ and $v$ are strongly antiadjacent: then either $v \in B_1 \cup C_1$, in which case the same kind of argument applies to prove that $\beta'_r(v)$ is strongly anticomplete to $\beta'_r(a_2)$; or $v = b_2$, in which case we conclude as follows: since the 2-join is even, there exists no odd path between $A_2$ and $B_2$ in $T$, in particular no edge between $A_2$ and $B_2$. So $A_2$ is strongly anticomplete to $B_2$ in $T$, and thus $\beta'_r(a_2) = \beta_r(A_2)$ is strongly anticomplete to $\beta'_r(b_2) = \beta_r(B_2)$ in $T_0$.
- if $u = b_2$: by symmetry, the same argument applies for $b_2$.




(a) Case 1: even 2-join.




(b) Case 2: odd 2-join.

FIGURE 3.11: Illustration for the proof of Lemma 3.35. The little boxes show how $\beta$ partition $V(T_0)$, witnessing that $(T, w)$ is a model of $(T_0, w_0)$. Boxes with bold font show groups of teams that are merged together by $\beta'$, witnessing that the contraction $(T', w')$ of $(T, w)$ still is a model of $(T_0, w_0)$. For case 2, red boxes highlight the most tricky part of the proof concerning the extra-complete and extra-anticomplete teams of the new switchable pair $a_2b_2$. The extra-complete teams are depicted on the left of dotted line $L_1$, extra-anticomplete teams are depicted on the right of dotted line $L_2$, and real teams are in between. For a better drawing, adjacencies assumed for the extra-condition are implied but not depicted. Grey lines indicate that there may or may not be some edges. Dashed lines link strongly anticomplete teams, and straight lines link strongly complete teams.

- if $u = c_2$: since $c_2$ has no strong neighbor in $T'$, we are left with the case where $v$ is strongly antiadjacent to $c_2$ and thus $v \in X_1$. By definition of a 2-join, $v$ is strongly anticomplete to $C_2$ in $T$, and thus $\beta'_r(v) = \beta_r(v)$ is strongly anticomplete to $\beta'_r(c_2) = \beta_r(C_2)$ in $T_0$, since $(T, w)$ is a model of $(T_0, w_0)$.

Consequently, the *strong adjacency condition* holds. It is rather easy to see that the *extra-condition* is also ensured, because the new extra-complete (resp. extra-anticomplete) teams are obtained by merging former extra-complete (resp. extra-anticomplete) teams. Let us describe the argument on an example (the other cases are quite similar). Let $v \in \beta'_c(a_2c_2)$, by definition there exists $ac \in \sigma(T)$ such that $a \in A_2$, $c \in C_2$ and $v \in \beta_c(ac)$. Since $(T, w)$ is a model of $(T_0, w_0)$, $v$ is strongly complete to every other extra-complete teams of $\beta$ except the one it belongs to (and thus to every extra-complete teams of $\beta'$ except $\beta'_c(a_2c_2)$, which it belongs to), and $v$ is also strongly complete to every real team except maybe $\beta_r(a)$ and $\beta_r(c)$. But $a \in A_2$ and $c \in C_2$ so $a \in \beta'_r(a_2)$ and $c \in \beta'_r(c_2)$. Consequently, $v$ is strongly complete to every real teams, except maybe $\beta'_r(a_2)$ and $\beta'_r(c_2)$: this is what we require for a member of $\beta'_c(a_2c_2)$.

We now have to see if $w'$ is balanced. First of all,

$$w'_c(T') + w'_{\overline{c}}(T') = w_c(T) + w_{\overline{c}}(T) \leqslant \frac{7}{55} \cdot w_t(T) = \frac{7}{55} \cdot w'_t(T') \,.$$

Moreover, observe that

$$w_t(X_1) = w_r(A_1) + w_r(B_1) + w_r(C_1) + w_c(X_1) + w_{\overline{c}}(X_1) \,.$$

But $w_t(X_1) \geqslant w_t(T)/2$ and

$$w_c(X_1) + w_{\overline{c}}(X_1) \leqslant w_c(T) + w_{\overline{c}}(T) \leqslant \frac{7}{55} \cdot w_t(T)$$

so

$$\max(w_r(A_1), w_r(B_1), w_r(C_1)) \geqslant \frac{1}{3}\left(\frac{1}{2} - \frac{7}{55}\right) w_t(T) \geqslant \frac{1}{55} \cdot w_t(T) \,.$$

Since the other cases are handled similarly, we assume that $w_r(A_1) \geqslant \frac{1}{55} \cdot w_t(T)$. Each of $\beta'_r(a_2)$, $\beta'_r(b_2)$ and $\beta'_r(c_2)$ is either strongly complete or strongly anticomplete to $\beta_r(A_1)$ whose weight is $w_0(\beta_r(A_1)) = w_r(A_1) \geqslant \frac{1}{55} \cdot w_t(T)$, consequently if $\max(w'_r(a_2), w'_r(b_2), w'_r(c_2)) \geqslant \frac{1}{55} \cdot w_t(T)$, we find a biclique or a complement biclique of large enough weight in $T_0$, and the first item holds.

Otherwise, observe that every extra-complete team among $\beta'_c(a_2)$, $\beta'_c(b_2)$, $\beta'_c(c_2)$, $\beta'_c(a_2c_2)$, $\beta'_c(b_2c_2)$ is strongly complete to all the real teams $\beta'_r(x)$ for $x \in X_1$, thus if one of them has weight $\geqslant \frac{1}{55} \cdot w_t(T)$ in $(T_0, w_0)$, we find a biclique in $T_0$ and the first item holds. Thus $w'_c(a_2)$, $w'_c(b_2)$, $w'_c(c_2)$, $w'_c(a_2c_2)$, $w'_c(b_2c_2) \leqslant \frac{1}{55} \cdot w_t(T)$. By similar arguments, $w'_{\overline{c}}(a_2)$, $w'_{\overline{c}}(b_2)$, $w'_{\overline{c}}(c_2)$, $w'_{\overline{c}}(a_2c_2)$, $w'_{\overline{c}}(b_2c_2) \leqslant \frac{1}{55} \cdot w_t(T)$ otherwise we find a large complement biclique in $T_0$ and conclude with the first item. Otherwise, $w'$ is balanced and we conclude with the second item.

**Case 2:** $(X_1, X_2)$ **is an odd 2-join.**

As in the previous case, we begin with proving that $(T', w')$ is a model of $(T_0, w_0)$. Since $(T, w)$ is a model of $(T_0, w_0)$, there exists a partition mapping $\beta$ that witnesses it. Let us build $\beta'$ a partition mapping for $(T', w')$ in the following way (see Figure 3.11.(b)):

- for every $v \in X_1$, we set $\beta'(v) = \beta(v)$ and for every $u, v \in X_1$, $uv \in \sigma(T')$, we set $\beta'(uv) = \beta(uv)$.
- $\beta'(a_2) = \beta(A_2)$, $\beta'(b_2) = \beta(B_2)$.
- For the switchable pair $a_2b_2$, remember that the weight given to it was slightly more complicated because the marker vertex $c_2$ does not exist. Following the same approach, we do not want to loose track from the teams of type $\beta_t(c)$ for $c \in C_2$ or $\beta_t(vc)$ for $c \in C_2$, $vc \in \sigma(T)$. Their extra-complete teams are thus merged with the extra-complete team $\beta_c(A_2, B_2)$, and their extra-anticomplete teams are merged with the extra-anticomplete team $\beta_{\overline{c}}(A_2, B_2)$. But there is a remaining part $\beta_r(C_2)$: since $C_2$ is strongly anticomplete to $X_1$, we decide to merge also $\beta_r(C_2)$ with the extra-anticomplete team $\beta_{\overline{c}}(A_2, B_2)$ (in the case of a complement odd 2-join, it is merged in the extra-complete team $\beta_c(A_2, B_2)$). Formally, we define:

$$\begin{aligned}&\beta'(a_2b_2) = (\beta'_c(a_2b_2), \beta'_{\overline{c}}(a_2b_2)), \quad \text{where}\\ &\beta'_c(a_2b_2) = \beta_c(A_2, B_2) \cup \beta_c(A_2, C_2) \cup \beta_c(B_2, C_2) \cup \beta_c(C_2) \text{ and}\\ &\beta'_{\overline{c}}(a_2b_2) = \beta_{\overline{c}}(A_2, B_2) \cup \beta_{\overline{c}}(A_2, C_2) \cup \beta_{\overline{c}}(B_2, C_2) \cup \beta_{\overline{c}}(C_2) \cup \beta_r(C_2)\ .\end{aligned}$$

Once again, we easily see that the *weight condition* is ensured, and with the same arguments as in Case 1, we can check that the *strong adjacency condition* is also ensured. As for the *extra-condition*, the only interesting case is concerned with $\beta'_{\overline{c}}(a_2b_2)$: let $v \in \beta'_{\overline{c}}(a_2b_2) \subseteq V(T_0)$. The goal is to prove that $v$ is strongly anticomplete to every other extra-anticomplete team of $\beta'$, and to every real team of $\beta'$ except maybe the real team of $a_2$ and the real team of $b_2$. By definition of $\beta'_{\overline{c}}(a_2b_2)$, one of the following holds:

- $v \in \beta_{\overline{c}}(A_2, C_2)$: then there exists $ac \in \sigma(T)$ such that $a \in A_2$, $c \in C_2$ and $v \in \beta_{\overline{c}}(ac)$. Since $(T, w)$ is a model of $(T_0, w_0)$, $v$ is strongly anticomplete to every other extra-anticomplete team of $\beta$, thus of $\beta'$, and $v$ is also strongly anticomplete to every real teams except maybe $\beta_r(a)$ and $\beta_r(c)$. But $\beta_r(a) \subseteq \beta'_r(a_2)$ and $\beta_r(c) \subseteq \beta_r(C_2) \subseteq \beta'_{\overline{c}}(a_2b_2)$ so $v$ is strongly anticomplete to every real team except maybe $\beta'_r(a_2)$.
- the cases $v \in \beta_{\overline{c}}(B_2, C_2)$ and $v \in \beta_{\overline{c}}(A_2, B_2)$ are handled in the same fashion
- $v \in \beta_{\overline{c}}(C_2)$: then there exists $c \in C_2$ such that $v \in \beta_{\overline{c}}(c)$. Since $(T, w)$ is a model of $(T_0, w_0)$, $v$ is strongly anticomplete to every other extra-anticomplete team of $\beta$, thus of $\beta'$, and is also strongly anticomplete to every real teams except maybe

$\beta_r(c)$. But $\beta_r(c) \subseteq \beta_r(C_2) \subseteq \beta'_{\overline{c}}(a_2b_2)$ so $v$ is strongly anticomplete to every real team of $\beta'$.

- $v \in \beta_r(C_2)$: then there exists $c \in C_2$ such that $v \in \beta_r(c)$. By definition of a 2-join, $c$ is strongly anticomplete to $X_1$ in $T$, so since $(T, w)$ is a model of $(T_0, w_0)$, $v$ is strongly anticomplete to every real team $\beta_r(x)$ with $x \in X_1$, *i.e.* to every real team of $\beta'$ except maybe $\beta'_r(a_2)$ and $\beta'_r(b_2)$. Moreover, by the extra-condition on $(T, w)$, $\beta_r(C_2)$ is strongly anticomplete to every extra-anticomplete team of $\beta$ except those included in $\beta_{\overline{c}}(C_2)$, $\beta_{\overline{c}}(A_2, C_2)$ or $\beta_{\overline{c}}(B_2, C_2)$. But those three are all included in $\beta'_{\overline{c}}(a_2b_2)$, so $v$ is strongly anticomplete to every extra-anticomplete team different from $\beta'_{\overline{c}}(a_2b_2)$.

Let us now check that $w'$ is balanced. With the same argument as in Case 1, we obtain that $\max(w_r(A_1), w_r(B_1), w_r(C_1)) \geqslant \frac{1}{55} \cdot w_t(T)$. Consequently we also have $\max(w_r(a_2), w_r(b_2), w_r(C_2)) \leqslant \frac{1}{55} \cdot w_t(T)$, otherwise we find a biclique or complement biclique of large weight in $(T_0, w_0)$. Moreover, $\beta'_c(a_2)$, $\beta'_c(b_2)$, $\beta'_c(a_2b_2)$ and $\beta'_{\overline{c}}(a_2)$, $\beta'_{\overline{c}}(b_2)$, $\beta'_{\overline{c}}(a_2b_2)$ are each either strongly complete or strongly anticomplete to $\beta_r(X_1)$, so their respective weight $w'_c(a_2)$, $w_c(b_2)$, $w'_c(a_2b_2)$ and $w'_{\overline{c}}(a_2)$, $w'_{\overline{c}}(b_2)$, $w'_{\overline{c}}(a_2b_2)$ are at most $\frac{1}{55} \cdot w_t(T)$, otherwise we find a biclique or complement biclique of large weight in $(T_0, w_0)$.

Finally, we have $w'_c(T') + w'_{\overline{c}}(T') = w_c(T) + w_{\overline{c}}(T) + w_r(C_2)$. We want to prove that $w'_c(T') + w'_{\overline{c}}(T') \leqslant \frac{7}{55} \cdot w'_t(T')$. Assume not, then $w_c(T) + w_{\overline{c}}(T) \geqslant \frac{6}{55} \cdot w_t(T)$ since $w_r(C_2) \leqslant \frac{1}{55} \cdot w_t(T)$ and $w'_t(T') = w_t(T)$. Thus one of $w_c(T)$ or $w_{\overline{c}}(T)$, say $w_c(T)$, is at least $\frac{3}{55} \cdot w_t(T)$. Since every extra-complete team $\beta_c(x)$ for $x \in V(T)$ or $x \in \sigma(T)$ has weight at most $\frac{1}{55} \cdot w_t(T)$, we can split $\beta_c(T)$ into two parts $(X, Y)$ such that no extra-complete team intersects both $X$ and $Y$, and such that both $w_0(X)$ and $w_0(Y)$ are at least $\frac{1}{55} \cdot w_t(T)$. Since each extra-complete teams is strongly complete to every other extra-complete team, $(X, Y)$ is a biclique, and its weight is at least $\frac{1}{55} \cdot w_0(T_0)$: the first item of the lemma holds. □

**LEMMA 3.36**

> *Let $(T, w)$ be a weighted trigraph such that $T$ is a basic trigraph and $w$ is balanced. Then $T$ admits a biclique or a complement biclique $(X, Y)$ of weight* $\min(w_r(X), w_r(Y)) \geqslant \frac{1}{55} \cdot w_t(T)$.

Before going to the proof, we need a technical lemma that will be useful to handle the line trigraph case. A graph $G$ has $m$ *multi-edges* if its set of edges $E$ is a multiset of $\binom{V(G)}{2}$ of size $m$: there can be several edges between two distinct vertices. An edge $uv$ has two *extremities* $u$ and $v$. The *degree* of $v \in V(G)$ is counted with multiplicity, that is to say $d(v) = |\{e \in E \mid v \text{ is an extremity of } e\}|$.

**LEMMA 3.37**

> *Let $G$ be a bipartite graph $(A, B)$ with $m$ multi-edges and with maximum degree less than $m/3$. There exist two subsets $E_1, E_2$ of edges of $G$ such that $|E_1|, |E_2| \geqslant m/48$ and if $e_1 \in E_1, e_2 \in E_2$ then $e_1$ and $e_2$ do not have a common extremity.*

*Proof.* If $m \leqslant 48$, it is enough to find two edges with no common extremity. Such two edges always exists since the maximum degree is bounded by $m/3$ so no vertex can be a common extremity to every edge. Otherwise, assume $m > 48$ and let us consider a random uniform partition $(U, U')$ of the vertices. For every set of two distinct edges $\{e_1, e_2\} \subseteq E$, consider the random variable defined by $X_{\{e_1,e_2\}} = 1$ if $e_1 \in \binom{U}{2}$ and $e_2 \in \binom{U'}{2}$ (or vice-versa, since $\{e_1, e_2\} = \{e_2, e_1\}$), and 0 otherwise. In the following, we write simply $X_{e_1,e_2}$ instead of $X_{\{e_1,e_2\}}$. If $e_1$ and $e_2$ have at least one common extremity, then $\mathbb{P}(X_{e_1,e_2} = 1) = 0$, otherwise $\mathbb{P}(X_{e_1,e_2} = 1) = 1/8$. We define the following:

$$\begin{aligned} p &= \{\{e_1, e_2\} \subseteq E \mid e_1 \text{ and } e_2 \text{ do not have a common extremity}\}, \\ p_A &= \{\{e_1, e_2\} \subseteq E \mid e_1 \text{ and } e_2 \text{ do not have a common extremity in } A\}, \text{ and} \\ q_A &= \{\{e_1, e_2\} \subseteq E \mid e_1 \text{ and } e_2 \text{ have a common extremity in } A\}. \end{aligned}$$

We define similarly $p_B$ and $q_B$. Assume that $p \geqslant \frac{1}{3}\binom{m}{2}$. Then

$$\mathbb{E}\left(\sum_{\{e_1,e_2\}\subseteq E} X_{e_1,e_2}\right) = \sum_{\{e_1,e_2\}\subseteq E} \mathbb{P}(X_{e_1,e_2} = 1) = \frac{p}{8} \geqslant \frac{1}{24}\binom{m}{2}.$$

Thus there exists a partition $(U, U')$ such that

$$\sum_{\{e_1,e_2\}\subseteq E} X_{e_1,e_2} \geqslant \frac{1}{24}\binom{m}{2}.$$

Let $E_1 = E \cap \binom{U}{2}$ and $E_2 = E \cap \binom{U'}{2}$. Then $|E_1|, |E_2| \geqslant m/48$, otherwise

$$\sum_{\{e_1,e_2\}\subseteq E} X_{e_1,e_2} = |E_1| \cdot |E_2| < \frac{m}{48} \cdot \left(1 - \frac{1}{48}\right) m \leqslant \frac{1}{24}\binom{m}{2},$$

a contradiction. So $E_1$ and $E_2$ satisfy the requirements of the lemma. We finally have to prove that $p \geqslant \frac{1}{3}\binom{m}{2}$. The intermediate key result is that $p_A \geqslant 2q_A$. Number the vertices of $A$ from 1 to $|A|$ and recall that $d(i)$ is the degree of $i$. Then $\sum_{i=1}^{|A|} d(i) = m$ and

$$2p_A = \left(\sum_{i=1}^{|A|} d(i)(m - d(i))\right) = \left(\left(\sum_{i=1}^{|A|} d(i)\right)^2 - \sum_{i=1}^{|A|} (d(i))^2\right) = \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{|A|} d(i)d(j)\right)$$

$$2q_A = \left(\sum_{i=1}^{|A|} d(i)(d(i) - 1)\right) = \left(\sum_{i=1}^{|A|} (d(i))^2 - m\right)$$

Consequently,

$$
\begin{aligned}
2p_A - (4q_A + 2m) &= \sum_{i=1}^{|A|} d(i) \left( \sum_{\substack{j=1 \\ j \neq i}}^{|A|} d(j) - 2d(i) \right) \\
&= \sum_{i=1}^{|A|} d(i) \left( \sum_{j=1}^{|A|} d(j) - 3d(i) \right) \\
&= \sum_{i=1}^{|A|} d(i) \left( m - 3d(i) \right)
\end{aligned}
$$

But for every $i$, $d(i) < m/3$ thus $m - 3d(i) \geqslant 0$. So $2p_A - (4q_A + 2m) \geqslant 0$ and thus $p_A \geqslant 2q_A$. But $p_A + q_A = \binom{m}{2}$ so $q_A \leqslant \frac{1}{3}\binom{m}{2}$. Similarly, $p_B \geqslant 2q_B$ and $q_B \leqslant \frac{1}{3}\binom{m}{2}$. Finally,

$$
p \geqslant \binom{m}{2} - q_A - q_B \geqslant \binom{m}{2} - \frac{2}{3}\binom{m}{2} \geqslant \frac{1}{3}\binom{m}{2} .
$$

□

We can now give the proof for the case of basic trigraphs.

*Proof of Lemma 3.36.* Let us transform the weight $w$ into a virgin weight $w_0$ defined as $w_0(v) = (w_r(v), 0, 0)$ for every vertex $v$ and $w_0(uv) = (0, 0)$ for every $uv \in \sigma(T)$. In other words, all the non-real weight is deleted. The fact that $w$ is balanced ensures that $w_c(T) + w_{\bar{c}}(T) \leqslant \frac{7}{55} \cdot w_t(T)$ so

$$
w_0(T) = w_t(T) - (w_c(T) + w_{\bar{c}}(T)) \geqslant \left(1 - \frac{7}{55}\right) w_t(T) .
$$

Now it is enough to find a biclique or a complement biclique in $(T, w_0)$ with weight at least $\frac{1}{48} \cdot w_0(T)$ since $\frac{1}{48} \cdot w_0(T) \geqslant \frac{1}{55} \cdot w_t(T)$. Observe that every vertex still has weight at most $\frac{1}{55} \cdot w_t(T) \leqslant \frac{1}{48} \cdot w_0(T)$.

If $T$ is a bipartite graph, then $V(T)$ can be partitioned into two strong stable sets. One of them has weight at least $\geqslant \frac{w_0(T)}{2} \geqslant \frac{1}{16} \cdot w_0(T)$. Moreover, each vertex has weight at most $\frac{1}{48} \cdot w_0(T)$ so we can split the stable set into two parts of weight each $\frac{1}{48} \cdot w_0(T)$.

If $T$ is a doubled trigraph, then observe that $V(T)$ can be partitioned into two strong stable sets (the first side of the good partition) and two strong cliques (the second side of the good partition). Hence, one of these strong stable sets or cliques has weight $\geqslant w_0(T)/4$, and, by the same argument as above, we can split it in order to obtain a biclique or a complement biclique of weigth $\geqslant \frac{1}{48} \cdot w_0(T)$.

It becomes more complicated if $T$ is a line trigraph. If there exists a clique $K$ of weight $\frac{1}{16} \cdot w_0(T)$, then it is a strong clique: indeed, by definition of a line trigraph, every clique of size at least three is a strong clique; moreover, a clique of size at most two has weight at most $\frac{1}{24} \cdot w_0(T)$. Then we can split $K$ as above and get a biclique of weight $\frac{1}{48} \cdot w_0(T)$.

Let $F$ be the full realization of $T$ (the graph obtained from $T$ by replacing every switchable pair by an edge). Observe that a complement biclique in $F$ is also a complement biclique in $T$. By definition of a line trigraph, $F$ is the line graph of a bipartite graph $G$. Instead of keeping positive integer weight on the edges of $G$, we transform $G$ into a multigraph $G'$ by changing each edge $uv$ of weight $s$ into $s$ edges $uv$. The inequality $w_0(K) \leqslant 1/16 \cdot w_0(T)$ for every clique $K$ of $T$ implies on that the maximum degree of a vertex of $G'$ is at most $1/16 \cdot w_0(T)$. Lemma 3.37 proves the existence of two subsets $E_1, E_2$ of edges of $G'$ such that $|E_1|, |E_2| \geqslant w(V)/48$ and if $e_1 \in E_1, e_2 \in E_2$ then $e_1$ and $e_2$ do not have a common extremity. This corresponds to a complement biclique in $F$ and thus in $T$ of weight $\geqslant \frac{1}{48} \cdot w_0(T) \geqslant \frac{1}{55} \cdot w_t(T)$. □

We can now prove the main theorem of this section:

**THEOREM 3.38**

*Let $T_0$ be a trigraph of $\mathcal{F}$ with no balanced skew-partition, equipped with a virgin balanced weight $w_0$. Then $T_0$ admits a biclique or a complement biclique of size at least $\frac{1}{55} \cdot w_0(T_0)$.*

In particular, this proves that the class of Berge graphs with no balanced skew-partition has the Strong Erdős-Hajnal property:

**COROLLARY 3.39**

*Let $T$ be a trigraph of $\mathcal{F}$ with no balanced skew-partition and $w_0$ be the virgin weight defined by $w_0(v) = (1, 0, 0)$ for every vertex $v \in V(T)$. If $|V(T)| \geqslant 3$ then $T$ admits a biclique or a complement biclique of size at least $|V(T)|/55$.*

*Proof.* If $|V(T)| \geqslant 55$, then $w_0$ is balanced and $w_0(T) = |V(T)|$, so we apply Theorem 3.38. Otherwise, since $|V(T)| \geqslant 3$ and $T \in \mathcal{F}$, $T$ contains at least one strong edge or one strong antiedge: this gives a biclique or a complement biclique of size 1, and $1 \geqslant \frac{1}{55} \cdot |V(T)|$. □

*Proof of Theorem 3.38.* Start with $(T, w) = (T_0, w_0)$ and contract $(T, w)$ while the following conditions hold:

*(i)* $T$ is not basic.

*(ii)* $T$ is a trigraph of $\mathcal{F}$ with no balanced skew-partition.

*(iii)* $(T, w)$ is a balanced model of $(T_0, w_0)$.

According to Lemma 3.35, if condition (ii) or (iii) does not hold anymore, there exists a biclique or a complement biclique in $(T_0, w_0)$ of weight $\geqslant \frac{1}{55} \cdot w_0(T_0)$, which concludes the proof.

Otherwise, conditions (ii) and (ii) but not (i) hold when we stop: $T$ is basic, $(T, w)$ is a model of $(T_0, w_0)$ and $w$ is balanced. By Lemma 3.36, $T$ admits a biclique or a complement biclique, say a biclique, of weight $\geqslant \frac{1}{55} \cdot w_t(T)$. This means that there exists a pair $(X, Y)$ of disjoint subsets of vertices of $T$ such that $w_r(X), w_r(Y) \geqslant \frac{1}{55} \cdot w_t(T)$ and $X$ is strongly complete to $Y$. Since $(T, w)$ is a model of $(T_0, w_0)$, we transform $(X, Y)$ into a biclique of large weight in $T_0$ as follows: let $\beta$ be the partition mapping for $(T, w)$ and let $X' = \beta_r(X) \subseteq V(T_0)$ and $Y' = \beta_r(Y) \subseteq V(T_0)$. By the *strong adjacency condition* in the definition of a model, $X'$ is strongly complete to $Y'$ in $T_0$ since $X$ is strongly complete to $Y$ in $T$. Moreover, by the *weight condition*, $w_0(\beta_r(X)) = w_r(X)$ and $w_0(\beta_r(Y)) = w_r(Y)$. But then $w_0(X'), w_0(Y') \geqslant \frac{1}{55} \cdot w_t(T) = \frac{1}{55} \cdot w_0(T_0)$, which concludes the proof. □

### In the closure $\overline{\mathcal{C}^{\leqslant k}}$ of $\mathcal{C}$ by generalized $k$-join

In fact, the method of contraction of a 2-join used in the previous subsection can easily be adapted to a generalized $k$-join. We only require that the basic class $\mathcal{C}$ of graphs is hereditary and has the Strong Erdős-Hajnal property. We invite the reader to refer to Subsection 3.5.3 for the definitions of a generalized $k$-join and the classes $\mathcal{C}^{\leqslant k}$ and $\overline{\mathcal{C}^{\leqslant k}}$. Things are even much easier than for Berge trigraphs with no balanced skew-partition because there is no problematic case such as the odd 2-join, where no vertex keeps track of the deleted $C_2$ part. Consequently, there is no need to introduce extra-complete and extra-anticomplete weight, and from now on, we simply work with non-negative integer weight on the vertices. A biclique (resp. complement biclique) in $T$ is still a pair $(X, Y)$ of subsets of vertices such that $X$ is strongly complete (resp. strongly anticomplete) to $Y$. Its weight is defined as $\min(w(X), w(Y))$.

We now define the contraction of a weighted trigraph $(T, w)$ containing a generalized $k$-join. As announced, it is much simpler than for the 2-join because the weight function only maps a non-negative integer weight to every vertex. Assume that $T$ is the generalized $k$-join of $T_1$ and $T_2$. We follow the notations introduced in the definition of the generalized $k$-join, in particular the vertex set $V(T)$ is partitioned into $(A_1, \ldots, A_r, B_1, \ldots, B_s)$. Without loss of generality, assume $w(\cup_{j=1}^r A_j) \geqslant w(\cup_{i=1}^s B_i)$. Then the *contraction* of $T$ is the weighted trigraph $(T', w')$ with $T' = T_1$ and $w'$ defined by $w'(v) = w(v)$ if $v \in \cup_{j=1}^r A_j$, and $w'(b_i) = w(B_i)$ for $1 \leqslant i \leqslant s$.

Finally, the definition of *model* is also much simpler in this setting. Indeed, given a weighted trigraph $(T_0, w_0)$, we say that $(T, w)$ is a model of $(T_0, w_0)$ if the following conditions hold:

- *the partition condition:* there exists a *partition mapping* $\beta$ which assigns to every vertex $v \in V(T)$ a subset $\beta(v) \subseteq V(T_0)$ of vertices of $T_0$, called the *team* of $v$. Moreover, any two teams are disjoint and the union of all teams is $V(T_0)$. Intuitively, the team of $v$ will contain all the vertices of $V(T_0)$ that have been contracted to $v$.

Similarly as before, for a subset $U \subseteq V(T)$ of vertices, we define

$$\beta(U) = \bigcup_{u \in U} \beta(u) .$$

- *the weight condition:* $w(T) = w_0(T_0)$ and for all $v \in V(T)$, $w(v) = w_0(\beta(v))$.
- *the strong adjacency condition:* if two vertices $u$ and $v$ are strongly adjacent in $T$, then $\beta(u)$ and $\beta(v)$ are strongly complete in $T_0$. If $u$ and $v$ are strongly antiadjacent in $T$, then $\beta(u)$ and $\beta(v)$ are strongly anticomplete in $T_0$.

Here are two last definitions before going to the proof. Given $0 < c < 1/2$ and a trigraph $T$, a weight function $w : V(T) \mapsto \mathbb{N}$ is *c-balanced* if for every vertex $v \in V(T)$, $w(v) \leqslant c \cdot w(T)$. A hereditary class $\mathcal{C}$ of graphs is said *c-good* if the following holds: for every $G \in \mathcal{C}$ with at least 2 vertices and for every $c$-balanced weight function $w$ on $V(G)$, $G$ admits a biclique or a complement biclique of weight $\geqslant c \cdot w(G)$. We are now ready to obtain the following result:

**THEOREM 3.40**

*Let $k \geqslant 1$, $0 < c < 1/2$ and assume that $\mathcal{C}$ is a ck-good class of graphs. Then for every $T_0 \in \overline{\mathcal{C}^{\leqslant k}}$ containing at least one strong edge or one strong antiedge, and for every c-balanced weight function $w_0$, the weighted trigraph $(T_0, w_0)$ has a biclique or a complement biclique of weight $c \cdot w_0(T_0)$.*

In particular:

**COROLLARY 3.41**

*Let $k \geqslant 1$, $0 < c < 1/2$ and $\mathcal{C}$ be a ck-good class of graphs. Then every weighted trigraph $(T_0, w_0)$ such that $w_0(v) = 1$ for every $v \in V(T_0)$ and $T_0 \in \overline{\mathcal{C}^{\leqslant k}}$ admits a biclique or a complement biclique of size $c \cdot |V(T_0)|$, provided $T_0$ has at least one strong edge or one strong antiedge.*

*Proof.* If $V(T_0) < 1/c$, then one strong edge or one strong antiedge is enough to form a biclique or a complement biclique of size $1 \geqslant c \cdot |V(T_0)|$. Otherwise, $w_0$ is $c$-balanced so we apply Theorem 3.40. □

To begin with, we need a counterpart of Lemma 3.35 to prove that the contraction of a model is still a model:

**LEMMA 3.42**

*Let $\mathcal{C}$ be a class of graphs, $k \geqslant 1$, and $0 < c < 1/2k$. Let $(T_0, w_0)$ be a weighted trigraph such that $T_0 \in \overline{\mathcal{C}^{\leqslant k}}$ and $w_0$ is c-balanced. Then if $(T, w)$ is a model of $(T_0, w_0)$ with $T \in \overline{\mathcal{C}^{\leqslant k}}$ but $T \notin \mathcal{C}^{\leqslant k}$ and if $w$ is c-balanced, one of the following holds:*

(i) *There exists a biclique or a complement biclique in $T_0$ of weight $\geqslant c \cdot w_0(T_0)$.*

(ii) *The contraction $(T', w')$ of $(T, w)$ is also a model of $(T_0, w_0)$. Furthermore, $T' \in \overline{\mathcal{C}^{\leqslant k}}$ and $w'$ is $c$-balanced.*

*Proof.* Since $T \notin \mathcal{C}^{\leqslant k}$, the trigraph $T$ is the generalized $k$-join between two trigraphs, say $T_1 = (A_1, \dots A_r, \{b_1, \dots, b_s\})$ and $T_2 = (\{a_1, \dots a_r\}, B_1, \dots, B_s)$ with $r, s \leqslant k$ (using the same notations as in the definition). Without loss of generality, we can assume that $w(\cup_{j=1}^r A_j) \geqslant w(\cup_{i=1}^s B_i)$. Since $r \leqslant k$, there exists $j_0$ such that $w(A_{j_0}) \geqslant \frac{1}{2k} \cdot w(T)$. Now if there exists $i_0$ such that $w(B_{i_0}) \geqslant c \cdot w(T)$, then $(A_{j_0}, B_{i_0})$ is a biclique or a complement biclique, by definition of a generalized $k$-join, and its weight is at least $c \cdot w(T) = c \cdot w_0(T_0)$, thus item (i) holds. Otherwise, the goal is to prove that the contraction $(T', w')$ of $(T, w)$ is also a model of $(T_0, w_0)$, where $T' = T_1 \in \overline{\mathcal{C}^{\leqslant k}}$ and $w'$ defined as above by $w'(v) = w(v)$ if $v \in \cup_{j=1}^r A_j$, and $w'(b_i) = w(B_i)$ for $1 \leqslant i \leqslant s$. Observe that $w'(T') = w(T)$ and that $w'$ is $c$-balanced. Moreover, let $\beta$ be the partition mapping witnessing $(T, w)$ being a model of $(T_0, w_0)$. We can easily see that $(T', w')$ is a model of $(T_0, w_0)$ by defining $\beta'(v) = \beta(v)$ if $v \in \cup_{j=1}^r A_j$, and $\beta'(b_i) = \beta(B_i)$ for every $1 \leqslant i \leqslant s$. We can check that all the conditions are ensured. This concludes the proof. $\square$

For the *basic* case, we need to adapt our assumption on $\mathcal{C}$ to make it work on $\mathcal{C}^{\leqslant k}$:

**LEMMA 3.43**

*Let $k \geqslant 1$, $0 < c < 1/2$ and $\mathcal{C}$ be a $ck$-good class of graphs. Let $(T, w)$ be a weighted trigraph such that $T \in \mathcal{C}^{\leqslant k}$, $w$ is $c$-balanced and $T$ contains at least one strong edge or one strong antiedge. Then $T$ admits a biclique or a complement biclique of weight $c \cdot w(T)$.*

*Proof.* For every switchable component of $T$, select the vertex with the biggest weight and delete the others. We obtain a graph $G \in \mathcal{C}$ and define $w_G(v) = w(v)$ on its vertices. Observe that $w_G(G) \geqslant w(T)/k$ since every switchable component has size at most $k$, and that $w_G(v) = w(v) \leqslant c \cdot w(T) \leqslant ck \cdot w_G(G)$ for every $v \in V(G)$. Moreover, $G$ has at least 2 vertices since $T$ has at least two different switchable components. Since $\mathcal{C}$ is $ck$-good, there exists a biclique or complement biclique$(V_1, V_2)$ in $G$ such that $w_G(V_1), w_G(V_2) \geqslant ck \cdot w(G)$. Then $(V_1, V_2)$ is also a biclique or complement biclique in $T$ with the same weight $\geqslant ck \cdot w_G(G) \geqslant c \cdot w(T)$ . $\square$

*Proof of Theorem 3.40.* Let $(T_0, w_0)$ be a weighted trigraph such that $T_0 \in \overline{\mathcal{C}^{\leqslant k}}$ has at least one strong edge or one strong antiedge, and such that $w_0$ is $c$-balanced. Start with $(T, w) = (T_0, w_0)$ and keep contracting $(T, w)$ while the following three conditions hold:

*(i)* $T \notin \mathcal{C}^{\leqslant k}$

*(ii)* $T \in \overline{\mathcal{C}^{\leqslant k}}$

*(iii)* $(T, w)$ is a model of $(T_0, w_0)$ and $w$ is $c$-balanced.

By Lemma 3.42, if condition (ii) or (iii) does not hold anymore, then $(T_0, w_0)$ has a biclique or a complement biclique of weight $c \cdot w_0(T_0)$. Otherwise, $T \in \mathcal{C}^{\leqslant k}$, $w$ is $c$-balanced and $(T, w)$ is a model of $(T_0, w_0)$. By definition of a contraction, $T$ has at least one strong edge or one strong antiedge. Since $\mathcal{C}$ is $ck$-good, apply Lemma 3.43 to get a biclique or complement biclique $(V_1, V_2)$ in $T$ of weight at least $c \cdot w(T)$. Let $\beta$ be a partition mapping that witness $(T, w)$ being a model of $(T_0, w_0)$. Then $(\beta(V_1), \beta(V_2))$ is a biclique or complement biclique in $T_0$ according to the strong adjacency condition. Moreover, by the weight condition,

$$\min(w_0(\beta(V_1)), w_0(\beta(V_2))) = \min(w(V_1), w(V_2)) \geqslant c \cdot w(T) = c \cdot w_0(T_0)\ .$$

This concludes the proof. $\square$

# Chapter 4

# Extended formulations

LET us now describe the initial motivation for Clique-Stable Set Separation. We somehow leave the setting of pure graph theory to enter the world of combinatorial optimization. The reader may not be familiar with this other field, for this reason we take a particular care to start from scratch, and recall basic notions on linear programming. Furthermore, this chapter does not contain any new result; it is fully devoted to giving all the necessary definitions and hopefully some intuition on the different objects and results that appeared in this prolific area since the middle of last century. This chapter is written keeping in mind our final goal, which is an understanding of the ins and the outs of the Clique-Stable Set Separation. However it is a long and windy road and, starting from the Maximum Weighted Stable Set problem, we will have to go through a lot of various concepts: polytopes and polyhedral combinatorics; then a study of the stable set polytope and its particular properties in bipartite graphs and perfect graphs; we then define extended formulations and extension complexity, which leads us to define the slack matrix and state the *Factorization Theorem*; we end up this survey by explaining the well-known links with communication complexity, which in particular gave birth to the Clique-Stable Set Separation.

We assume some basic knowledge in algebra. Let us recall the most useful definitions before starting: given $n$ vectors $x^{(1)}, \dots, x^{(n)} \in \mathbb{R}^n$ (also called *points*),

- a *linear combination* of them is obtained as $\sum_i \lambda_i x^{(i)}$ where $\lambda_i \in \mathbb{R}$ for all $i$,
- an *affine combination* of them is obtained as $\sum_i \lambda_i x^{(i)}$ where $\lambda_i \in \mathbb{R}$ for all $i$, and $\sum_i \lambda_i = 1$,
- a *convex combination* of them is obtained as $\sum_i \lambda_i x^{(i)}$ where $\lambda_i \geqslant 0$ for all $i$, and $\sum_i \lambda_i = 1$.

The set of all convex combinations (respectively affine combinations) of a set of points $X$ is called the *convex hull* (respectively *affine hull*) of $X$ and is denoted by $\mathrm{conv}(X)$ (respectively $\mathrm{aff}(X)$).

# 4.1 Linear Programming

Let us start with a short story: M. Yog owns a yogurt factory. He has two different yogurt recipes using three different ingredients in total: milk, sugar and strawberry, which are all delivered in packs. To make 1 kg of *standard yogurt*, he needs one pack of sugar, one pack of strawberry and two packs of milk. For 1 kg of *Superfruit yogurt*, he needs two packs of strawberry, one pack of milk and no sugar. The standard yogurt is sold 4€/kg, and the Superfruit yogurt is sold 5€/kg. M. Yog has just received his ingredient delivery and his stock currently contains 800 packs of strawberry, 700 packs of milk and 300 packs of sugar. He would like to know how to use his stock in order to earn the maximum amount of money.

M. Yog was a good student at maths so he decides to use *variables* $x_1$ and $x_2$ to denote the respective quantities of standard yogurt and Superfruit yogurt that he should make. Then the recipes impose that:

$$
\begin{array}{rcll}
2x_1 + \phantom{2}x_2 & \leqslant 700 & & \text{for the constraints imposed on the milk quantity,} \\
x_1 + 2x_2 & \leqslant 800 & & \text{for the constraints imposed on the strawberry quantity,} \\
x_1 \phantom{+2x_2} & \leqslant 300 & & \text{for the constraints imposed on the sugar quantity.}
\end{array}
$$

In addition to that, he obviously wants a non-negative quantity of each, so he sets $x_1, x_2 \geqslant 0$. Finally the goal is to maximize the income, which is given by

$$\textbf{Objective function:} \quad \text{maximize } 4x_1 + 5x_2 \,.$$

He has thus modeled his problem as a *linear program*.

More generally, a *Linear Program* (*LP* for short) is given by a set $\{x_1, \ldots, x_d\}$ of variables (often written as a column vector $x \in \mathbb{R}^d$) to which we must assign some real values such that:

- the variables must satisfy some linear *constraints*, expressed by a linear system $Ax \leqslant b$ where $A$ is an $m \times d$ matrix and $b \in \mathbb{R}^d$.
- the goal is to minimize or maximize an *objective function*, which is a linear function of the variables written $w^\intercal x$ with $w \in \mathbb{R}^d$.

A vector that satisfies all the constraints is called a *feasible solution*, and the goal is thus to optimize the objective function over the set of feasible solutions. Linear programs are a major tool in operational research because on the one hand, it can model many real-life problems and on the other hand, there exist efficient algorithms to find an optimal solution (discussed in Subsection 4.3).

Although the word *program* can mislead us, a linear program is not a piece of code written in a programming language such as C, C++, Java, ... In fact, solving a system of linear inequalities is a problem much prior to the appearance of the first computers, since Fourier already published in 1827 a method for deciding the existence of a solution (the Fourier-Motzkin elimination, [87, 140]). The first linear program formulation in the general setting is due to Kantorovich [130] in 1939, for military applications. He developed during World War II another method for solving it. In fact, World War II seems to have been a catalyst in terms of finding common frameworks to efficiently express and solve a whole range of problems. In particular, linear programming also appeared independently in Koopmans' work in economics [141], for which he shared the 1975 Nobel prize in economics with Kantorovich, and it also appeared in Hitchcock's work [119] for transportation problems. After the war, Dantzig published [60] in 1947 a method for solving LPs called the *simplex method* (still valuable nowadays because efficient in practice and rather easy to understand). The field was renewed by von Neumann in 1948 (his works are collected in [208]) with his groundbreaking theory of duality, which we will not investigate here. The community had to wait for a few decades until the first polynomial-time algorithm was found by Khachiyan [135] in 1979, now known as the *ellipsoid algorithm*. This algorithm was however lacking practical efficiency and it was followed in 1984 by a great theoretical and practical breakthrough by Karmakar [131] who introduced the *interior-point method* for solving linear programs.

Although many problems can be expressed as a linear program, most combinatorial problems do not fit in this setting because they require the variables to be assigned integer values. These can be expressed as a variant of a linear program, called an *integer linear program* (*ILP* or *IP* for short): the model is the same as a linear program, except that the values assigned to the variables must be taken in $\mathbb{Z}$ (and most of the time, even in $\{0,1\}$). Let us illustrate this on two well-known problems, the *Maximum Weighted Stable Set* and the *Traveling Salesman Problem*. The former will be of particular interest in the following, and the latter has a strong historical role.

**Problem 1: Maximum Weighted Stable Set** (**MWSS** for short) (see *e.g.* [182])
*Instance:* A graph $G = (V, E)$ and a weight function $w : V \to \mathbb{R}$.
*Goal:* Find a stable set of maximum weight.

The weight of a subset $V' \subseteq V$ is defined as usual by $w(V') = \sum_{v \in V'} w(v)$. This optimization problem can be modeled with the IP described below, on variable $x \in \mathbb{R}^{|V|}$, where each coordinate $x_v$ is associated to a vertex $v \in V$. Moreover, the weight function $w$ is seen as a column vector in $\mathbb{R}^{|V|}$.

**Objective function:** maximize $w^\intercal x$

Subject to the following constraints:

$$\begin{aligned} x_u + x_v \leqslant 1 \quad &\text{for every } uv \in E \qquad &&\text{(called the \textit{edge constraints})} \\ x_v \in \{0,1\} \quad &\text{for every } v \in V \qquad &&\text{(called the \textit{integrality contraints})} \end{aligned}$$

The integrality constraints express that there are only two options for every vertex $v$: either $v$ is taken in the candidate solution $x$ ($x_v = 1$) and then its weight $w_v$ is added, or not ($x_v = 0$ and then $w_v x_v = 0$). This is because of this binary alternative that we use an IP and not a LP. The edge constraints express the fact that, for every edge, one can take at most one of its two endpoints to ensure that the selected subset of vertices forms a stable set.

There is a one-to-one correspondence between subsets $V' \subseteq V$ and 0/1 vectors $x \in \{0,1\}^{|V|}$ given by the *characteristic vector* $\chi^{V'}$ of $V'$: number the vertices of $V$ with $\{1, \dots, |V|\}$, then the $i$-th coordinate of the vector $\chi^{V'} \in \{0,1\}^{|V|}$ is equal to 1 if $i \in V'$, and 0 otherwise. It is easy to see that for every stable set $S$, $\chi^S$ is a solution to the above system and $w(S) = w^\intercal \chi^S$. Conversely, any solution $x$ to the above system is the characteristic vector of a stable set $S$ of $G$ (ensured by the edge constraints) whose weight is $w^\intercal x$. This proves that the IP formulation indeed models the MWSS problem.

**Problem 2: Traveling Salesman Problem** (**TSP** for short)

The original problem comes from the following question: imagine that a Traveling Salesman wants to visit every city of his country to sell his goods. What is the shortest tour to go through every city exactly once and return to his own city? More formally, the problem is described as follows:

*Instance:* A positive integer $n$ and a weight function $w : E(K_n) \to \mathbb{R}$ on the edges of the complete graph $K_n = (V, E)$.

*Goal:* Find a *tour* of minimum length, *i.e.* a subset of edges that form a cycle going through every vertex exactly once - also called a *Hamiltonian cycle*- whose length is defined as the sum of the weight of its edges.

Let us number the vertices with $\{1, \dots, n\}$. For convenience, we consider $K_n$ as a directed graph containing every arc $(i,j)$ with $i \neq j$ (every edge is transformed into two arcs in opposite direction). In the following, we write $\vec{ij}$ instead of $(i,j)$ to emphasize the difference between $\vec{ij}$ and $\vec{ji}$. Let us denote by $A$ the set of arcs. The TSP can be model by an IP on variable vector $(x,u) \in \mathbb{R}^{|A|+|V|}$. Each of the $|A|$ first coordinates $x_{\vec{ij}}$ of $x$ is a 0/1-variable associated to the arc $\vec{ij}$; intuitively, it must translate the fact that the arc $\vec{ij}$ is selected ($x_{\vec{ij}} = 1$) or not ($x_{\vec{ij}} = 0$). Each of the $|V|$ last coordinates $u_i$ is an integer variable that must indicate that $i$ is the $u_i$-th vertex to be visited on the cycle when starting at vertex 1. The weight function $w$ is once again seen as a column vector in $\mathbb{R}^{|A|}$ (the weight of an edge $ij$ is given to both arcs $\vec{ij}$ and $\vec{ji}$). Let us now write the IP [159]:

**Objective function:** minimize $w^\intercal x$

Subject to the following constraints:

$$\sum_{j \in V \setminus i} x_{\vec{ij}} = 1 \quad \text{for every } i \in V \qquad \text{(called the \textit{out constraints})}$$

$$\sum_{j \in V \setminus i} x_{\vec{ji}} = 1 \quad \text{for every } i \in V \qquad \text{(called the \textit{in constraints})}$$

$$u_i - u_j + n x_{\vec{ij}} \leqslant n - 1 \quad \text{for every } \vec{ij} \in A,\ i,j \neq 1 \qquad \text{(called the \textit{unique cycle constr.})}$$

$$x_{\vec{ij}} \in \{0,1\} \quad \text{for every } \vec{ij} \in A \qquad \text{(called the \textit{integrality constr.})}$$

$$u_i \in \mathbb{Z} \quad \text{for every } i \in V$$

Let us check that a solution of the IP gives an Hamiltonian cycle and vice-versa. More precisely, we need to show that a subset $A'$ of arcs forms an oriented Hamiltonian cycle with $w(A') = r$ if and only if there exists a vector $\mu \in \mathbb{R}^n$ (an assignment of the dummy variables $u_i$) such that the vector $y$ defined as $y = \begin{pmatrix} \chi^{A'} \\ \mu \end{pmatrix} \in \mathbb{R}^{|A|+|V|}$, where $\chi^{A'} \in \{0,1\}^{|A|}$ is the characteristic vector of $A'$, satisfies the above IP and $w^\intercal \chi^{A'} = r$. Let $y = \begin{pmatrix} x \\ u \end{pmatrix}$ be a solution of the above system. The integrality constraints ensures that $x$ is a $0/1$-vector and thus $x$ is the characteristic vector of a subset $A'$ of arcs, for which $w^\intercal x = w(A')$. The *in* (resp. *out*) constraints ensure that exactly one arc of $A'$ goes in (resp. out of) each vertex $i \in V$, so $A'$ is a vertex-disjoint union of oriented cycles that covers all the vertices. There remains to prove that it contains in fact a unique cycle, which we achieve by showing that no cycle can avoid vertex 1. Assume by contradiction that there is a cycle $C \subseteq A'$ not passing through vertex 1. Summing all the *unique cycle constraints* for $\vec{ij} \in C$ (possible because $C$ does not go through vertex 1) gives:

$$\sum_{\vec{ij} \in C} u_i - u_j + n \underbrace{x_{\vec{ij}}}_{=1} \leqslant \sum_{\vec{ij} \in C} (n-1)$$
$$\text{thus} \quad n|C| \leqslant (n-1)|C|, \quad \text{a contradiction.}$$

Conversely, let $C$ be a Hamiltonian cycle in $K_n$, then we arbitrarily choose an orientation of the cycle to get an oriented Hamiltonian cycle $A'$. We set $\mu_i = t$ if vertex $i$ is visited as the $t$-th vertex on the oriented cycle, starting at vertex 1 (in particular $\mu_1 = 1$). Then the vector $y = \begin{pmatrix} \chi^{A'} \\ \mu \end{pmatrix}$ is a solution of the system: trivially, the *in* and the *out* constraints are satisfied, as well as the integrality constraints. As for the *unique cycle constraints*, we can observe that if $x_{\vec{ij}} = 0$ then $u_i - u_j \leqslant n-1$ because $u_i \leqslant n$ and $u_j \geqslant 1$. Moreover, if $x_{\vec{ij}} = 1$ and $j \neq 1$ then $u_j = u_i + 1$, so $u_i - u_j + n x_{\vec{ij}} = n-1$. Consequently, all the constraints are satisfied and $w(A') = w^\intercal \chi^{A'}$.

Presented in such a way, the reader may be tempted to assume that the triangle inequality holds (i.e. $w(\vec{ij}) + w(\vec{jk}) \leqslant w(\vec{ik})$ for every $i,j,k \in V$), but in fact some other real-life problems can be modeled as a TSP where the triangle inequality does not hold. In both cases, it is well-known that the problem is NP-hard[1][99].

We can infer from the two above modelizations that solving an IP is a NP-hard problem. To get around this difficulty, one can try to remove the integrality constraints and replace the $x_i \in \{0,1\}$ constraints by the linear ones $0 \leqslant x_i \leqslant 1$ for every variable $x_i$ (for convenience, we will call such a constraint a *trivial constraint*). The

[1] However, the triangle inequality makes a difference for approximation: when it holds, Christofides' algorithm gives a 3/2-approximation in poly-time, whereas the general case is NPO-complete, which basically means that there is no non-trivial approximation.

LP obtained in this way is called the *LP-relaxation* of the original IP. One could think of using a polynomial-time algorithm to solve this LP-relaxation and hope that the optimal value of the LP-relaxation and the optimal value of the original IP are equal, or at least reasonably close. However, the gap between the two of them can be arbitrarily large. Consider for example the MWSS problem on the complete graph $K_n$ with weight $w$ identically equal to 1. Then the optimal value of the IP is of course 1, because singletons are the only stable sets. However, the LP-relaxation has optimal value $n/2$ since we can set $x_v = 1/2$ for every vertex $v$ and still satisfy all the edge constraints and the trivial constraints. Nonetheless, for some problems this method can give approximate solutions up to a constant factor, for instance Vertex Cover[2] admits such a 2-approximation algorithm (see *e.g.* [99]).

In order to get rid of these undesirable solutions, one could think of adding extra constraints to the IP to make the LP-relaxation more *precise*, in some sense. As suggested by the $u_i$ variable in the TSP example, we can also try to take advantage of new variables, enabling us to formulate more constraints. In this perspective, in 1986-87 there were attempts to prove P=NP by giving a polynomial-size LP that would solve the TSP [193]. Each of them had to be carefully read by the experts to find a mistake, and this was very time-consuming for sophisticated LPs. Yannakakis tried to find a meta-argument that would refute all such attempts: he proved that every *symmetric* LP for the TSP has exponential size [211]. *Symmetric* means that every permutation of the cities can be turned into a permutation of the variables of the LP that preserves the constraints. Since all the LPs given so far for TSP were symmetric, he managed to reach his goal. However, he left as a big open question the case of asymmetric LPs. Fiorini, Massar, Pokutta, Tiwary and de Wolf proved [83], more than 20 years later, that every *extended formulation*, *i.e.* every LP expressing the TSP, symmetric or not, has a superpolynomial size.

## 4.2 Polytopes

Before explaining what is an extended formulation, we provide here some definitions about polytopes (see Schrijver's book for more details [182]). Let us first describe the connection between polytopes and IPs. First, keep in mind that we are interested in better understanding the set of feasible solutions. Observe now that a convex combination of feasible solutions is a vector that satisfies all the linear constraints of the IP, but obviously not always the integrality constraints. We can afford to consider these convex combinations as somehow admissible solutions because none of them can beat the objective function: the optimal value of the linear objective function over the set $P$ of convex combinations of feasible solutions is equal to the optimal value over only the set of feasible solutions. Thus it makes sense to optimize over $P$. This can be viewed in a geometrical point of view by taking the convex hull $P \subseteq \mathbb{R}^d$ of

[2] A vertex cover is a subset of vertices which contains at least one endpoint of every edge. The goal of Vertex Cover problem is to minimize the weight of such a subset.

the set $\mathcal{S}$ of feasible solutions to the IP. If $\mathcal{S}$ is finite (which is the case in the classical restriction of IP to binary variables, i.e. $x \in \{0,1\}^d$ instead of $x \in \mathbb{Z}^d$), $P$ is the convex hull of a finite set of points in $\mathbb{R}^d$, which is the definition of a *polytope*.

Let us illustrate this on an example: let us consider the Maximum Weighted Stable Set problem on a graph $G$ and construct the associated polytope called the *stable set polytope* (or *vertex packing polytope* in some older literature). It is denoted $\textsc{Stab}(G)$ and defined as (see *e.g.* [111]):

$$\textsc{Stab}(G) = \operatorname{conv}(\{\chi^S \in \{0,1\}^{|V|} \mid S \text{ is a stable set of } G\})\ .$$

This polytope is a *0/1-polytope*, which means that it is the convex hull of a set of points that all belong to $\{0,1\}^d$. An equivalent definition of a polytope is a set $P \subseteq \mathbb{R}^d$ that is the intersection of a finite collection of closed halfspaces, and that is bounded. Since every closed halfspace can be expressed by a linear inequality, it is also equivalent to saying that $P$ is bounded and is the set of solutions of a finite system of linear inequalities and possibly equalities (each of which can be represented by a pair of inequalities). If we drop the boundedness condition, it is called a *polyhedron*. Given a polyhedron $P$, a linear system $Ax \leqslant b$ for which $P = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$ is said to *express*[3] $P$. Given a polytope $P \subseteq \mathbb{R}^d$, a closed halfspace $H^+$ that contains $P$ is said to be *valid* for $P$. In this case, the hyperplane $H$ that bounds $H^+$ is also said to be *valid* for $P$. A *face* of $P$ is either $P$ itself or the intersection of $P$ with a valid hyperplane. Every face of a polytope is again a polytope (it can be expressed by the original linear program with one additional equality coming from the hyperplane description). A face is called *proper* if it is not the polytope itself. A *vertex* is a minimal non-empty face, and in the case of $\textsc{Stab}(G)$, it stands for (the characteristic vector of[4]) a stable set of $G$. A *facet* is a maximal proper face. An inequality $a^\intercal x \leqslant \delta$ is said to be *valid* for $P$ if it is satisfied by all points of $P$. The face it defines is $F = \{x \in P \mid a^\intercal x = \delta\}$. The inequality is called *facet-defining* if $F$ is a facet. If $P$ is expressed by the set of inequalities $Ax \leqslant b$, then for every face there exists a set $I$ of inequalities of $Ax \leqslant b$ such that the face can be expressed as the set of points $x$ for which $Ax \leqslant b$ and the inequalities from $I$ are tight[5].

The *dimension* of a polytope $P$ is the dimension of its affine hull, that is to say the maximum number of affinely independent points in $P$ minus 1. If the dimension of $P \subseteq \mathbb{R}^d$ is $d$, then $P$ is *full-dimensional*. In particular, $\textsc{Stab}(G)$ is full-dimensional because it contains the $|V(G)|$ vectors $\chi^{\{v\}}$ for every $v \in V(G)$ (singletons are stable sets), and the zero vector $\chi^{\emptyset}$. If $P$ is full-dimensional, then every (finite) system $Ax \leqslant b$ such that $P = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$ contains all the facet-defining inequalities of $P$, up to scaling by positive numbers. Conversely, $P$ is expressed by its facet-defining inequalities.

Optimization on a polytope $P$ can now be seen as follows: let $w^\intercal x$ be the objective

[3] *define* or *describe* are also used in the literature.

[4] From now on, we may make a slight abuse of notation: a subset $V' \subseteq V$ refers both to the subset itself and to its characteristic vector $\chi^{V'} \in \{0,1\}^{|V|}$, depending on the context.

[5] An inequality is *tight* on a point $x$ if the equality holds.

function we want to minimize or maximize. The goal is to push the hyperplane $H$ whose normal vector is $w$, in the direction of $w$ (to maximize the objective function) or in the opposite direction (to minimize the objective function), as much as we can until we reach the extreme position of the hyperplane that gives a non-empty intersection with $P$. It is rather easy to convince oneself that, in this extreme position, the intersection between $P$ and $H$ contains a vertex of the polytope (otherwise, we can push the hyperplane a little bit further). If $P$ is the convex hull of feasible solutions of an IP, this means that there always exists an integral solution that hits the optimal value of the objective function over $P$.

# 4.3 Algorithms for LP

Let us briefly mention the three main algorithms to solve LPs, although we are not entering into much details.

**The simplex algorithm (Dantzig 1947 [60])** The algorithm makes in fact no effective use of a *simplex* (at least in its most common form). However, it consists on a walk on the vertices of the polytope along the edges, which suggests the concept of a simplex[6]. Let $P$ be the polyhedron given by the set of feasible solutions of a LP. The initialization of the algorithm consists in computing a vertex of $P$. Then, the algorithm iterates a procedure which can be informally summed up as follows: find an edge of $P$ whose direction improves the objective function. If there is none, then the current vertex of $P$ is an optimal solution and we stop. Otherwise, move along the edge until reaching a vertex of $P$, which thus has a better value on the objective function, and repeat from this new vertex. If the edge is infinite, answer that the problem is unbounded. The algorithm terminates in finite time because it visits each vertex of the polytope at most once (except in some degenerated cases), however the worst-case complexity is exponential [202]. The reason why this algorithm still has a great importance is that its running time is quite efficient in practice (see [188]). In fact, the average-case complexity is polynomial [14], and if we apply a small perturbation on the input, the worst-case complexity also gets down to polynomial-time [192].

**Ellipsoid algorithm (Khachiyan 1979 [135])** This was the first polynomial-time algorithm for solving LPs, which was a great theoretical breakthrough. Inspired by previous work from Shor in 1970 [191], and Nemirovski and Yudin [212] in 1976, the algorithm is rather complicated and, unfortunately, it is much less efficient in practice than the simplex algorithm. However, besides its historical impact, it still has a great importance for the following reason: many optimization problems (*e.g.* graph problems) have an exponential number of constraints (*e.g.* in the number of vertices of the graph), consequently we cannot afford to list them all. The ellipsoid algorithm does not need an explicit listing of all the constraints, and can work with an implicit formulation of the constraint called a *separation oracle* which is an algorithm that, given a point $x$, can either ensures that $x$ is a feasible solution or can exhibit a

[6] Historical note: this name was given by Motzkin and not by Dantzig himself.

violated constraint. It is far from easy to find an efficient separation oracle in general, but this enabled Grötchel, Lovász and Schrijver [110] to solve in 1988 the MWSS in polynomial time in perfect graphs (see Subsection 4.5 for more details).

**Interior-point method (Karmarkar 1984 [131])** The simplex method is efficient in practice but not in theory, and the ellipsoid algorithm is efficient in theory but not in practice. Karmarkar's algorithm reconciles both sides: it is fast both in practice and in theory. Its complexity is polynomial in the number of variables and the size of the input. The idea is, contrary to the simplex algorithm, not to follow the boundary of the polytope $P$, but to move through the interior of $P$ along a direction that improves the objective function. Besides, it is simpler to describe than the ellipsoid algorithm. However, it needs explicit constraints and cannot be used with a separation oracle, contrary to the ellipsoid algorithm. Several other algorithms use the interior-point method[7] (path-following approach [107], potential reduction method [197]).

# 4.4 Total unimodularity: when IP meets LP

Since we have polynomial-time algorithms to solve LPs, we would like to use the LP-relaxation to solve an IP problem. As previously observed, in the general case the LP-relaxation can admit an optimal solution that is better than all the feasible solutions of the IP. This is the case for the MWSS problem on the complete graph $K_n$ with weight identically equal to 1. The optimal solution of the IP is of course 1, whereas the LP-relaxation has optimal value $n/2$ (given by the assignment $x_v = 1/2$ for every $v \in V$).

However, we can wonder whether the LP-relaxation gives the right answer for *some* instances. The answer is *yes*, and in particular this the case for the MWSS problem on bipartite graphs, which we are going to investigate. Is there a general setting that ensures that the LP-relaxation is good? To answer this question, we introduce the definition of a *totally unimodular* matrix: it is a matrix $A$ in which each square submatrix has determinant belonging to $\{-1, 0, 1\}$. In particular, each entry of $A$ must be -1, 0 or 1. The link between this algebraic property and linear programming was showed by Hoffman and Kruskal:

**THEOREM 4.1** [121]

*Let $A \in \mathbb{R}_{m\times d}$ be a totally unimodular matrix, and $b \in \mathbb{Z}^m$. Let*

$$P = \{x \in \mathbb{Z}^d \mid Ax \leqslant b\} \text{ and } Q = \operatorname{conv}(\{x \in \mathbb{R}^d | Ax \leqslant b\}) .$$

*Assume moreover that $P$ is bounded, then $P = Q$,* i.e. *the LP-relaxation gives the same polytope as the original IP.*

[7] Sometimes even the ellipsoid algorithm is included in the interior-point algorithms, since it does not follow the boundary of the polytope. However the methods are quite different.

It turns out that bipartite graphs fit into this setting for MWSS, let us show it. For convenience, let us define

$$\text{ESTAB}(G) = \left\{ x \in \mathbb{R}^{|V(G)|} \;\middle|\; \begin{array}{l} x \text{ satisfies the edge constraints} \\ \text{and } 0 \leqslant x_v \leqslant 1 \text{ for every } v \in V(G)\,. \end{array} \right\}.$$

The incidence matrix of a graph $G = (V, E)$ is the $|V| \times |E|$ matrix $\text{INC}(G)$ where each row stands for a vertex $v$, each column stands for an edge $e$, and $\text{INC}(G)_{v,e} = 1$ if $v$ is an endpoint of $e$, 0 otherwise. In particular, the $v$-row contains exactly $d(v)$ 1's, and each column contains exactly two 1's. The linear system expressing the edge constraints and the trivial constraints of the MWSS problem for $G$ is given by:

$$\begin{pmatrix} \text{INC}(G)^\intercal \\ I_n \\ -I_n \end{pmatrix} \cdot x \leqslant \begin{pmatrix} 1_{|E|} \\ 1_{|V|} \\ 0_{|V|} \end{pmatrix}$$

where $n = |V|$, $I_n$ is the $n \times n$ identity matrix, $1_r$ (resp $0_r$) is the column vector containing $r$ 1's (resp. $r$ 0's) for any integer $r$, and $x \in \mathbb{R}^n$.

If $G$ is bipartite with bipartition $V_1 \uplus V_2$, we can easily prove by induction on $k$ that every $k \times k$ submatrix of $\text{INC}(G)$ has determinant equal to $-1$, 0 or 1: indeed the only interesting case is when the submatrix $M$ contains exactly two 1-entries per column (otherwise we can develop along some column and apply the induction hypothesis). Let $v_1 \in V_1$ and replace the row $M_{v_1}$ by the combination $\sum_{v \in V_1} M_v - \sum_{v \in V_2} M_v$. This gives a row with only 0-entries and thus $\det(M) = 0$. Hence the incidence matrix of a bipartite graph is totally unimodular, and we thus obtain the following:

**COROLLARY 4.2** (see *e.g.* [183])

$\text{ESTAB}(G) = \text{STAB}(G)$ *if and only if $G$ is bipartite.*

*Proof.* The *if* part is given by Theorem 4.1.

The *only if* part can be proved as follows: assume that $G = (V, E)$ is not bipartite, then there exists an odd cycle $C$. Let $x \in \mathbb{R}^{|V|}$ be the vector that gives to every vertex of $C$ the weight $1/2$, and 0 to every other vertex. Then $x$ satisfies all the edge constraints and all the trivial constraints. However, $x$ is not a convex combination of stable sets of $G$: assume by contradiction that $x$ can be written as

$$x = \sum_{S \in \mathcal{S}} \lambda_S \chi^S \text{ with } \sum_{S \in \mathcal{S}} \lambda_S = 1 \text{ and } \lambda_S \geqslant 0 \text{ for every } S \in \mathcal{S}\,,$$

where $\mathcal{S}$ denotes the set of all stable sets of $G$. Let $b = (1, \ldots, 1)^\intercal \in \mathbb{R}^{|V|}$.

$$\text{On the one hand} \quad x^\intercal b = \sum_{v \in V(G)} x_v = \frac{|C|}{2}\,,$$

$$\text{on the other hand} \quad x^\intercal b = \sum_{S \in \mathcal{S}} \lambda_S (\chi^S)^\intercal b = \sum_{\substack{S \in \mathcal{S} \\ \lambda_S \neq 0}} \lambda_S |S| \leqslant \sum_{S \in \mathcal{S}} \lambda_S \frac{|C| - 1}{2}\,.$$

The last inequality is obtained by observing that every stable set $S$ having a positive coefficient $\lambda_S$ must be included in $C$, and that none of them can contain more than $\lfloor |C|/2 \rfloor$ vertices of the odd cycle $C$. Moreover $\sum_{S \in \mathcal{S}} \lambda_S = 1$ which yields a contradiction about the value of $x^\intercal b$. We have thus found a vector $x$ which is in $\mathrm{ES{\small TAB}}(G)$ but not in $\mathrm{S{\small TAB}}(G)$. □

A characterization of totally unimodular matrices can be found in [182], as well as some examples of such matrices, including network matrices.

# 4.5 STAB(G) in perfect graphs

We have seen in the previous subsection that the edge constraints together with the trivial constraints are not enough to express $\mathrm{S{\small TAB}}(G)$ if $G$ is not bipartite, or equivalently that $\mathrm{ES{\small TAB}}(G)$ is too large compared to $\mathrm{S{\small TAB}}(G)$. We must add some more valid inequalities to our LP-relaxation in order to describe more precisely $\mathrm{S{\small TAB}}(G)$.

First of all, the proof of Corollary 4.2 suggests that we must control the behavior on the odd cycles. For this, we add the *odd cycle constraints*:

$$\sum_{v \in C} x_v \leqslant \frac{|C|-1}{2} \quad \text{for every odd cycle } C\,.$$

Chvátal [48] studied the graphs for which these odd cycle constraints, together with the trivial constraints and the edge constraints, express $\mathrm{S{\small TAB}}(G)$: such graphs are now called *$t$-perfect* graphs. Yannakakis proved [211] that we can express $\mathrm{S{\small TAB}}(G)$ with a polynomial-size LP in this case, by adding some dummy variables. The class of $t$-perfect graph contains all the bipartite graphs (of course), the odd holes, the series-parallel graphs[8] [18], the almost bipartite graphs (Fonlupt and Urhy, 1982 [85]), and the strongly $t$-perfect graphs (Gerards and Schrijver 1986 [102]). However, even some small and well-structured graphs, for instance $K_4$, are not $t$-perfect.

The case of complete graphs can easily be fixed by considering the *clique constraints*:

$$\sum_{v \in K} x_v \leqslant 1 \quad \text{for every clique } K \subseteq V(G)\,.$$

Even though there is no complete description so far of a linear system that would express $\mathrm{S{\small TAB}}(G)$ in the general case, this set of constraints appears to be of major importance, as we will exhibit in the remainder of this section. First observe that it contains the edge constraints (for cliques of size 2) and one part of the trivial constraint ($x_v \leqslant 1$ for every clique $\{v\}$). It is easy to see that all the clique inequalities

[8] Series-parallel graphs is a class of graphs with two distinguished vertices, a source $s$ and a target $t$, which contains $K_2$ and is closed under series composition of two such graphs $G_1, G_2$ (consists in merging the source of $G_1$ with the target of $G_2$) and parallel composition (consists in merging both sources together and both targets together).

are indeed valid since a clique can intersect a stable set on at most one vertex. However, there are an exponential number of them in general, so unfortunately, even if it expressed $\mathrm{STAB}(G)$, it would not immediately give a polynomial-time algorithm for MWSS.

Let us study the polytope of feasible solutions for this new LP:

$$\mathrm{QSTAB}(G) = \left\{ x \in \mathbb{R}^{|V(G)|} \;\middle|\; \begin{array}{l} x \text{ satisfies the clique constraints} \\ \text{and } x_v \geqslant 0 \text{ for every } v \in V(G)\,. \end{array} \right\}.$$

In fact, we only have to focus on (inclusion-wise) maximal cliques:

**LEMMA 4.3** [164]

*The clique constraint associated with clique $K$ defines a facet of* $\mathrm{STAB}(G)$ *is and only if $K$ is a maximal clique of $G$.*

We trivially have $\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G) \subseteq \mathrm{ESTAB}(G)$. We have seen that, in bipartite graphs, $\mathrm{ESTAB}(G) = \mathrm{STAB}(G)$ and thus $\mathrm{QSTAB}(G) = \mathrm{STAB}(G)$. It seems reasonable to hope that in some more cases, we could have only the latter equality $\mathrm{QSTAB}(G) = \mathrm{STAB}(G)$. Let us investigate on those cases. Theorem 4.1 implies that $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ if the matrix expressing the clique constraints is totally unimodular. Such a graph is called *unimodular*. In addition to bipartite graphs, we can observe that line graphs of bipartite graphs are also unimodular: indeed, the matrix expressing the clique constraints of $G$ is in fact the incidence matrix of the root bipartite graph $R(G)$, which is totally unimodular, as already observed. The evocation of those two classes may ring a bell about the basic classes for Berge graphs decomposition theorem (Theorem 1.6). Indeed, $\mathrm{QSTAB}(G)$ has a very good behavior in Berge graphs: it is equal to $\mathrm{STAB}(G)$. More precisely, the statement holds for perfect graphs and was much prior to the Strong Perfect Graph Theorem (Theorem 1.5) which proves that those two classes are equal. Our goal is now to understand why $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$ for perfect graphs.

Let us introduce some notations about the weighted version of the clique number and of the chromatic number: given a graph $G$ and a weight function $w : V \to \mathbb{N}$, we denote by $\omega(G, w)$ the *weighted clique number*, *i.e.* the maximum weight of a clique of $G$. The *weighted chromatic number* $\chi(G, w)$ is the least number of stables sets of $G$ such that each vertex $v$ is covered by exactly $w(v)$ stable sets. If $w$ is identically equal to 1, then $\omega(G, w)$ is the classical clique number $\omega(G)$ and $\chi(G, w)$ is the classical chromatic number $\chi(G)$. The following theorem comes from several distinct contributions:

**THEOREM 4.4** [150, 93, 48, 164]

*For any graph $G = (V, E)$, the following conditions are equivalent:*

(i) $\omega(G') = \chi(G')$ *for every induced subgraph* $G' \subseteq G$.

(ii) $\omega(G, w) = \chi(G, w)$ *for every weight function* $w : V \to \mathbb{N}$.

(iii) $\text{STAB}(G) = \text{QSTAB}(G)$.

(iv) $\overline{G}$ *satisfies (i).*

(v) $\overline{G}$ *satisfies (ii).*

(vi) $\overline{G}$ *satisfies (iii).*

The original proof of (i) $\Rightarrow$ (ii) and (i) $\Leftrightarrow$ (iv) is due to Lovász in 1972 [150],
the proof of (ii) $\Rightarrow$ (v) $\Rightarrow$ (iv) is due to Fulkerson in 1971 [93], and
the proof (i) $\Leftrightarrow$ (iii) is independently due to Chvátal [48] and Padberg [164] in 1974-75.

*Proof. (i) $\Rightarrow$ (ii) by Lovász via the Replication Lemma.* The Replication Lemma (Theorem 1.2) states that a graph obtained from a perfect graph by replication of a vertex $v$ (*i.e.* a copy of $v$ is added, adjacent to $v$ and to all of its neighbors) is still perfect. Thus if $G$ is perfect, we can construct for every weight function $w : V \to \mathbb{N}$ a perfect graph $G_w$ obtained by replicating $(w(v) - 1)$ times every node $v$ (if $w(v) = 0$, we delete $v$). Then $\omega(G_w) = \chi(G_w)$ because $G_w$ is perfect, and we also have $\omega(G_w) = \omega(G, w)$ and $\chi(G_w) = \chi(G, w)$, which proves (ii).

*(ii) $\Rightarrow$ (iii) by Grötchel, Lovász and Schrijver.* Since the inequalities describing the polytope $\text{QSTAB}(G)$ all have only rational coefficients, we only have to prove that every point of $\text{QSTAB}(G)$ with rational coordinates belongs to $\text{STAB}(G)$. So let $y \in \mathbb{Q}^{|V|}$ be an arbitrary vector from $\text{QSTAB}(G)$ and let us prove that $y \in \text{STAB}(G)$. Let $q \in \mathbb{N}$ be the least common denominator of the entries in $y$. Then $qy \in \mathbb{N}^{|V|}$ and, since $y$ satisfies the clique constraints:

$$\sum_{i \in K} qy_i = q \sum_{i \in K} y_i \leqslant q \cdot 1 \quad \text{for every clique } K .$$

But then

$$\omega(G, qy) = \max_{K \text{ clique}} \sum_{i \in K} qy_i \leqslant q .$$

By (ii), we deduce that $\chi(G, qy) \leqslant q$, thus there exists at most $q$ stable sets $S_1, \ldots, S_q$ such that each node $i \in V(G)$ is covered exactly $qy_i$ times. In other words,

$$qy = \sum_{j=1}^{q} \chi^{S_j} \quad \text{resp. } y = \frac{1}{q} \sum_{j=1}^{q} \chi^{S_j} .$$

This proves that $y$ can be written as a convex combination of stable sets of $G$, and thus $y \in \text{QSTAB}(G)$.

*(iii) ⇒ (iv) by Grötchel, Lovász and Schrijver.* If $\text{STAB}(G)$ is expressed by the clique constraints and the non-negativity constraints only, then the same holds for every induced subgraph $G'$ of $G$. Thus we only need to prove that $G$ itself can be partitioned into $\alpha(G)$ cliques. We prove it by induction on $|V|$. Let $F$ be a face of $\text{STAB}(G)$ spanned by all stable sets of size $\alpha(G)$, *i.e.* $F$ contains all the convex combinations of the characteristic vectors of the maximum stable sets of $G$. Observe that the maximum stable sets all belong to the same hyperplane because the constraint of *having cardinality equal to* $\alpha(G)$ can be expressed by a linear equality. Thus there exists a facet-defining clique inequality of $\text{STAB}(G)$ that contains $F$, *i.e.* there is a clique $K$ whose associated inequality is tight on all the maximum stable sets (otherwise, $F$ would be the intersection of some non-negativity facets, a contradiction to $\chi^{\emptyset} \notin F$). But then $\alpha(G \setminus K) = \alpha(G) - 1$. Apply the induction hypothesis to $G \setminus K$ to partition it into $\alpha(G) - 1$ cliques. Adding $K$ gives a partition of $G$ into $\alpha(G)$ cliques. □

Theorem 4.4 in particular the following powerful corollary:

**COROLLARY 4.5**

$\text{STAB}(G) = \text{QSTAB}(G)$ *if and only if G is perfect.*

Let us now investigate a bit further about $\text{STAB}(G)$ in perfect graphs: as announced in Chapter 1, there is a non-combinatorial algorithm that computes a Maximum Weighted Stable Set in perfect graphs. At this point of the reading, a tempting explanation would be a result of type *we can optimize over* $\text{QSTAB}(G)$, *i.e.* the LP given by the exponentially many clique constraints (together with the non-negativity constraints) can be solved in polynomial-time. However, Grötschel, Lovász and Schrijver proved in 1981 [110] that the optimization problem over $\text{QSTAB}(G)$ in NP-hard in general.

Thus we need to go a little bit further and talk about the *Theta body* introduced by Lovász in 1979 [151]. We do not provide a formal definition here because it would need to introduce *semi-definite programming*, a powerful generalization of linear programming which gave rise to a full branch of combinatorial optimization. The *Theta body* of a graph $G$ is a convex set $\text{TH}(G)$ defined by a specific set of infinitely many non-linear inequalities (one for each so-called *orthonormal representation* of the graph). Since it is not defined by linear inequalities, it is not a polytope in general. Nonetheless, it has two very great properties, the first one is:

**THEOREM 4.6** [111]

$\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G)$ *for every graph G.*

In particular, Corollary 4.5 implies that for every perfect graph, the equality holds. Grötchel, Lovász and Schrijver [111] even proved that $\text{TH}(G)$ is a polytope if and only if $G$ is perfect. They also proved the second very great property of $\text{TH}(G)$: we can optimize over it (which means *approximate*, since the optimal value is not necessarily rational) in polynomial time.

**THEOREM 4.7** [110]

*Let $G = (V, E)$ be a graph and $w : V \to \mathbb{Z}_+$ be a weight function. Then there exists a polynomial-time algorithm that computes*

$$\vartheta(G, w) = \max\{w^\intercal x \mid x \in \mathrm{TH}(G)\}$$

*with arbitrary precision. Formally, this means that for every $\varepsilon > 0$, the algorithm returns a rational number at distance less than $\varepsilon$ from $\vartheta(G, w)$, in time polynomial in $|V|$, $\log(\max_{v \in V} w(v))$, and $\log(1/\varepsilon)$.*

The function $\vartheta$ is often referred to as the *$\vartheta$-function of Lovász*, because it was first defined in a prior work of Lovász in 1979 [151]. Note that the algorithm they designed deeply relies on the ellipsoid algorithm. Finally, Corollary 4.5 and Theorem 4.6 prove that $\vartheta(G, w)$ is an integer if $G$ is perfect, which together with Theorem 4.7 give:

**COROLLARY 4.8** [110]

*The Maximum Weighted Stable Set problem can be solved in polynomial time in perfect graphs.*

*Proof.* Observe that computing $\vartheta(G, w)$ with precision $\varepsilon = 1/2$ in a perfect graph $G$ and rounding the output to the closest integer gives exactly $\alpha(G, w)$. By Theorem 4.7, this can be done in polynomial time. We now describe an algorithm that actually computes a stable set of weight $\alpha(G, w)$:

*Input:* a perfect graph $G = (V, E)$ and a weight function $w : V \to \mathbb{Z}_+$.
*Output:* a stable set $S$ of $G$ of weight $\alpha(G, w)$.

Compute $s = \alpha(G, w)$, and then set $S \leftarrow \emptyset$, $G' \leftarrow G$, and $w' \leftarrow w$.
Then repeat while $V(G') \setminus S \neq \emptyset$:

*(i)* Pick a vertex $v \in V(G') \setminus S$ and compute $t = \alpha(G' \setminus v, w'_{G' \setminus v})$ where $w'_{G' \setminus v}$ is the function $w'$ restricted to $V(G') \setminus v$.

*(ii)* If $s = t$, then set $G' \leftarrow G' \setminus v$ and $w' \leftarrow w'_{G' \setminus v}$.

*(iii)* Otherwise, $v$ belongs to all stable sets of weight $s$ in $G'$, so we set $S \leftarrow S \cup \{v\}$.

When the loop is over, return $S$.

At each step, $|V(G') \setminus S|$ decreases by 1 so the number of iterations is $|V(G)|$. Moreover, each iteration as well as the initialization is performed in time polynomial time according to the above discussion. The correctness of the algorithm can be easily checked with the following loop-invariant: *$\alpha(G', w') = s$ and for every stable set $S_0$ of weight $s$ in $G'$, we have $S \subseteq S_0$.* In particular, $S$ is a stable set at each step, and at the end, $V(G') = S$ so $S$ is a stable set of weight $\alpha(G', w') = s$. $\square$

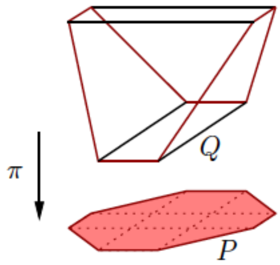



FIGURE 4.1: The polytope $P \subseteq \mathbb{R}^2$ with 8 facets admits an extension $Q \subseteq \mathbb{R}^3$ with only 6 facets. $\pi$ is the orthogonal projection such that $\pi(Q) = P$.

# 4.6 Extended Formulation

After this digression on $\text{STAB}(G)$ that will be useful later on, let us go back to the main subject of this chapter: extended formulations. The reader should now be convinced of the importance of expressing a polytope (especially coming from the feasible solutions of an IP) with a linear system, in order to use LP solvers. However, we have seen that the number of inequalities in such a linear system may be exponential in the dimension. Although the ellipsoid method may in some cases exempt us from an explicit listing of the constraints, it would be much better to have only a polynomial number of constraints: first, we would not have to find a separation oracle which can be very hard; second, we could use Karmakar's interior point method [131], which is much faster both in theory and in practice. This is the target behind the concept of extended formulations.

Imagine that we want to solve a LP on variable vector $x$ with a huge number of constraints. We can try to add some additional variables (such as the $u_i$ in the TSP) in order to get rid of many constraints, and such that every feasible solution of the new LP gives a feasible solution to the previous LP, when projecting on the $x$ coordinates (we simply forget about the existence of the additional variables), and vice versa. In such a case, we can optimize over the new LP with much fewer constraints, and project the optimal solution that we get to obtain an optimal solution to the initial LP. Let us see this in a polytope point of view (see Figure 4.1 for an illustration[9]): imagine that we have a polytope $P \subseteq \mathbb{R}^d$ we want to optimize over. Adding some additional variables means moving to a space of larger dimension $\mathbb{R}^{d+r}$. The goal is to find a polytope $Q \subseteq \mathbb{R}^{d+r}$ such that $Q$ projects exactly on $P$ (via a linear projection). If $Q$ has much fewer facets than $P$, then it is easier to optimize over $Q$ and project the optimal solution to $\mathbb{R}^d$ to get an optimal solution on $P$. A polytope $Q$ which projects on $P$ is called an *extension* of $P$ and its *size* is the number of facets of $Q$. The *extension complexity* of $P$ is the minimum size of an extension of $P$.

There is another equivalent definition of the extension complexity: it is the min-

[9]This picture is taken from slides by S. Pokutta.

imum size of an extended formulation of $P$, where an *extended formulation* (*EF* for short) of the polytope $P \subseteq \mathbb{R}^d$ is a linear system

$$Ex + Fy \leqslant g, \quad E'x + F'y = g'$$

in variables $(x, y) \in \mathbb{R}^{d+r}$ such that $x \in P$ if and only if there exists $y$ such that $(x, y)$ satisfies the above system. The *size* of an extended formulation is the number of *inequalities* in the system.

If the extension complexity of $P \subseteq \mathbb{R}^d$ is polynomial in $d$, we say that $P$ has a *compact formulation*. One could imagine that the *size* of an extended formulation is rather the sum of the number of variables and total number of constraints (equalities plus inequalities) defining the extended formulation. This sometimes occurs as an alternative definition in the literature, but the difference is nearly insignificant for our purpose. Indeed if $P \subseteq \mathbb{R}^d$ has an extended formulation with $r$ inequalities, then it has an extended formulations with $d + r$ variables, $r$ inequalities and at most $d + r$ equalities (see Observation 4.16 for a proof). Moreover if $P$ is full-dimensional, then $d \leqslant r$ so both definitions lead to essentially the same number up to a constant factor.

We can now state properly Yannakakis' result on the TSP, where the *TSP polytope* $\mathrm{TSP}(n) \subseteq \mathbb{R}^{|E(K_n)|}$ is defined as the convex hull of the characteristic vectors of Hamiltonian cycles in $K_n$. In fact for every classical combinatorial problem (*e.g.* perfect matching or spanning tree) we can define a polytope (*e.g.* the *perfect matching polytope* or the *spanning tree polytope*) which is the convex hull of the characteristic vectors of the solutions.

**THEOREM 4.9** [211]

*The TSP polytope* $\mathrm{TSP}(n)$ *cannot be expressed by a symmetric LP of subexponential size.*

In fact, he proved that a face of the TSP polytope can be projected on another polytope $\mathrm{MATCH}(n)$, which itself cannot be described by a compact symmetric LP. In some sense, this method is the counterpart of usual NP-completeness reductions, but in polyhedral terms. The polytope $\mathrm{MATCH}(n)$ is the perfect matching polytope of the complete graph $K_n$ with $n$ even.

**THEOREM 4.10** [211]

*The matching polytope* $\mathrm{MATCH}(n)$ *cannot be expressed by a symmetric LP of subexponential size.*

He left as an open question the case of asymmetric EFs, but he suspected that asymmetry could not help much. However, it was shown by Kaibel, Pashkovich and Theis [128] that the symmetry condition indeed matters: they proved that the convex hull of all $\log n$-size matchings has a compact asymmetric formulation, but no symmetric one.

Figure 4.1 shows an example where one can save 2 facets out of 8 by moving to a higher dimensional space. It is certainly a gain, but a very small one. One could wonder whether we can really save some significant amount of facets by moving to a higher dimensional space. The study of the spanning tree polytope in the complete graph $K_n$ puts an end to this doubt: it has $2^{\Omega(n)}$ many facets [66], but its extension complexity is only $\mathcal{O}(n^3)$ [156]. Other examples of polytopes admitting a compact formulation were found in the 1990's, we can mention the permutahedron[10] [105], the parity polytope[11] (see *e. g.* [55]) , the matching polytope in planar graphs [11] and more generally the matching polytope in graphs with bounded genus [101].

For almost 20 years after Yannakakis' paper, only upper bound results had been proved in this field. However, the combinatorial optimization community has recently made great effort in trying to develop tools to obtain lower bound results. It has been very successful for the past 5 years, where a series of beautiful results appeared: Rothvoß led the way in 2010 by proving with a counting argument that a random 0/1-polytope would have extension complexity that is exponential in the dimension [176]. However, this result is not constructive. This existential technique was extended to polygons by Fiorini et al. [84] and to the semi-definite programming extension complexity by Briët, Dadush and Pokutta [23]. These promising but somehow frustrating results were followed in 2012 by this groundbreaking result by Fiorini, Massar, Pokutta, Tiwary and de Wolf, finally answering Yannakakis' question about asymmetric compact formulation for the TSP polytope, and even for the stable set polytope:

**THEOREM 4.11** [83]

*The extension complexity of the TSP polytope* $\mathrm{TSP}(n)$ *is* $2^{\Omega(n^{1/2})}$.
*Moreover, there exists graphs* $(G_n)$ *with* $n$ *vertices such that the extension complexity of* $\mathrm{STAB}(G_n)$ *is* $2^{\Omega(n^{1/2})}$.

In fact, their result ruled out any attempt of compact formulation for a whole series of well-studied polytopes (not only $\mathrm{TSP}(n)$ and $\mathrm{STAB}(G)$ but also the cut polytope and the correlation polytope, see [83] for definitions), but they could not manage to extend their result to the matching polytope, which was yet in the scope of Yannakakis' bound for symmetric LPs (Theorem 4.10). The case of the matching polytope is particularly interesting for one more reason: by Edmonds' algorithm [65], we can compute in polynomial time with the help of augmenting paths the maximum matching for any input graph. This algorithm can even be adapted to the weighted case [154], which proves that we *can* optimize over the matching polytope in polynomial time. Moreover, Edmonds provided a full description of the polytope (with exponentially many constraints, one for each odd-cardinality subset of vertices, and one for each vertex). That is why a superpolynomial lower bound for the matching polytope could be somehow counter-intuitive: all the superpolynomial extension complexities known so far were concerned with NP-hard problems (TSP, MWSS, ...). However, it would prove that the extension complexity and the classical computa-

[10] Convex hull of all the vectors given by a permutation of $\{1, \dots, n\}$.
[11] Convex hull of the 0/1-vectors in $\mathbb{R}^d$ with an odd number of 1's.

tional complexity are really different measures of the hardness of a problem. Rothvoß achieved such a result in 2013:

**THEOREM 4.12** [177]

*For every even $n$, the extension complexity of the perfect matching polytope in $K_n$ is $2^{\Omega(n)}$.*

By Yannakakis' reduction between the matching polytope and the TSP polytope, Rothvoß' result implies a better exponential lower bound for $\mathrm{TSP}(n)$:

**COROLLARY 4.13** [177]

*For all $n$, the extension complexity of the TSP polytope $\mathrm{TSP}(n)$ is $2^{\Omega(n)}$.*

The recently developed techniques have then been adapted in terms of approximation. But *what is approximation in the polyhedral world?* When facing an NP-complete problem, it is natural to look for a polynomial time algorithm that gives a solution which is *close* to the optimal. When facing a polytope $P$ with exponential extension complexity, one can try to find a polytope $Q$ which is *close* to $P$, but easier, *i.e.* admitting a compact formulation. Inspired by the recent results, several researchers [19, 22, 20] were interested in the complexity of approximating the correlation polytope, which can be interpreted in terms of approximation of the maximum clique problem. They obtained that any linear program approximating the clique problem within $n^{1-\varepsilon}$ has exponential extension complexity. With a completely different approach, Chan *et al.* [27] proved (using method from Fourier analysis) a superpolynomial lower bound on the extension complexity of any polytope approximating the (submissive of the) CUT polytope within $2-\varepsilon$. The most recent result in this direction is again about the matching polytope, for which Braun and Pokutta [21] (improving Rothvoß [177]) ruled out any compact approximation within $(1+\varepsilon)$. For more details, see the recent book by Conforti, Cornuéjols and Zambelli dedicated to *Integer Programming* [56].

# 4.7 The Slack Matrix and the Factorization Theorem

One could easily imagine how the proof for an upper bound result should start: design a LP that expresses the polytope. However, a lower bound is a non-existence result so it seems much harder to know where to start. Fortunately, Yannakakis provided the so-called *Factorization Theorem* which restates the extension complexity of a polytope in terms of a measure on its *slack matrix*.

Let us define the slack matrix of a polytope $P$ (which in fact is not unique). Let us number the vertices of $P$ with $v_1, \ldots, v_\gamma$. Let $Ax \leqslant b$ be a linear system expressing $P$. Then the *slack matrix* of $P$ with respect to $Ax \leqslant b$ is the matrix $M(P)$ every row of which stands for an inequality of $Ax \leqslant b$, every column of which stands for a vertex of $P$, and the $(i,j)$-entry measures the *slack* of the $i$-th inequality on the $j$-th vertex, that is to say $M(P)_{i,j} = b_i - A_i v_j$ where the $i$-th inequality is written $A_i x \leqslant b_i$. In particular if $Ax \leqslant b$ contains exactly one inequality per facet, $M(P)_{i,j}$ measures somehow the distance between the $j$-th vertex and the $i$-th facet (up to scaling). Since every vertex of $P$ satisfies all the inequalities, the slack matrix contains only non-negative entries.

To understand Yannakakis' equivalence theorem, we also need to define the *non-negative rank* of a matrix. Let $M$ be a $m \times m'$ non-negative matrix (*i.e.* $M$ contains only non-negative entries), the *non-negative rank* of $M$, denoted $\mathrm{rank}_+(M)$, is the smallest integer $r$ such that $M$ can be written as the product $M = TU$ of a $m \times r$ matrix $T$ and a $r \times m'$ matrix $U$, with the additional condition that $T$ and $U$ are both non-negative matrices. We say that $TU$ is a *non-negative factorization* of $M$. Observe that if we drop the non-negativity condition, this is just the definition of the usual rank of a matrix. In his seminal paper, Yannakakis proved the following theorem:

**THEOREM 4.14** (The *Factorization Theorem*) [211]

*Let $P \subseteq \mathbb{R}^d$ be a polytope expressed by $Ax \leqslant b$ and $V$ be the set of vertices of $P$. Assume that $\dim(P) \geqslant 1$, then for every positive integer $r$, the following conditions are equivalent:*

(i) *The slack matrix $M(P)$ of $P$ with respect to $Ax \leqslant b$ has non-negative rank at most $r$.*

(ii) *$P$ has an extension of size at most $r$,* i.e. *with at most $r$ facets.*

(iii) *$P$ has an extended formulation of size at most $r$,* i.e., *with at most $r$ inequalities.*

The proof uses the following Corollary of Farkas' Lemma (see *e. g.* [56]):

**LEMMA 4.15** [83]

*Let $P \subseteq \mathbb{R}^d$ be a polyhedron expressed by $Ax \leqslant b$ that admits a direction $u \in \mathbb{R}^d$ such that $u^\intercal x$ is not constant and both upper-bounded and lower-bounded when $x$ ranges over $P$. Let $c^\intercal x \leqslant \delta$ be a valid inequality for $P$. Then there exists non-negative multipliers $\lambda \in \mathbb{R}^d$ such that $\lambda^\intercal A = c^\intercal$ and $\lambda^\intercal b = \delta$,* i.e. *$c^\intercal x \leqslant \delta$ can be derived as a non-negative combination from $Ax \leqslant b$.*

Observe in particular that the above lemma holds whenever $P$ is a polytope of dimension at least 1 or whenever $P$ is an unbounded polyhedron that linearly projects to a polytope of dimension at least 1.

*Proof of Theorem 4.14. (i) $\Rightarrow$ (ii)* Let $M(P) = TU$ be a non-negative factorization of $M(P)$ of rank $r$. We prove that

$$Q = \{(x,y) \in \mathbb{R}^{d+r} \mid Ax + Ty = b, y \geqslant 0\}$$

is an extension of $P$ with the projection $\pi$ on variable $x$. Obviously, since all entries of $T$ are non-negative, $\pi(Q) \subseteq P$. Let us prove that $P \subseteq \pi(Q)$, for this we prove that every vertex of $P$ is in $\pi(Q)$. Let $v_j$ be the $j$-th vertex of $P$, and let $U^j$ (resp. $M(P)^j$) be the $j$-th column of $U$ (resp. $M(P)$). Then we have $TU^j = M(P)^j = b - Av_j$, consequently $Av_j + TU^j = b$ and $U^j \geqslant 0$ by definition, so $(v_j, U^j) \in Q$, hence $v_j \in \pi(Q)$.

*(ii) ⇒ (iii)* This should be clear because each polytope in $\mathbb{R}^{d+r}$ with $r$ facets can be expressed by a linear system with $r$ inequalities.

*(iii) ⇒ (i)* Let

$$Ex + Fy = g,\ E'x + F'y \leqslant g'$$

be an EF of $P$ with $r$ inequalities and let $Q$ be the (not necessarily bounded) polyhedron expressed by this system. By definition of an EF, for every vertex $v_j$ of $P$, there exists $w_j \in \mathbb{R}^r$ such that $(v_j, w_j) \in Q$. In particular, each inequality of $Ax \leqslant b$ is valid for any vertex of $Q$. Let $S_Q$ be the matrix whose $j$-th column records the slack of inequalities of $E'x + F'y \leqslant g'$, and then of $Ax \leqslant b$, on $(v_j, w_j)$. In particular, $M(P)$ is the submatrix of $S_Q$ obtained by deleting the $r$ first rows so $\text{rank}_+(M(P)) \leqslant \text{rank}_+(S_Q)$. By Lemma 4.15, any valid inequality $c^\intercal x \leqslant \delta$ is a non-negative combination of inequalities of the system $E'x + F'y \leqslant g'$ (and of equalities of $Ex + Fy = g$, but the slack on the equalities is null so their contribution to the slack will be null). Consequently, every row of $S_Q$ is a non-negative combination of the first $r$ rows of $S_Q$. Consequently, $\text{rank}_+(S_Q) \leqslant r$ and thus (i) holds. □

**OBSERVATION 4.16**

*By the Factorization Theorem, if a polytope $P = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$ has an EF with $r$ inequalities, then its slack matrix has a non-negative factorization $TU$ of rank at most $r$, and thus $Ax + Ty = b$, $y \geqslant 0$ with $y \in \mathbb{R}^r$ is an EF of $P$ with $d + r$ variables, $r$ inequalities and $m$ equalities, where $m$ is the number of rows in $A$. But if $m > d + r$, some of these equalities will be redundant, so there always exists a subset of at most $d + r$ equalities expressing the same subspace. So $P$ has an EF with $d + r$ variables, at most $d + r$ equalities and $r$ inequalities.*

Because of the Factorization Theorem, we now have an insight on how to bound the extension complexity: we can work with a matrix, a much more familiar object. Moreover, we might make a slight abuse of notation and say *the non-negative rank of **the** slack matrix*, even if the slack matrix is not unique. Indeed, the Factorization Theorem (Theorem 4.14) ensures that *all* the slack matrices of $P$ have the same non-negative rank, because $\text{xc}(P)$ depends only on $P$.

Although the rank of a matrix is related to many well-known algebraic tools, the non-negative rank is not so standard. The first way of getting a lower bound was described by Yannakakakis and is called the *rectangle covering bound*. The idea is the following: a non-negative factorization $TU$ of rank $r$ of a matrix $M$ is a covering of its positive entries by $r$ rectangles: indeed, $M = \sum_{i=1}^r T^i U_i$ where $T^i$ is the $i$-th column of $T$ and $U_i$ is the $i$-th row of $U$. Moreover, the positive entries of $T^i U_i$ is a rectangle up to permuting the order of the rows and the columns (positive entries of $T^i$ at the top;

positive entries of $U_i$ on the left-most part). Those $r$ rectangles covers all the positive entries of $M$ and no more (because of the non-negativity constraints on $T$ and $U$), and the overlapping is controlled by the summing condition. Imagine that, for a start, we forget about this overlapping condition: we just want to cover exactly all the positive entries with rectangles. Then the exact value of each entry is not important anymore in itself, so just replace every positive entry by 1, and keep the zero entries as they are. In such a way, we obtain the *support* of the matrix $M$, denoted by $\mathrm{supp}(M)$. Now we want to cover all the 1-entries with as few rectangles as we can, provided that all the rectangles contain only 1-entries. Formally, a *combinatorial rectangle* $\mathcal{R}$ is a set of entries $(i, j)$ of $M$ defined by a pair $(R, C)$ where $R$ is a subset of rows of $M$, $C$ is a subset of rows of $M$, and $(i, j) \in \mathcal{R}$ if and only if $i \in R$ and $j \in C$. If $M_{i,j} = 1$ (resp. $M_{i,j} = 0$) for every $(i, j) \in \mathcal{R}$, we say that the rectangle $\mathcal{R}$ is *1-monochromatic* (resp. *0-monochromatic*). A set of 1-monochromatic combinatorial rectangles that cover all the 1-entries of $M$ (*i.e.* every 1-entry of $M$ is contained in at least one of the rectangles) is called a *rectangle covering* of $M$, and the minimum number of such a set is called the *rectangle covering number* of $M$, and is denoted by $\mathrm{rc}(M)$. As argued above, this gives a lower bound on the extension complexity:

**THEOREM 4.17** [211]

*Let $P$ be a polytope and $M(P)$ be one of its slack matrices. Then*

$$\mathrm{rc}(M(P)) \leqslant \mathrm{rank}_+(M(P)) \quad \textit{thus} \quad \mathrm{rc}(M(P)) \leqslant \mathrm{xc}(P)\,.$$

The so-called *rectangle covering bound* refers to the method of lower bounding the extension complexity by the rectangle covering of a slack matrix. There are mainly two different approaches leading to such a lower bound: the *fooling set method* and the *rectangle size method*. Moreover, the *rectangle covering bound* is not always enough to obtain interesting bounds on $\mathrm{rank}_+(M)$. In particular, Yannakakis proved that $\mathrm{rc}(M) = \mathcal{O}(n^4)$ for a slack matrix $M$ of the matching polytope (the slack matrix with respect to Edmond's linear description), although we now know that the extension complexity is exponential: Rothvoß [177] used another method, first developed by Fiorini [82], called the *hyperplane separation method*. Let us briefly give some hints on these three methods.

**Fooling set method** A *fooling set* is a very combinatorial object that can be used to lower bound the rectangle covering number. For $b \in \{0, 1\}$, a *b-fooling set* $\mathcal{F}$ in a matrix $M$ is a set of $b$-entries of the matrix that cannot be pairwise put in the same $b$-monochromatic combinatorial rectangle, that is to say for every $(i, j), (i', j') \in \mathcal{F}$, $M_{i,j} = M_{i',j'} = b$ but $M_{i',j} \neq b$ or $M_{i,j'} \neq b$. In addition to being a very simple object, the size of a 1-fooling set trivially provides a lower bound on $\mathrm{rc}(M)$. Interestingly, fooling sets will also appear quite naturally in Chapter 5 in the study of the biclique partition number of $K_n$ (see Theorem 5.10). However, the power of fooling sets are rather limited in general.

**Rectangle size method** Initiated by a result of Razborov [170] and improved by de

Wolf [61], this method was used by Fiorini *et. al* for the exponential lower bound on $\mathrm{xc}(\mathrm{TSP}(n))$. The key part is a weight function (in fact, a probabilistic distribution) over the entries of the matrix $M$ such that each combinatorial rectangle containing a substantial amount of weight coming from 1-entries must also contain a non-zero amount of weight coming from 0-entries. However, the 1-monochromatic rectangles in a rectangle covering do not contain any 0-entry, so each of them can contain only a very small amount of weight coming from 1-entries: consequently, there must be a lot of them.

**Hyperplane separation method** As already mentioned, this more sophisticated method does not give a lower bound on $\mathrm{rc}(M)$ but directly on $\mathrm{rank}_+(M)$, which can be very useful when the the rectangle covering is too small (*e.g.* the matching polytope). Intuitively, the idea is to find a linear function $\lambda$ that, applied to the slack matrix $M$, gives a large real value $\lambda(M)$, but applied to any 1-monochromatic rectangle, gives only a small real value. By linearity, $\lambda(M)$ is the sum of $\lambda(R)$ for all the rectangles $R$, and thus the number of rectangles must be large. For further details on this technique, see [177].

# 4.8 Link with communication complexity

Some tools about the slack matrix, including the rectangle covering number and fooling sets, come in fact from the world of *communication complexity*. Let us rephrase some of the above concepts in those terms: let $M$ be a matrix that will be called the *communication matrix*. Let Alice and Bob be two characters that both know the matrix. At the beginning, they can pre-compute as much as they want and freely exchange messages. Now Alice is secretly given a row $i$ and Bob is secretly given a column $j$. They have to compute $M_{i,j}$, by minimizing the number of bits exchanged between them. We can apply this setting to the particular case where $M$ is the support of a slack matrix of a polytope. Since each row of the matrix stands for a constraint and each column stands for a vertex, this means that they have to decide whether Alice's constraint is tight on Bob's vertex; in particular if Alice's row is a facet-defining inequality, the question is to know whether Bob's vertex lies on the hyperplane supporting Alice's facet.

Let us describe the three main types of protocols (for more details, see the book by Kushilevitz and Nisan dedicated to *Communication Complexity* [144]).

**Deterministic protocol** After the pre-processing step, Alice and Bob can successively send messages one to each other, and when one of them has decided an answer, he/she outputs it. The protocol has to be *correct*, that is to say that the answer they output should be the right answer.

**Probabilistic protocol** At each turn, Alice or Bob draws a random bit with probability depending on her/his input and on the history of previously exchanged messages. This random bit is sent to the other character, and they keep going until one of them decide to stop and outputs a solution. The protocol does not have to be correct, but they need to compute the right value *in expectation* (which differs from most usual probabilistic settings where it is required to compute the right value with high probability). See [75] for more details.

**Non-deterministic protocol** Assume that $M$ is a 0/1-matrix. We introduce a third character called the Prover. He takes part in the pre-processing step, so at this time the three characters can agree on a protocol. When Alice is given a row $i$, and Bob is given a column $j$, the Prover knows both of them. The Prover can write a message on a blackboard, that both Alice and Bob can read. The Prover could just send the answer but Alice and Bob do not trust him: the Prover has to write a *certificate* to convince them of the right answer. Each of Alice and Bob must then choose between *Accepting the certificate* or *Rejecting the certificate*, and the certificate is accepted if both Alice and Bob accept it. Observe that, when dealing with usual complexity, a decision problem is in NP if there is a polynomial-time non-deterministic algorithm for deciding *Yes*, whereas the problem is in co-NP is there exists such an algorithm for deciding *No*. The same distinction exists here, so we assume that the Prover wants to convince Alice and Bob that the answer is 1. The protocol is correct is the following holds: the Prover can find a certificate that will be accepted by Alice and Bob if and only if the correct answer is 1. The cost of the protocol is the number of bits written on the blackboard in the worst-case. The non-deterministic communication complexity of a problem is the minimum cost of a correct protocol. The co-nondeterministic communication complexity of a problem is defined as the non-deterministic communication complexity of the complementary problem, where 0-entries in the matrix are replaced by 1-entries and vice-versa. In other words, it is the minimum cost of a correct protocol when the Prover wants to convince Alice and Bob that the answer is 0.

Let us now try to understand the relationship between the protocols and the communication matrix $M$.

**Non-determinism and rectangle covering** In the non-deterministic case, there is a one-to-one correspondence between correct protocols and rectangle coverings: indeed, suppose there is a rectangle covering with $r$ monochromatic rectangles numbered $R_1, \dots R_r$. The three characters can agree on such a numbering during the pre-processing step. Given Alice's input $i$ and Bob's input $j$, the Prover can find a rectangle $R_k$ such that $M_{i,j} \in R_k$. He writes $k$ on $\lceil \log r \rceil$ bits, then Alice (resp. Bob) can check that row $i$ (resp. column $j$) has non-empty intersection with $R_k$. If they both agree, by definition of a combinatorial rectangle it shows that $M_{i,j} \in R_k$ and since the rectangles are monochromatic, they know that the right answer is the value of the matrix on the rectangle. On the contrary, given a correct non-deterministic protocol, we can partition the entries of $M$ depending on the certificate that the Prover would send. In fact, it is not exactly a partition since there can be several good certificates for the same entry, thus this forms a covering of all the entries. Since each of Alice and Bob decides to accept the certificate on the sole basis of the certificate and her/his own input, we can prove that one certificate forms a monochromatic rectangle (the argument is essentially the same as in Theorem 3.1). The number of rectangles is bounded by the number of different certificates, at most $2^{\ell}$ where $\ell$ is the number of bits exchanged, *i.e.* the communication complexity.

**Determinism and protocol partition** For a deterministic protocol, we can also partition the entries of the matrix into parts that lead to the same history of exchanged messages. In this case, it is really a partition because of determinism, and this will lead to monochromatic parts. Moreover, each time Alice (resp. Bob) sends a bit to Bob (resp. Alice), this discards a set of rows (resp. columns) among those remaining: thus the entries that lead to exactly the same exchanged messages are rectangles. However, not every partition into monochromatic rectangles can be obtained in such a way because it may not be structured enough. A partition is called a *protocol partition* if it comes from a deterministic protocol.

**Probability and non-neg. factorization** This non-classical probabilistic setting based on *computation in expectation* was introduced recently by Faenza *et. al* [75] because they noticed that this was the good setting for capturing exactly the non-negative rank of the communication matrix. Indeed, we have seen above that a non-deterministic protocol captures the hardness of the rectangle covering of the matrix, *i.e.* gives a lower bound on the non-negative rank; and that a deterministic protocol gives some specific monochromatic partition, *i.e.* gives an upper bound on the non-negative rank. They guessed that there must be some in-between communication setting that translates exactly into a non-negative factorization of the matrix, and they proved it for protocols computing in expectation.

# 4.9 The birth of CS-Separation

The reader may now recognize some similarities with the beginning of Chapter 3. Indeed, CS-Separation comes from the study of $\text{STAB}(G)$ in perfect graphs with the help of non-deterministic communication complexity. Let us now describe this a little more precisely.

We have seen in Subsection 4.5 that, for perfect graphs, we have the equality $\text{STAB}(G) = \text{QSTAB}(G)$, meaning that the clique inequalities (together with the non-negativity constraints) are enough to express $\text{STAB}(G)$. In particular, for every perfect graph $G = (V, E)$ on $n$ vertices, the slack matrix of $\text{STAB}(G)$ with respect to this system is a $0/1$-matrix $\begin{pmatrix} \text{CS}(G) \\ Z \end{pmatrix}$ where $Z$ is the matrix whose columns are the characteristic vectors of stable sets of $G$ (corresponding to non-negativity constraints) and $\text{CS}(G)$ is the Clique vs. Stable Set *matrix*: it contains one row for each clique $K$, one column for every stable set $S$, and $\text{CS}(G)_{K,S} = 1$ if $K$ and $S$ are disjoint, 0 otherwise. We can now see that $\text{CS}(G)$ is the communication matrix of the *Clique versus Independent Set* problem, described at the beginning of Chapter 3: Alice has a clique, Bob has a stable set, and they want to decide if the clique and the stable set intersect or not. Note that, in Chapter 3, we said that Alice and Bob output *Yes* if the intersection is not empty, although the slack in this case is 0 (because it made more sense at the time to answer to the question *do K and S intersect?* rather than the contrary).

Because of the Factorization Theorem, Yannakakis tried to study the extension complexity of STAB$(G)$ in perfect graphs by focusing on the non-negative rank of the slack matrix. He tried to get a lower bound on the rectangle covering number, which is exactly the non-deterministic communication complexity of *Clique versus Independent Set*. The one-to-one correspondence between a CS-Separator and a rectangle covering is given in Theorem 3.1: a 1-monochromatic rectangle is a cut that separates all the cliques (rows) from all the stable sets (columns) involved in the rectangle, and vice versa.

Yannakakis' initial goal was to provide a lower bound on the extension complexity with the help of lower bounds on Clique-Stable Set separators. However, the results we obtained in Chapter 3 for CS-Separation provide *upper bounds*: this has no direct implication on the extension complexity. One can still wonder whether we can *extend the results and the tools used for CS-Separation to get bounds on the extension complexity?* This seems to be an interesting direction for further work.

# Chapter 5

# Alon-Saks-Seymour conjecture

AFTER trying to bound the chromatic number $\chi(G)$ in terms of the clique number $\omega(G)$ in Chapter 1, we are interested here in bounding $\chi(G)$ in terms of a new measure of the graph, called the *biclique partition number*[1] of $G$ and denoted by $\mathbf{bp}(G)$ (see Figure 5.1 for an example): it is the minimum number of edge-disjoint complete bipartite graphs needed to partition the edges of $G$. Observe that an edge is itself a complete bipartite graph, so $\mathbf{bp}(G)$ is well-defined. However, Alon, Saks and Seymour conjectured that the inequality $\chi(G) \leqslant \mathbf{bp}(G) + 1$ may be true. We give in Section 5.1 a brief overview of the origins of this conjecture which was disproved in 2012 by Huang and Sudakov [122]. We then wonder whether a polynomial upper-bound $\mathbf{bp}(G)^c$ could hold for $\chi(G)$.

The latter is particularly interesting because we prove that it is equivalent to having a polynomial-size CS-Separator for every graph $G$ (see Chapter 3 for more details about the Clique-Stable Set Separation). To obtain such a result, we introduce in Section 5.2 an oriented version of the biclique partition number, that we denote $\mathbf{bp}_{or}(G)$, and which captures exactly the difficulty of the CS-Separation. In Section 5.3, we generalize further the Alon-Saks-Seymour conjecture by asking for $\chi(G)$ to be polynomially bounded in terms of another measure of the graph, called the *t-biclique covering number* $\mathbf{bp}_t(G)$: instead of partitioning the edges of $G$ (as in the definition of $\mathbf{bp}(G)$), we now ask for a covering of the edges with complete bipartite graphs, provided that the overlapping is controlled. We prove that polynomially bounding $\chi(G)$ in terms of $\mathbf{bp}_t(G)$ for every $G$ is equivalent to polynomially bounding $\chi(H)$ in terms of $\mathbf{bp}(H)$ for every $H$. Combining this with Section 5.2, we prove the equivalence between those statements and a polynomial upper-bound for the CS-Separation in the general case.

Note that the content of this chapter is covered in the following paper:

[III] **Clique versus Independent Set**, with N. BOUSQUET and S. THOMASSÉ, *European Journal of Combinatorics*, 40:73–92, 2014.

It should be noticed that this work is prior to Göös' result [108] providing a su-

[1]Called *bipartite packing* in [III].

perpolynomial lower bound for the general case of the Clique-Stable set Separation. Consequently, this chapter is written with the point of view of establishing the equivalence between two questions that were opened at that time. Nonetheless, we mention the consequences of Göös' result when appropriate.

# 5.1 From origins to disproval

The trivial bound $\mathbf{bp}(G) \leqslant |E(G)|$ is only useful to straightforwardly convince oneself that it is always possible to partition $E(G)$ into complete bipartite graphs. However, a much better upper bound can always be achieved:

**OBSERVATION 5.1**

*For every graph $G$, $\mathbf{bp}(G) \leqslant |V(G)| - 1$.*

*Proof.* If the graph is not a stable set, select one vertex $x$, and consider the edges whose one endpoint is $x$: they form a complete bipartite graph between $x$ and $N(x)$. Delete them and repeat until $E(G)$ is empty. □

Graham and Pollak [109] proved in 1972 that this bound is optimal for the complete graph $K_n$:

**THEOREM 5.2** [109]

*For every $n$, $\mathbf{bp}(K_n) = n - 1$.*

Some simpler proofs of this theorem were found later on, among them [165] and a simple algebraic proof by Tverberg [201], but none of them are combinatorial. Observe that this result still generates some interest (see for instance the recent alternative proofs [205, 206, 210], and also the generalization to hypergraphs by Cioabă and Tait [50]).

At the same time, the Erdős-Faber-Lovász conjecture [67] dating from 1972, caught a lot of attention: it states that the union of $k$ pairwise edge-disjoint complete graphs on $k$ vertices is $k$-colorable (an alternative statement exists in terms of hypergraphs). This conjecture is still open but some partial results were achieved: it was asymptotically proven by Kahn [126], and a fractional version was proved by Kahn and Seymour [127]; see also [123, 29].

Inspired by both the Erdős-Faber-Lovász conjecture and the Graham-Pollak theorem, Alon, Saks and Seymour conjectured in 1991 the following:

**CONJECTURE 5.3** (The *Alon-Saks-Seymour conjecture*, cited in [125])

*For every graph $G$, $\chi(G) \leqslant \mathbf{bp}(G) + 1$.*

Several partial results were achieved in this direction, in particular it was proved to be true [98, 173] for extremal values of $\mathbf{bp}(G)$, namely either if $\mathbf{bp}(G) \leqslant 9$ or if $\mathbf{bp}(G) \geqslant |V(G)| - 3$. However, it was finally disproved by Huang and Sudakov in 2012, twenty years after its statement:

**THEOREM 5.4** (Counterexample to the Alon-Saks-Seymour conjecture [122])

*There exists graphs $G$ with arbitrarily large biclique partition number such that $\chi(G) = \Omega(\mathbf{bp}(G)^{6/5})$.*

Their proof is constructive and based on a study of the 7-dimensional cube $Q_7$ (vertex set $\{0,1\}^7$ and edges between two vertices if and only if they differ in exactly one coordinate). Their construction is inspired by Razborov's counterexample [169] to another conjecture, called the *Rank-Coloring conjecture* and first proposed in 1976 by Van Nuffelen [203] (see also Fajtlowicz [76]): it asserts that the chromatic number of a graph $G$ would be upper bounded by the rank of its adjacency matrix. This conjecture was first disproved in 1989 by Alon and Seymour [7], but some further work tried to improve the gap between $\chi(G)$ and the rank ([179] based on Kasami graphs; in [169], the first superlinear gap, by Razborov; in [163] the largest gap so far, by Nisan and Wigderson). Recently, Cioabă and Tait [50] generalized Huang and Sudakov's counterexample to produce a family of graphs that are counterexamples to both the Alon-Saks-Seymour conjecture and the Rank-Coloring conjecture.

Now, the Alon-Saks-Seymour conjecture is disproved, however the counterexample did not kill any attempt of bounding $\chi(G)$ in terms of $\mathbf{bp}(G)$. In particular, Mubayi and Vishwanathan proved the following:

**THEOREM 5.5** [161]

*For every graph $G$, $\chi(G) \leqslant \mathbf{bp}(G)^{\frac{1}{2}(\log(\mathbf{bp}(G))+1)}(1+o(1))$.*

Given such a quasi-polynomial upper-bound, a natural question is: *can we polynomially upper bound $\chi(G)$ in terms of $\mathbf{bp}(G)$, for every graph $G$?* Huang and Sudakov did not think so, and conjectured the existence of graphs such that $\chi(G) \geqslant 2^{c\log^2(\mathbf{bp}(G))}$ for some constant $c > 0$ [122]. We will prove that the former is equivalent to having a polynomial-size CS-Separator for every graph $G$ (however, the equivalence does not hold for each graph individually). In a very recent work [108], Göös provides a family of graphs admitting no polynomial-size CS-Separator. This, combined with Theorem 5.16, implies that it is not possible to polynomially bound $\chi(G)$ in terms of $\mathbf{bp}(G)$. In fact, Alon and Haviv already observed (private communication, but described by Huang and Sudakov in [122]) that the existence for some graphs $G$ of a gap between $\chi(G)$ and $\mathbf{bp}(G)$ implies a lower bound on the size of a CS-Separator in some graphs. However, the reverse direction was an open problem in [122]. They still obtained a weaker reversed implication, by switching from biclique partition $\mathbf{bp}(G)$ to the *2-biclique covering number* $\mathbf{bp}_2(G)$: it is the minimum number of edge-disjoint complete bipartite graphs needed to cover the edges of $G$, with the additional condition that each edge is covered at most twice. They obtained the following (slightly restated):

**THEOREM 5.6** (Alon and Haviv; see [122])

*For every non-decreasing function $f$ and $f'$:*

- *If there exists a family of graphs $H$ such that $\chi(H) \geqslant f(\mathbf{bp}(H))$, then there exists graphs $G$ such that the size of any CS-Separator is at least $f(|V(G)|)$.*
- *If there exists a family of graphs $G$ for which the size of any CS-Separator is at least $f'(|V(G)|)$, then there exists graphs $H$ such that $\chi(H) \geqslant f'(\mathbf{bp}_2(H))$.*

The proof of this statement is very similar to the one of Theorem 5.7 to be stated in Section 5.2. As a consequence of this result, Huang and Sudakov's construction gives a family of graphs with a $\Omega(|V(G)|^{6/5})$ lower bound for the size of a CS-Separator. This lower bound was improved to $\Omega(|V(G)|^{3/2})$ [8] and then to $\Omega(|V(G)|^{2-\varepsilon})$ (for any $\varepsilon > 0$) [190] by introducing (independently from us) an *oriented* version of the biclique partition number. Such an oriented variant is the subject of the next Section.

## 5.2 An oriented version & Equivalence with the CS-Separation

We introduce here a variant of the biclique partition number. Note that it is the same notion as *ordered biclique covering*, denoted by $\mathbf{bp}_{1.5}$, which has been independently introduced in [8]. An *oriented* complete bipartite graph is a complete bipartite graph $(A, B)$ where each arc goes from $A$ to $B$. The *oriented biclique partition number* $\mathbf{bp}_{or}(G)$ of a non-oriented graph $G$ is the minimum number of oriented complete bipartite graphs such that each edge is covered by an arc in at least one direction (it can be in both directions), but it cannot be covered twice in the same direction (see Figure 5.1 for an example). An *oriented biclique partition*[2] of size $k$ is a set $\{(A_1, B_1), \dots, (A_k, B_k)\}$ of $k$ oriented bipartite subgraphs of $G$ that fulfill the above conditions restated as follows: for each edge $xy$ of $G$, free to exchange $x$ and $y$, there exists $i$ such that $x \in A_i, y \in B_i$, but there do not exist distinct $i$ and $j$ such that $x \in A_i \cap A_j$ and $y \in B_i \cap B_j$.

The main result of this Section is the following:

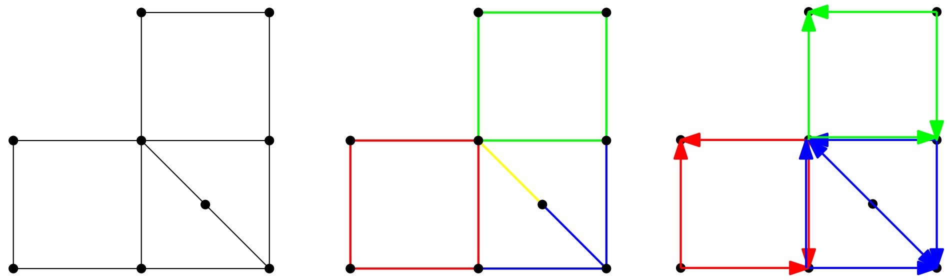
**THEOREM 5.7**

*For every non-decreasing function $f$, the following are equivalent:*

- *For every graph $G$, there exists a CS-Separator of size $f(|V(G)|)$.*
- *For every graph $H$, the inequality $\chi(H) \leqslant f(\mathbf{bp}_{or}(H))$ holds.*

As already mentioned, the proof is quite similar to Alon and Haviv's observation (Theorem 5.6). We decompose it into two lemmas, one for each direction of the equivalence.

[2] Called *packing certificate* in [III].



(a) A graph $G$ (b) $\mathbf{bp}(G) = 4$ (c) $\mathbf{bp}_{or}(G) = 3$

FIGURE 5.1: A graph $G$ such that $\mathbf{bp}(G) = 4$ and $\mathbf{bp}_{or}(G) = 3$. The colors on the edges give the (oriented or not) biclique partition. In (c), two edges are covered once in each direction.

**LEMMA 5.8**

*For every non-decreasing function $f$, if $\chi(H) \leqslant f(\mathbf{bp}_{or}(H))$ for every graph $H$ then for every graph $G$, there exists a CS-Separator of size $f(|V(G)|)$.*

*Proof.* Let $G$ be a graph on $n$ vertices. We want to build a CS-Separator of $G$. Consider all the pairs $(K, S)$ such that the clique $K$ does not intersect the stable set $S$. Construct an auxiliary graph $H$ as follows. The vertices of $H$ are the pairs $(K, S)$ and there is an edge between a pair $(K, S)$ and a pair $(K', S')$ if and only if there is a vertex $x \in S \cap K'$ or $x \in S' \cap K$. For every vertex $x$ of $G$, let $(A_x, B_x)$ be the oriented bipartite subgraph of $H$ where $A_x$ is the set of pairs $(K, S)$ for which $x \in K$, and $B_x$ is the set of pairs $(K, S)$ for which $x \in S$. By definition of the edges, $(A_x, B_x)$ is complete. Moreover, every edge is covered by such an oriented bipartite subgraph: if $(K, S)(K', S') \in E(H)$ then there exists $x \in S \cap K'$ or $x \in S' \cap K$ thus the corresponding arc is in $(A_x, B_x)$. Finally, an arc $(K, S)(K', S')$ cannot appear in both $(A_x, B_x)$ and $(A_y, B_y)$ otherwise the clique $K$ and the stable set $S'$ intersect on two vertices $x$ and $y$, which is impossible. Hence the oriented biclique partition number of this graph is at most $n$.

By assumption, $\chi(H) \leqslant f(\mathbf{bp}_{or}(H))$ so $\chi(H) \leqslant f(n)$. Consider a color of this $f(n)$-coloring. Let $A$ be the set of vertices of this color, so $A$ is a stable set. Let us now define

$$U_{\mathcal{K}} = \bigcup_{(K,S) \in A} K \quad \text{and} \quad U_{\mathcal{S}} = \bigcup_{(K,S) \in A} S\,.$$

Then $U_{\mathcal{S}}$ and $U_{\mathcal{K}}$ are disjoint: otherwise, there are two vertices $(K, S)$ and $(K', S')$ of $A$ such that $K$ intersects $S'$, thus $(K, S)(K', S')$ is an edge. This is impossible since $A$ is a stable set. Consequently, the cut $(U_{\mathcal{K}}, V(G) \setminus U_{\mathcal{K}})$ separates every $(K, S) \in A$. The same can be done for every color. This is a CS-Separator of size $\chi(H) \leqslant f(n)$ cuts, which achieves the proof. $\square$

**LEMMA 5.9**

*For every non-decreasing function $f$, if every graph $G$ admits a CS-Separator of size $f(|V(G)|)$, then for every graph $H$, the inequality $\chi(H) \leqslant f(\mathbf{bp}_{or}(H))$ holds.*

*Proof.* Let $H = (V, E)$ be a graph with $\mathbf{bp}_{or}(H) = k$. Construct an auxiliary graph $G$ as follows. The vertices are the elements of an oriented biclique partition of size $k$. There is an edge between two elements $(A_1, B_1)$ and $(A_2, B_2)$ if and only if there is a vertex $x \in A_1 \cap A_2$. Hence the set of all $(A_i, B_i)$ such that $x \in A_i$ is a clique of $G$ (say the clique $K_x$ *associated to* $x$). The set of all $(A_i, B_i)$ such that $y \in B_i$ is a stable set in $G$ (say the stable set $S_y$ *associated to* $y$). It is indeed a stable set, otherwise there are two vertices $(A_1, B_1), (A_2, B_2) \in S_y$ such that $y \in B_1 \cap B_2$ and there is an edge between them resulting from some $x \in A_1 \cap A_2$, but then the arc $xy$ is covered twice which is impossible. Note that a clique or a stable set associated to a vertex can be empty, but this does not trigger any problem. By assumption, there is a CS-Separator with $f(k)$ cuts. In particular, for every $x \in V$, the pair $(K_x, S_x)$ is separated.

Associate to each cut a color, and let us now color the vertices of $H$ with them. We color each vertex $x$ by the color of the cut separating $(K_x, S_x)$. Let us finally prove that this coloring is proper. Assume there is an edge $xy \in E$ such that $x$ and $y$ are given the same color. Then there exists a bipartite graph $(A, B) \in V(G)$ that covers the edge $xy$, hence $(A, B)$ is in both $K_x$ and $S_y$. Since $x$ and $y$ are given the same color, then the corresponding cut separates both $K_x$ from $S_x$ and $K_y$ from $S_y$. This is impossible because $K_x$ and $S_y$ intersect in $(A, B)$. Then we have a coloring of $H$ with at most $f(k)$ colors. □

*Proof of Theorem 5.7.* This is straightforward using Lemmas 5.8 and 5.9. □

The second half of this section proves that upper-bounding $\mathbf{bp}_{or}(K_m)$ is deeply linked with the maximum size of a fooling set for the *Clique versus Independent Set Problem*. Recall the definitions of Chapter 4: in the *Clique vs. Stable Set* matrix $\textsc{Cs}(G)$, each row corresponds to a clique $K$, each column corresponds to a stable set $S$, and $\textsc{Cs}(G)_{K,S} = 1$ if $K$ and $S$ are disjoint, 0 otherwise. A 1-fooling set (or simply *fooling set* here) $\mathcal{F}$ is a set of pairs $(K, S)$ such that $K$ and $S$ do not intersect, and for all $(K, S), (K', S') \in \mathcal{C}$, $K$ intersects $S'$ or $K'$ intersects $S$ (consequently $\textsc{Cs}(G)_{K,S'} = 0$ or $\textsc{Cs}(G)_{K',S} = 0$). Thus $\mathcal{F}$ is a set of 1-entries of the matrix that pairwise cannot be put together into the same combinatorial 1-rectangle. The maximum size of a fooling set consequently is a lower bound on the non-deterministic communication complexity for *Clique versus Independent Set* on 1-entries (up to a log application), and consequently on the size of a CS-separator.

**THEOREM 5.10**

*Let $n, m$ be positive integers. There exists a fooling set $\mathcal{F}$ of size $m$ on $\textsc{Cs}(G)$ for some graph $G$ on $n$ vertices if and only if $\mathbf{bp}_{or}(K_m) \leqslant n$.*

The proof of this theorem is very close to the proof of Theorem 5.7. In the same fashion, we decompose it into two lemmas.

**LEMMA 5.11**

> *Let $n, m$ be positive integers. If there exists a fooling set $\mathcal{F}$ of size $m$ on $\mathrm{CS}(G)$ for some graph $G$ on $n$ vertices then $\mathbf{bp}_{or}(K_m) \leqslant n$.*

*Proof.* Consider all pairs $(K, S)$ of cliques and stable sets of $G$ in the fooling set $\mathcal{F}$, and construct an auxiliary graph $H$ in the same way as in the proof of Lemma 5.8: the vertices of $H$ are the $m$ pairs $(K, S)$ of the fooling set and there is an edge between $(K, S)$ and $(K', S')$ if and only if there is a vertex in $S \cap K'$ or in $S' \cap K$. By definition of a fooling set, $H$ is a complete graph. For $x \in V(G)$, let $(A_x, B_x)$ be the oriented bipartite subgraph of $H$ where $A_x$ is the set of pairs $(K, S)$ for which $x \in K$, and $B_x$ is the set of pairs $(K, S)$ for which $x \in S$. This defines an oriented biclique partition of size $n$ on $H$ : first of all, by definition of the edges, $(A_x, B_x)$ is complete. Moreover, every edge is covered by such an oriented bipartite subgraph: if $(K, S)(K', S') \in E(H)$ then there exists a vertex $x \in S \cap K'$ or $x \in S' \cap K$ thus the corresponding arc is in $(A_x, B_x)$. Finally, an arc $(K, S)(K', S')$ cannot appear in both $(A_x, B_x)$ and $(A_y, B_y)$ otherwise the stable set $S$ and the clique $K'$ intersect on two vertices $x$ and $y$, which is impossible. Hence $\mathbf{bp}_{or}(H) \leqslant n$. $H$ being a complete graph on $m$ vertices proves the lemma. $\square$

**LEMMA 5.12**

> *Let $n, m$ be positive integers. If $\mathbf{bp}_{or}(K_m) \leqslant n$ then there exists a fooling set of size $m$ on $\mathrm{CS}(G)$ for some graph $G$ on $n$ vertices.*

*Proof.* Construct an auxiliary graph $G$: the vertices are the elements of an oriented biclique partition of $K_m$ of size $n$, and there is an edge between $(A_1, B_1)$ and $(A_2, B_2)$ if and only if there is a vertex $x \in A_1 \cap A_2$. Then for all $x \in V(K_m)$, the set of all bipartite graphs $(A, B)$ with $x \in A$ form a clique called $K_x$, and the set of all bipartite graphs $(A, B)$ with $x \in B$ form a stable set called $S_x$. $S_x$ is indeed a stable set, otherwise there are $(A_1, B_1)$ and $(A_2, B_2)$ in $S_x$ (implying $x \in B_1 \cap B_2$) linked by an edge resulting from a vertex $y \in A_1 \cap A_2$, then the arc $yx$ is covered twice. Consider all pairs $(K_x, S_x)$ for $x \in V(K_m)$: this is a fooling set of size $m$. Indeed, on the one hand $K_x \cap S_x = \emptyset$. On the other hand, for all $x, y \in V(K_m)$, the edge $xy$ is covered by a complete bipartite graph $(A, B)$ with $x \in A$ and $y \in B$ (or conversely). Then $K_x$ and $S_y$ (or $K_y$ and $S_x$) intersects in $(A, B)$. $\square$

*Proof of Theorem 5.10.* Lemmas 5.11 and 5.12 conclude the proof. $\square$

In particular, Amano proved that $\mathbf{bp}_{or}(K_n) = \mathcal{O}(n^{1/2+\varepsilon})$ (for arbitrarily small $\varepsilon > 0$), which implies the existence of a fooling set of size $\Omega(|V(G)|^{2-\varepsilon'})$ for some graph $G$. This gives consequently a $\Omega(|V(G)|^{2-\varepsilon'})$ lower bound for the size of a CS-Separator. However this fooling set technique has arrived to its limit: indeed, the rank of $\mathrm{CS}(G)$ is at most $|V(G)| + 1$ since $\mathrm{CS}(G)_{K,S} = 1 - (\chi^K)^\top \chi^S$. Moreover it is known that, for every matrix $M$, the maximum size of a fooling set in $M$ is upper bounded by $\mathcal{O}(\mathrm{rk}(M)^2)$ [63]. Hence Amano's lower bound is almost optimal.

# 5.3 Generalization to $t$-biclique covering

This section is concerned with a natural generalization of the Alon-Saks-Seymour conjecture, studied by Huang and Sudakov in [122]. While the biclique partition number $\mathbf{bp}(G)$ deals with partitioning the edges, we weakened the condition in the previous Section in such a way that an edge could be covered twice (but not twice in the same direction). We relax here further in the undirected case to a covering of the edges by complete bipartite graphs, meaning that an edge can be covered several times. Formally, a *$t$-biclique covering* of an undirected graph $G$ is a collection of complete bipartite graphs that covers every edge of $G$ at least once and at most $t$ times. The minimum size of such a covering is called the *$t$-biclique covering number*, and is denoted by $\mathbf{bp}_t(G)$. In particular, $\mathbf{bp}_1(G)$ is the usual biclique partition number $\mathbf{bp}(G)$.

In addition to being an interesting parameter to study in its own right (it has even be studied in the list version $\mathbf{bp}_L(G)$, where the number of times an edge can be covered is described by the list $L$ [50]), the $t$-biclique covering number of complete graphs is also closely related to a question in combinatorial geometry about neighborly families of boxes. It was studied by Zaks [213] and by Alon [4], who proved that $\mathbb{R}^d$ has a $t$-neighborly family of $k$ standard boxes if and only if the complete graph $K_k$ has a $t$-biclique covering of size $d$ (see [122] for definitions and further details). Alon also gives asymptotic bounds for $\mathbf{bp}_t(K_k)$, then slighty improved by Huang and Sudakov [122] (see the work by Cioabă and Tait for further investigation [50]):

$$(1+o(1))(t!/2^{t-1})^{1/t}k^{1/t} \leqslant \mathbf{bp}_t(K_k) \leqslant (1+o(1))tk^{1/t} \ .$$

Our results are concerned not only with $K_k$ but for every graph $G$. It is natural to ask the same question for $\mathbf{bp}_t(G)$ as for $\mathbf{bp}(G)$, namely *can we polynomially bound $\chi(G)$ in terms of $\mathbf{bp}_t(G)$, for every graph $G$?* Observe that a $t$-biclique covering is *a fortiori* a $t'$-biclique covering for all $t' \geqslant t$. Moreover, an oriented biclique partition of size $\mathbf{bp}_{or}(G)$, which covers each edge at most once in each direction can be seen as a non-oriented biclique covering which covers each edge at most twice. Hence, we have the following inequalities:

**Observation 5.13**

*For every graph $G$:*

$$\ldots \leqslant \mathbf{bp}_{t+1}(G) \leqslant \mathbf{bp}_t(G) \leqslant \ldots \leqslant \mathbf{bp}_2(G) \leqslant \mathbf{bp}_{or}(G) \leqslant \mathbf{bp}(G) \ .$$

Observation 5.13 and bounds on $\mathbf{bp}_2(K_n)$ [4] give the following lower bound: $\mathbf{bp}_{or}(K_n) \geqslant \mathbf{bp}_2(K_n) \geqslant \Omega(\sqrt{n})$. This bound together with Theorem 5.10 show, as discussed at the end of previous section, the almost optimality of Amano's $\Omega(n^{2-\varepsilon})$ lower bound for the size of a fooling set on a graph on $n$ vertices.

We prove that finding a universal polynomial bound for $\chi(G)$ in terms of $\mathbf{bp}(G)$ or in terms $\mathbf{bp}_t(G)$ comes in fact to the same thing:

**THEOREM 5.14**

*Let $t$ be a positive integer. The following are equivalent:*

- *There exist two postive constants $\lambda$ and $c$ such that for every graph $G$, $\chi(G) \leqslant \lambda \cdot \mathbf{bp}(G)^c$.*
- *There exist two positive constants $\lambda_t$ and $c_t$ such that for every graph $G$, $\chi(G) \leqslant \lambda_t \cdot \mathbf{bp}_t(G)^{c_t}$.*

*Proof.* Assume that the second item is true. Then $\chi(G) \leqslant \lambda_t \cdot \mathbf{bp}_t(G)^{c_t}$ and, according to Observation 5.13, $\mathbf{bp}_t(G) \leqslant \mathbf{bp}(G)$ so $\chi(G) \leqslant \lambda_t \cdot \mathbf{bp}(G)^{c_t}$. Hence the first item holds.

Now we focus on the other direction, and assume that the first item holds, *i.e* $\chi(G) \leqslant \lambda \cdot \mathbf{bp}(G)^c$ for every $G$, for some $\lambda$, $c > 0$. Let us prove the existence of the constants $\lambda_t$ and $c_t$ by induction on $t$, initialization for $t = 1$ being obvious. Let $G = (V, E)$ be a graph and let $\mathcal{B} = (B_1, ..., B_k)$ be a $t$-biclique covering. Then $E$ can be partitioned into $E_t$ the set of edges that are covered exactly $t$ times in $\mathcal{B}$, and $E_{<t}$ the set of edges that are covered at most $t-1$ times in $\mathcal{B}$. Construct an auxiliary graph $H$ with the same vertex set $V$ as $G$ and with edge set $E_t$.

**LEMMA 5.15**

$\mathbf{bp}(H) \leqslant (2k)^t$.

The proof of Lemma 5.15 is postponed at the end of the current proof. By assumption, $\chi(H) \leqslant \lambda \cdot \mathbf{bp}(H)^c$ thus by Lemma 5.15, $\chi(H) \leqslant \lambda \cdot (2k)^{ct}$. Consequently $V$ can be partitioned into $(S_1, \ldots, S_{\lambda(2k)^{ct}})$ where $S_i$ is a stable set in $H$. In particular, the induced graph $G[S_i]$ contains no edge of $E_t$. Consequently $(B_1 \cap S_i, \ldots, B_k \cap S_i)$ is a $(t-1)$-biclique covering of $G[S_i]$, where $B_j \cap S_i$ is the bipartite graph $B_j$ restricted to the vertices of $S_i$. Thus $\mathbf{bp}_{t-1}(G[S_i]) \leqslant k$. By the induction hypothesis, there exists two constants $\lambda_{t-1}, c_{t-1} > 0$ such that $\chi(G[S_i]) \leqslant \lambda_{t-1} \cdot \mathbf{bp}_{t-1}(G[S_i])^{c_{t-1}} \leqslant \lambda_{t-1} \cdot k^{c_{t-1}}$. Let us now color the vertices of $G$ with at most $(\lambda(2k)^{ct}) \cdot (\lambda_{t-1}k^{c_{t-1}}) = \lambda_t \cdot \mathbf{bp}(G)^{c_t}$ with $\lambda_t = \lambda\lambda_{t-1} \cdot 2^{ct}$ and $c_t = ct + c_{t-1}$. Each vertex $v \in S_i$ is given color $(\alpha, \beta)$, where $\alpha$ is the color of $S_i$ in $H$ and $\beta$ is the color of $x$ in $G[S_i]$. This is a proper coloring of $G$, thus $\chi(G) \leqslant P_t(\mathbf{bp}(G))$: the second item holds. □

*Proof of Lemma 5.15.* For each $B_i$, let $(B_i^-, B_i^+)$ be its partition into a complete bipartite graph. We number $x_1, \ldots, x_n$ the vertices of $H$. Let $x_i x_j$ be an edge, with $i < j$, then $x_i x_j$ is covered by exactly $t$ bipartite graphs $B_{i_1}, \ldots, B_{i_t}$. We give to this edge the label $((B_{i_1}, \ldots, B_{i_t}), (\varepsilon_1, \ldots, \varepsilon_t))$, where $\varepsilon_l = -1$ if $x_i \in B_{i_l}^-$ (then $x_j \in B_{i_l}^+$) and $\varepsilon_l = +1$ otherwise (then $x_i \in B_{i_l}^+$ and $x_j \in B_{i_l}^-$). For each such label $\mathcal{L}$ appearing in $H$, call $E_{\mathcal{L}}$ the set of edges labeled by $\mathcal{L}$ and define a set of edges $B_{\mathcal{L}} = E(B_{i_1}) \cap E_{\mathcal{L}}$. Observe that $B_{\mathcal{L}}$ forms a bipartite graph. The goal is to prove that the set of every $B_{\mathcal{L}}$ is a 1-biclique covering of $H$. Since there can be at most $(2k)^t$ different labels, this will conclude the proof.

Let us first observe that each edge appears in exactly one $B_{\mathcal{L}}$ because each edge has exactly one label. Let $\mathcal{L}$ be a label, and let us prove that $B_{\mathcal{L}}$ is a complete bipartite graph. If $x_i x_{i'} \in B_{\mathcal{L}}$ and $x_j x_{j'} \in B_{\mathcal{L}}$, with $i < i'$ and $j < j'$ then these two edges have the same label $\mathcal{L} = ((B_{i_1}, \dots, B_{i_t}), (\varepsilon_1, \dots, \varepsilon_t))$. If $\varepsilon_l = -1$ (the other case in handle symmetrically), then $x_i$ and $x_j$ are in $B_{i_l}^-$ and $x_{i'}$ and $x_{j'}$ are in $B_{i_l}^+$. As $B_{i_l}$ is a complete bipartite graph, then the edges $x_i x_{j'}$ and $x_j x_{i'}$ appear in $E(B_{i_l})$. Thus these two edges have also the label $\mathcal{L}$, so they are in $B_{\mathcal{L}}$: as a conclusion, $B_{\mathcal{L}}$ is a complete bipartite graph. □

We can conclude this chapter by the following corollary:

**THEOREM 5.16**

*The following are equivalent:*

- *There exists two positive constants $\lambda$, $c$ such that every graph $G$ admits a CS-Separator of size $\lambda \cdot |V(G)|^c$.*
- *There exists two positive constants $\lambda'$, $c'$ such that $\chi(H) \leqslant \lambda' \cdot \mathbf{bp}(H)^{c'}$ for every graph $H$.*

*Proof.* Assume that the first item is true. Then by Theorem 5.7, for every graph $H$ we have $\chi(H) \leqslant \lambda \cdot \mathbf{bp}_{or}(H)^c \leqslant \lambda \cdot \mathbf{bp}(H)^c$ where the second inequality is obtained by Observation 5.13: consequently the second item holds.

Assume that the second item is true. By Theorem 5.14, there exist two positive constants $\lambda_2$ and $c_2$ such that for every graph $H'$, we have $\chi(H') \leqslant \lambda_2 \cdot \mathbf{bp}_2(H')^{c_2}$ Since $\mathbf{bp}_2(H') \leqslant \mathbf{bp}_{or}(H')$, we have $\chi(H') \leqslant \lambda_2 \cdot \mathbf{bp}_{or}(H')^{c_2}$. By Theorem 5.7, every graph $G$ admits a CS-Separator of size $\lambda_2 \cdot |V(G)|^{c_2}$. □

As already mentioned, Göös proved very recently [108] that the first item in the statement of Theorem 5.16 is false, consequently both items are false: there is a polynomial upper bound neither for the general case of the CS-Separation nor for the generalized Alon-Saks-Seymour conjecture.

# Chapter 6

# Constraint Satisfaction Problems

We now enter the world of *Constraint Satisfaction Problems*. We first define the general setting and give some examples of combinatorial problems that can be modeled in such a way. We then briefly survey the major results in this field and state the well-known *Dichotomy Conjecture*. In a second time, we focus in particular on one Constraint Satisfaction Problem called the *Stubborn Problem* that has raised a lot of interest in the past few years because it was a big open question to know whether it was polynomial-time solvable or not. A quite classical approach for this type of problem consists in reducing it to several instances of 2-SAT. It is known that we can reduce the Stubborn Problem to a quasi-polynomial number of instances of 2-SAT, which gives a quasi-polynomial algorithm, thus it is a natural question to wonder whether we could do so with a polynomial number of them. Interestingly, we prove in Section 6.2 that this question is equivalent to finding a polynomial CS-Separator in every graph. The Stubborn Problem was finally proved to be polynomial-time solvable with a reduction to another problem called *3-Compatible-Coloring problem*, for which Cygan *et al.* designed a polynomial-time algorithm. That is why we also investigate the 3-Compatible-Coloring problem and also prove that reducing it to a polynomial number of 2-SAT instances is equivalent to finding polynomial CS-Separators for all graphs.

Note that the content of this chapter is covered in the following paper:

[III] **Clique versus Independent Set**, with N. Bousquet and S. Thomassé, *European Journal of Combinatorics*, 40:73–92, 2014.

Similarly to Chapter 5, it should be noticed that this work is prior to Göös' result providing a superpolynomial lower bound for the general case of the Clique-Stable set Separation [108]. Consequently, this chapter is written with the point of view of establishing the equivalence between two questions that were opened at that time. Nonetheless, we mention the consequences of Göös' result when appropriate.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 8 | | | | | | | |
| | 9 | 3 | | | 5 | | | 4 |
| 4 | | | | | | 5 | 1 | |
| | 1 | 9 | 4 | | | 2 | 3 | 6 |
| | | | 7 | 2 | 9 | | | |
| 8 | 2 | 4 | | | 3 | 9 | 7 | |
| | 4 | 6 | | | | | | 8 |
| 3 | | | 9 | | | 1 | 6 | |
| | | | | | | | 4 | |

FIGURE 6.1: A Sudoku grid.

# 6.1 Context

Let us introduce *Constraint Satisfaction Problem* with an example. Consider a Sudoku grid (see Figure 6.1), which is a $9 \times 9$ square grid where a few entries are pre-filled with digits from 1 to 9. The goal is to fill all blanks such that each row, each column, and each framed $3 \times 3$ square subgrid contains every digit exactly once. This very common game can be model in the following way: consider the set $X = \{x_{i,j} \mid 1 \leqslant i, j, \leqslant 9\}$ of variables and the domain $D = \{1, \ldots, 9\}$. The goal is to assign to each variable $x_{i,j}$ (standing for the $(i, j)$-entry of the grid) a value taken in $D$ such that some constraints are satisfied. If we denote $\textsc{AllDifferent}_9 \subseteq \{1, \ldots, 9\}^9$ to be the set of vectors $x$ such that the nine coordinates of $x$ are all different, the constraint can be expressed as follows: for each row $i$, $(x_{i,1}, \ldots, x_{i,9}) \in \textsc{AllDifferent}_9$; for each column $j$, $(x_{1,j}, \ldots, x_{9,j}) \in \textsc{AllDifferent}_9$; for each framed $3 \times 3$ square $S$, the vector $x$ containing the 9 variables $x_{i,j}$ with $(i, j) \in S$ satisfies $x \in \textsc{AllDifferent}_9$. Moreover, we cannot change the value of a pre-filled entry, so we have one more set of constraints: for each $(i, j)$-entry pre-filled with digit $k$, $x_{i,j} \in \{k\}$ *i.e.* $x_{i,j} = k$.
Formally, given a set $D$, a *k-ary relation* $R$ over $D$ is a subset of $D^k$. Let $\Gamma$ be a set of relations over $D$, then the *Constraint Satisfaction Problem* associated to $\Gamma$, denoted CSP($\Gamma$), is described as follows (see the dedicated book [103]): an instance of CSP($\Gamma$) is given by

- A set of variables $X = \{x_1, \ldots, x_n\}$,
- For each $1 \leqslant i \leqslant n$, a domain $D_i \subseteq D$ of values that can be given to variable $x_i$,
- And a set $C = \{C_1, \ldots, C_m\}$ of constraints where constraint $C_j$ is given by an ordered set of $k$ variables $t_j = (x_{i_1}, \ldots, x_{i_k})$ and a $k$-ary relation $R \in \Gamma$.

A *solution* to this instance is an assignment $s$ of values to the variables such that $s(x_i) \in D_i$, and for each constraint $C_j$, the vector $s(t_j) = (s(x_{i_1}), \ldots, s(x_{i_k}))$ belongs to $R$. If the instance has a solution, then it is said to be *satisfiable*. The goal of CSP($\Gamma$) is to compute a solution to the given instance, and the decision problem associated to it is the problem of computing whether the instance is satisfiable or not.

CSP can model a great amount of problems, both practical and theoretical ones: on the one hand, they are widely used in operational research and artificial intelligence; on the other hand, many combinatorial problems can be modeled with a CSP, for instance the $k$-coloring problem for any integer $k$: given a graph $G = (V, E)$, the decision problem of knowing whether $\chi(G) \leqslant k$ can be modeled by a CSP as follows:

- The set of variables $X$ contains one variable $x_v$ for each vertex $v \in V$.
- The domain $D_i$ of $x_i$ is the same for all variables, it is just the set of colors $D = \{1, \ldots, k\}$.
- For each edge $uv \in E$, there is one constraint $(x_u, x_v) \in \textsc{NotEqual}$ where $\textsc{NotEqual} \subseteq D^2$ contains all pairs of different integers in $D$.

Every instance of $k$-coloring can thus be transformed into an equivalent instance of CSP($\textsc{NotEqual}$). In particular, CSP can model NP-complete problems so there is no hope to find a polynomial-time algorithm to solve CSP($\Gamma$) for all set of relations $\Gamma$ (unless P=NP). However, one could wonder whether all CSP that are not polynomial-time solvable are NP-complete. This is a famous conjecture by Feder and Vardi stated in 1993:

**Conjecture 6.1** (The *Dichotomy conjecture*) [81]

*Let $\Gamma$ be a set of relations over a set $D$. Then CSP($\Gamma$) is either polynomial-time solvable or NP-complete.*

If the conjecture is true, then CSP provide one of the largest known subsets of NP which avoids NP-intermediate problems, whose existence was demonstrated by Ladner's theorem [145] under the assumption that $P \neq NP$. To attack this conjecture, one could try to restrict oneself to specific classes of CSP. In particular, Schaeffer managed to prove the conjecture for every *Boolean CSP*, where *Boolean* means that the domain $D_i$ is $\{0, 1\}$ for every variable $x_i$:

**Theorem 6.2** [180]

*Let $\Gamma$ be a set of relations over $\{0, 1\}$. Then CSP($\Gamma$) is either polynomial-time solvable or NP-complete, and we can decide in polynomial-time which case occurs.*

This result was generalized in 2002 by Bulatov [25] for a three-element domain instead of the Boolean domain. Another generalization of Schaefer's dichotomy theorem was proved in 2013 by Bodirsky and Pinsker [13] who proved it for a larger class of relations called *propositional logic of graphs*, where basically $\Gamma$ is assumed to contain only Boolean combinations of an antireflexive symmetric binary relation $E$, which thus stands for an *edge* relation. Besides, several results give sufficient algebraic or combinatorial conditions for CSP($\Gamma$) to be tractable, for instance when the hypergraph of constraints has bounded treewidth [92] (see [103] for more examples).

## 6.2 The Stubborn Problem and its link with the CS-Separation

Let us now focus on some specific classes of CSP called *list-M partition problems*, which have been widely studied in the last decades (see [181] for an overview). $M$ stands for a fixed $k \times k$ symmetric matrix filled with $0, 1$ and $*$. The input is a graph $G = (V, E)$ together with a list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, \dots, A_k\})$ and the question is to determine whether the vertices of $G$ can be partitioned into $k$ sets $A_1, \dots, A_k$ respecting two types of requirements. The first one is given by the list assignments, that is to say $v$ can be put in $A_i$ only if $A_i \in \mathcal{L}(v)$. The second one is described in $M$, namely: if $M_{i,i} = 0$ (resp. $M_{i,i} = 1$), then $A_i$ is a stable set (resp. a clique), and if $M_{i,j} = 0$ (resp. $M_{i,j} = 1$), then $A_i$ is complete (resp. anticomplete) to $A_j$. If $M_{i,i} = *$ (resp. $M_{i,j} = *$), then $A_i$ can be any set (resp. $A_i$ and $A_j$ can have any kind of adjacency).

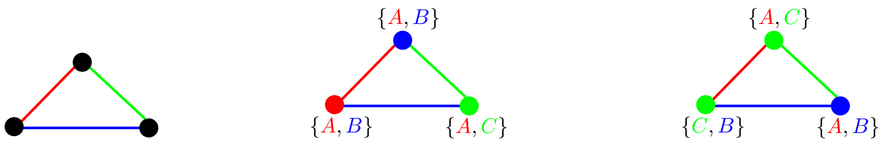
Those list-$M$ partition problems were studied by Feder et al. [78, 80] who proved in 2003 a *quasi-dichotomy theorem*: each of them is either NP-complete or quasi-polynomial time solvable (i.e. time $\mathcal{O}(n^{c \log n})$ where $c$ is a constant). Moreover, many investigations have been made about small matrices $M$ ($k \leqslant 4$) to get a *dichotomy theorem*, *i.e.* a special case of Conjecture 6.1. Cameron et al. [26] reached in 2007 such a dichotomy for $k \leqslant 4$, except for one special case (and its complement) then called the *Stubborn Problem* which remained only quasi-polynomial time solvable. The matrix of the Stubborn Problem is the following:

$$
\begin{array}{c c}
 & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 1 \end{pmatrix}
\end{array}
$$

In other words, the Stubborn Problem can be stated as follows:

STUBBORN PROBLEM
**Input:** A graph $G = (V, E)$ and a list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$.
**Question:** Can $V$ be partitioned into four sets $A_1, \dots, A_4$ such that $A_4$ is a clique, both $A_1$ and $A_2$ are stable sets, $A_1$ is anticomplete to $A_3$, and the partition is compatible with $\mathcal{L}$?

Cygan *et al.* [59] closed the complexity question by finding a polynomial time algorithm solving the Stubborn Problem. More precisely, the Stubborn Problem was reduced by Feder and Hell [77] in 2006 to another problem that they call 3-COMPATIBLE COLORING, for which Cygan *et al.* designed a polynomial time algorithm. Note that 3-COMPATIBLE COLORING has also been introduced and studied in [142] under the name ADAPTED LIST COLORING, and was proved to be a model for some strong scheduling problems. It is defined in the following way:




(a) An instance of 3-CCP

(b) The vertex coloring is a solution to the instance. The set of 2-constraints on each vertex is a 2-list assignment compatible with the solution.

(c) Another solution to the instance with a compatible 2-list assignment.

FIGURE 6.2: Illustration of the definitions relating to 3-CCP. Color are named as follows: A=red; B=blue; C=green. Both 2-list assignments together form a 2-list covering because any solution is compatible with at least one of them.

3-COMPATIBLE COLORING PROBLEM (3-CCP)
**Input:** An edge coloring $f_E$ of the complete graph on $n$ vertices with 3 colors $\{A, B, C\}$.
**Question:** Is there a coloring of the vertices with $\{A, B, C\}$, such that no edge has the same color as both its endpoints?

We are now going to highlight the links between the Clique-Stable Set separation problem and both the Stubborn Problem and 3-CCP, some of which were already observed in [80]. The quasi-dichotomy theorem for list-$M$ partitions proceeds by covering all the solutions by $\mathcal{O}(n^{\log n})$ particular instances of 2-SAT, called 2-list assignments. A natural extension would be a covering of all the solutions with a polynomial number of 2-list assignments. We prove that the existence of a polynomial covering of all the maximal solutions (to be defined later) for the Stubborn Problem is equivalent to the existence of such a covering for all the solutions of 3-CCP, which in turn is equivalent to the CS-Separation.

The following definitions are illustrated on Figure 6.2 and deal with list coloring. Let $G$ be a graph and COL a set of $k$ colors. A set of possible colors, called *constraint*, is associated to each vertex. If the set of possible colors is COL then the constraint on this vertex is *trivial*. A vertex has an *$l$-constraint* if its set of possible colors has size at most $l$. An *$l$-list assignment* is a function $\mathcal{L} : V \to \mathcal{P}(\text{COL})$ that gives each vertex an $l$-constraint. A solution $\mathcal{S}$ is a coloring of the vertices $\mathcal{S} : V \to \text{COL}$ that respects some requirements depending on the problem. We can equivalently consider $\mathcal{S}$ as a partition $(A_1, \ldots, A_k)$ of the vertices of the graph with $x \in A_i$ if and only if $\mathcal{S}(x) = A_i$ (by abuse of notation $A_i$ denotes both the color and the set of vertices having this color). An $l$-list assignment $\mathcal{L}$ is *compatible* with a solution $\mathcal{S}$ if for each vertex $x$, $\mathcal{S}(x) \in \mathcal{L}(x)$. A set of $l$-list assignment *covers* a solution $\mathcal{S}$ if at least one of the $l$-list assignment is compatible with $\mathcal{S}$.

Given an edge coloring $f_E$ on $K_n$, a set of 2-list assignments is a *2-list covering for 3-CCP on $(K_n, f_E)$* if it covers all the solutions of 3-CCP on this instance. Its *size* is the number of 2-list assignments it contains.

**Observation 6.3**

> *Given a 2-list assignment for 3-CCP, it is possible to decide in polynomial time if there exists a solution covered by it.*

*Proof.* Any 2-list assignment can be translated into an instance of 2-SAT. Each vertex has a 2-constraint $\{\alpha, \beta\}$ from which we construct two variables $x_\alpha$ and $x_\beta$ and a clause $x_\alpha \vee x_\beta$. Turn $x_\alpha$ to true will mean that $x$ is given color $\alpha$. Then we also need the clause $\neg x_\alpha \vee \neg x_\beta$ saying that only one color can be given to $x$. Finally for all edge $xy$ colored with $\alpha$, we add the clause $\neg x_\alpha \vee \neg y_\alpha$ if both variables exists, and no clause otherwise. □

Therefore, given a polynomial 2-list covering, it is possible to decide in polynomial time if the instance of 3-CCP has a solution. Observe nevertheless that the existence of a polynomial 2-list covering does not imply the existence of a polynomial algorithm. Indeed, such a 2-list covering may not be computable in polynomial time. Feder and Hell proved an intermediate result towards this attempt of polynomial 2-list covering:

**Theorem 6.4** [77]

> *There exists an algorithm giving a 2-list covering of size $O(n^{\log n})$ for 3-CCP. By Observation 6.3, this gives an algorithm in time $O(n^{\log n})$ for 3-CCP.*

Symmetrically, we want to define a *2-list covering for the Stubborn Problem*. However, there is no hope to cover all the solutions of the Stubborn Problem on each instance with a polynomial number of 2-list assignments. Indeed if $G$ is a stable set of size $n$ and if every vertex has the trivial 4-constraint, then for any partition of the vertices into 3 sets $(A_1, A_2, A_3)$, there is a solution $(A_1, A_2, A_3, \emptyset)$. Since there are $3^n$ partitions into 3 sets, and since every 2-list assignment covers at most $2^n$ solutions, all solutions cannot be covered with a polynomial number of 2-list assignments.

Thus we need a notion of maximal solutions. This notion is extracted from the notion of domination coming from the language of general list-$M$ partition problem (see [80]). Intuitively, if $\mathcal{L}(v)$ contains both $A_1$ and $A_3$ and $v$ belongs to $A_1$ in some solution $\mathcal{S}$, we can build a simpler solution by putting $v$ in $A_3$ and leaving everything else unchanged. A solution $(A_1, A_2, A_3, A_4)$ of the Stubborn Problem on $(G, \mathcal{L})$ is a *maximal solution* if no member of $A_1$ satisfies $A_3 \in \mathcal{L}(v)$. We say that $A_1$ *dominates* $A_3$. We may note that if $A_3$ is contained in every $\mathcal{L}(v)$ for $v \in V$, then every maximal solution of the Stubborn Problem on $(G, \mathcal{L})$ let $A_1$ empty. Now, a set of 2-list assignments is a *2-list covering for the Stubborn Problem on $(G, \mathcal{L})$* if it covers all the maximal solutions on this instance. Its *size* is the number of 2-list assignments it contains.

For edge-colored graphs, an *$(\alpha_1, ..., \alpha_k)$-clique* is a clique for which every edge has a color in $\{\alpha_1, ..., \alpha_k\}$. A *split* graph in this context is a graph in which vertices can be partitioned into an $\alpha$-clique and a $\beta$-clique. The *$\alpha$-edge-neighborhood* of $x$ is the set of vertices $y$ such that $xy$ is an *$\alpha$-edge*, i.e an edge colored with $\alpha$. The *majority color of $x \in V$* is the color $\alpha$ for which the $\alpha$-edge-neighborhood of $x$ is maximal in terms of cardinality (in case of ties, we arbitrarily cut them).

This section is devoted to the proof of the following result:

**THEOREM 6.5**

*The following are equivalent:*

(i) *There exists a polynomial $P$ such that for every graph $G$ and every list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a 2-list covering of size $P(|V(G)|$ for the Stubborn Problem on $(G, \mathcal{L})$.*

(ii) *There exists a polynomial $P'$ such that for every $n$ and every edge coloring $f : E(K_n) \to \{A, B, C\}$, there is a 2-list covering of size $P'(n)$ for 3-CCP on $(K_n, f)$.*

(iii) *There exists a polynomial $P''$ such that every graph $G$ has a CS-separator of size $P''(|V(G)|)$.*

As already mentioned, Göös proved very recently [108] the existence of a family of graphs admitting no polynomial-size CS-Separator. Consequently, this disproves all three statements in the theorem above.

We decompose the proof into three lemmas, each of which describing one implication.

**LEMMA 6.6**

*$(i) \Rightarrow (ii)$: Suppose that there exists a polynomial $P$ such that for every graph $G$ and every list assignment $\mathcal{L} : V \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a 2-list covering of size $P(|V(G)|)$ for the Stubborn Problem on $(G, \mathcal{L})$. Then for every integer $n$ and every edge coloring $f_E : E(K_n) \to \{A, B, C\}$, there is a 2-list covering of size $P'(n) = 3(P(n))^4$ for 3-CCP on $(K_n, f_E)$.*

*Proof.* Let $n \in \mathbb{N}$, $(K_n, f_E)$ be an instance of 3-CCP, and $x$ a vertex of $K_n$. The goal is to build a 2-list covering of size $P'(n)$ for 3-CCP on $(K_n, f_E)$, which we achieve by first producing $(P(n))^4$ 2-list assignments that cover all the solutions where $x$ is given color $A$, and conclude by symmetry on the colors. Let $(A, B, C)$ be a solution of 3-CCP where $x \in A$.

**OBSERVATION 6.7**

*Let $x$ be a vertex and $\alpha, \beta, \gamma$ be the three different colors. Let $U$ be the $\alpha$-edge-neighborhood of $x$. If there is a $\beta\gamma$-clique $Z$ of $U$ which is not split, then there is no solution where $x$ is colored with $\alpha$.*

*Proof.* Consider a solution in which $x$ is colored with $\alpha$. All the vertices of $Z$ are of color $\beta$ or $\gamma$ because they are in the $\alpha$-edge-neighborhood of $x$. The vertices of $Z$ colored with $\beta$ form a $\gamma$-clique, those colored by $\gamma$ form a $\beta$-clique. Hence $Z$ is split. □

A vertex $x$ is said *really 3-colorable* if for each color $\alpha$, every $\beta\gamma$-clique of the $\alpha$-edge-neighborhood of $x$ is a split graph. If a vertex $x$ is not really 3-colorable then there exists a color $\alpha$ such that no solution gives color $\alpha$ to $x$. Hence if the graph $K_n \setminus x$ has

| $f(v)$ | $f'(v)$ | $f''(v)$ |
|---|---|---|
| $A_2$ or $A_1, A_2$ | $*$ | $C$ |
| $A_3$ or $A_1, A_3$ | $*$ | $B, C$ |
| $A_4$ or $A_1, A_4$ | $*$ | $A$ |
| $A_2, A_4$ | $*$ | $A, C$ |
| $A_2, A_3$ | $*$ | $B, C$ |
| $A_3, A_4$ | $A'_2$ or $A'_1, A'_2$ | $B$ |
| $A_3, A_4$ | $A'_3$ or $A'_1, A'_3$ | $A, C$ |
| $A_3, A_4$ | $A'_4$ or $A'_1, A'_4$ | $C$ |
| $A_3, A_4$ | $A'_2, A'_4$ | $B, C$ |
| $A_3, A_4$ | $A'_2, A'_3$ | $A, B$ |
| $A_3, A_4$ | $A'_3, A'_4$ | $A, C$ |

FIGURE 6.3: This table describes the rules used in proof of lemma 6.6 to built a 2-list assignment $f''$ for 3-CCP from a pair $(f, f')$ of 2-list assignment for two instances of the Stubborn Problem. Symbol $*$ stands for any constraint. For simplicity, we write $X, Y$ (resp. $X$) instead of $\{X, Y\}$ (resp. $\{X\}$).

a 2-list covering of size $p$, the same holds for $K_n$ by assigning $\{\beta, \gamma\}$ to $x$ in each 2-list assignment.

Thus we can assume that $x$ is really 3-colorable, because otherwise there is a natural 2-constraint on it. Since we assume that the color of $x$ is $A$, we can consider that in all the following 2-list assignments, the constraint $\{B, C\}$ is given to the $A$-edge-neighborhood of $x$. Let us abuse notation and still denote by $(A, B, C)$ the partition of the $C$-edge-neighborhood of $x$, induced by the solution $(A, B, C)$. Since there exists a solution where $x$ is colored by $C$, and $C$ is a $AB$-clique, then Observation 6.7 ensures that $C$ is a split graph $C' \uplus C''$ with $C'$ a $B$-clique and $C''$ a $A$-clique. The situation is described in Figure 6.4(a). Let $H$ be the non-colored graph with vertex set the $C$-edge-neighborhood of $x$ and with edge set the union of $B$-edges and $C$-edges (see Figure 6.4(b)). Moreover, let $H'$ be the non-colored graph with vertex set the $C$-edge-neighborhood of $x$ and with edge set the $B$-edges (see Figure 6.4(c)). We consider $(H, \mathcal{L}_0)$ and $(H', \mathcal{L}_0)$ as two instances of the Stubborn Problem, where $\mathcal{L}_0$ is the trivial list assignment that gives each vertex the constraint $\{A_1, A_2, A_3, A_4\}$.

By assumption, there exists $\mathcal{F}$ (resp. $\mathcal{F}'$) a 2-list covering of size $P(|V(H)|)$ (resp. $P(|V(H')|)$) for the Stubborn Problem on $(H, \mathcal{L}_0)$ (resp. $(H', \mathcal{L}_0)$). We construct $\mathcal{F}''$ the set of all 2-list assignments $f''$ built from any pair of type $(f, f') \in \mathcal{F} \times \mathcal{F}'$ according to the rules described in Figure 6.3 (intuition for such rules is given in the next paragraph). $\mathcal{F}''$ aims at being a 2-list covering of size $\leqslant (P(n))^2$ for 3-CCP on the $C$-edge-neighborhood of $x$.

The following is illustrated on Figure 6.4(b) and 6.4(c). Let $\mathcal{S}$ be the partition defined by $A_1 = \emptyset$, $A_2 = C''$, $A_3 = B \cup C'$ and $A_4 = A$. We can check that $A_2$ is a stable set and $A_4$ is a clique (the others restrictions are trivially satisfied by $A_1$ being empty and $\mathcal{L}_0$ being trivial). In parallel, let $\mathcal{S}'$ be the partition defined by $A'_1 = \emptyset$,

$A'_2 = B$, $A'_3 = A \cup C''$ and $A_4 = C'$. We can also check that $A'_2$ is a stable set and $A'_4$ is a clique. Thus $\mathcal{S}$ (resp. $\mathcal{S}'$) is a maximal solution for the Stubborn Problem on $(H, \mathcal{L}_0)$ (resp. $(H', \mathcal{L}_0)$) inherited from the solution $(A, B, C = C' \uplus C'')$ for 3-CCP.

Let $f \in \mathcal{F}$ (resp. $f' \in \mathcal{F}'$) be a 2-list assignment compatible with $\mathcal{S}$ (resp. $\mathcal{S}'$). Then $f'' \in \mathcal{F}''$ built from $(f, f')$ is a 2-list assignment compatible with $(A, B, C)$.

Doing the same thing for the $B$-edge-neighborhood of $x$ gives also a set of at most $(P(n))^2$ 2-list assignments and then taking all possible combinations with the $C$-edge-neighborhood gives $(P(n))^4$ 2-list-assignments. Finally, pulling everything back together gives a 2-list covering of size $P'(n)$ for 3-CCP on $(K_n, f_E)$.

□

**LEMMA 6.8**

*$(ii) \Rightarrow (iii)$: Suppose that there exists a polynomial $P'$ such that for every $n$ and every edge coloring $f_E : E(K_n) \to \{A, B, C\}$, there is a 2-list covering of size $P'(n)$ for 3-CCP on $(K_n, f_E)$. Then for every graph $G$, there is a CS-separator of size $P''(|V(G)|)$ where $P''(x) = c \cdot xP'(x)$ for some constant $c > 0$.*

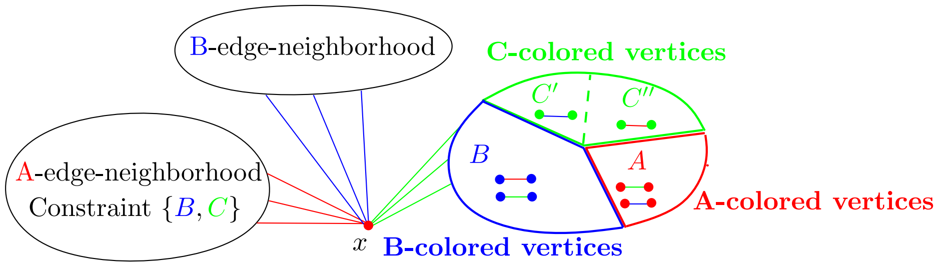
*Proof.* Let $G = (V, E)$ be a graph on $n$ vertices. Let $f_E$ be the coloring on $K_n$ defined by $f_E(e) = A$ if $e \in E$ and $f_E(e) = B$ otherwise. In the following $(K_n, f_E)$ is considered as a particular instance of 3-CCP with no $C$-edge. By hypothesis, there is a 2-list covering $\mathcal{F}$ for 3-CCP of size $P'(n)$ on $(K_n, f_E)$. Let us prove that we can derive from $\mathcal{F}$ a polynomial CS-separator $\mathcal{C}$.

Let $\mathcal{L} \in \mathcal{F}$ be a 2-list assignment. Denote by $X$ (resp. $Y$, $Z$) the set of vertices with the constraint $\{A, B\}$ (resp. $\{B, C\}$, $\{A, C\}$). Since no edge has color $C$, $X$ is split. Indeed, the vertices of color $A$ form a $B$-clique and conversely. Given a graph, there is a linear number of decompositions into a split graph [80]. Thus there are a linear number $c \cdot n$ (for some $c > 0$) of decomposition $(U_k, V_k)_{k \leqslant cn}$ of $X$ into a split graph where $U_k$ is a $B$-clique. For every $k$, the cut $(U_k \cup Y, V_k \cup Z)$ is added in $\mathcal{C}$. For each 2-list assignment we add a $c \cdot n$ cuts, so the size of $\mathcal{C}$ is at most $P''(n) = c \cdot nP'(n)$.

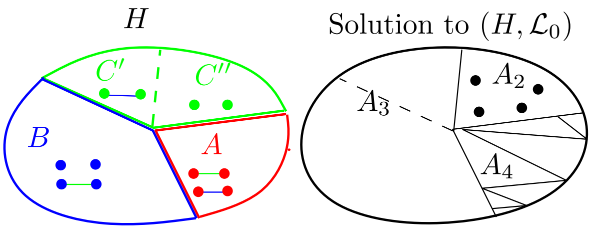
Let $K$ be a clique and $S$ a stable set of $G$ which do not intersect. The edges of $K$ are colored by $A$, and those of $S$ are colored by $B$. Then the coloring $\mathcal{S}(x) = B$ if $x \in K$, $\mathcal{S}(x) = A$ if $x \in S$ and $\mathcal{S}(x) = C$ otherwise is a solution of $(K_n, f_E)$. Left-hand side of Figure 6.5 illustrates the situation. There is a 2-list assignment $\mathcal{L}$ in $\mathcal{F}$ which is compatible with this solution. As before, let $X$ (resp. $Y$, $Z$) be the set of vertices which have the constraint $\{A, B\}$ (resp. $\{B, C\}$, $\{A, C\}$). Since the vertices of $K$ are colored $B$, we have $K \subseteq X \cup Y$ (see right-hand side of Figure 6.5). Likewise, $S \subseteq X \cup Z$. Then $(K \cap X, S \cap X)$ forms a split partition of $X$. So, by construction, there is a cut $((K \cap X) \cup Y, (S \cap X) \cup Z) \in \mathcal{C}$ which ensures that $(K, S)$ is separated by $\mathcal{C}$. □

**LEMMA 6.9**

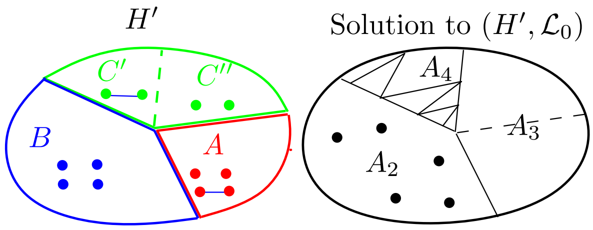
*$(iii) \Rightarrow (i)$: Suppose that there exists a polynomial $P''$ such that every graph $G$ on $n$ vertices admits a CS-separator of size $P''(n)$. Then for every graph $H$ and every list assignment $\mathcal{L} : V(H) \to \mathcal{P}(\{A_1, A_2, A_3, A_4\})$, there is a 2-list covering of size $P(|V(H)|)$ for the Stubborn Problem on $(H, \mathcal{L})$, where $P(x) = (P''(x))^2$.*




(a) Vertex $x$, its $A$-edge-neighborhood subject to the constraint $\{B, C\}$, and its $C$-edge-neighborhood separated in different parts.




(b) On the left, the graph $H$ obtained from the $C$-edge-neighborhood by keeping only $B$-edges and $C$-edges. On the right, the solution of the Stubborn Problem.




(c) On the left, the graph $H'$ obtained from the $C$-edge-neighborhood by keeping only $B$-edges. On the right,, the solution of the Stubborn Problem.

FIGURE 6.4: Illustration of the proof of Lemma 6.6. Color correspondence: A=red; B=blue; C=green. Cliques are represented by hatched sets, stable sets by dotted sets.

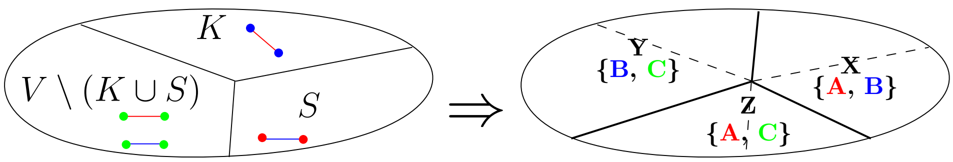



FIGURE 6.5: Illustration of the proof of Lemma 6.8. On the left-hand side, $G$ is separated in 3 parts: $K$, $S$, and the remaining vertices. Each possible configuration of edge- and vertex-coloring are represented. On the right-hand side, $(X, Y, Z)$ is a 2-list assignment compatible with the solution. $X$ (resp. $Y$, $Z$) has constraint $\{A, B\}$ (resp. $\{B, C\}$, $\{A, C\}$). Color correspondence: A=red ; B=blue ; C=green.

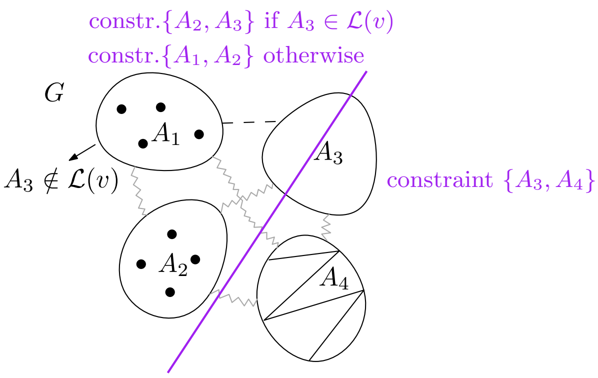



FIGURE 6.6: Illustration of the proof of Lemma 6.9. A solution to the Stubborn Problem together with the cut that separates $A_4$ from $A_1 \cup A_2$. The 2-list assignment built from this cut is indicated on each side.

*Proof.* Let $(G, \mathcal{L})$ be an instance of the Stubborn Problem. By assumption, there is a CS-separator of size $P''(n)$ for $G$ where $n = |V(G)|$.

**OBSERVATION 6.10**

*If there are $p$ cuts that separate all the cliques from the stable sets, then there are $p^2$ cuts that separate all the cliques from the unions $S \cup S'$ of two stable sets.*

*Proof.* Indeed, if $(V_1, V_2)$ separates $K$ from $S$ and $(V_1', V_2')$ separates $K$ from $S'$, then the new cut $(V_1 \cap V_1', V_2 \cup V_2')$ satisfies $K \subseteq V_1 \cap V_1'$ and $S \cup S' \subseteq V_2 \cup V_2'$. □

Let $\mathcal{F}_2$ be a family of $(P''(n))^2$ cuts that separate all the cliques from unions of two stable sets, which exists by Observation 6.10 and hypothesis. Then for every $(U, W) \in \mathcal{F}_2$, we build the following 2-list assignment $\mathcal{L}'$:

*(i)* If $v \in U$, let $\mathcal{L}'(v) = \{A_3, A_4\}$.

*(ii)* If $v \in W$ and $A_3 \in \mathcal{L}(v)$, then let $\mathcal{L}'(v) = \{A_2, A_3\}$.

*(iii)* Otherwise, $v \in W$ and $A_3 \notin \mathcal{L}(v)$, let $\mathcal{L}'(v) = \{A_1, A_2\}$.

Now the set $\mathcal{F}'$ of such 2-list assignment $\mathcal{L}'$ is a 2-list covering for the Stubborn Problem on $(G, \mathcal{L})$: let $\mathcal{S} = (A_1, A_2, A_3, A_4)$ be a maximal solution of the Stubborn Problem on this instance. Then $A_4$ is a clique and $A_1, A_2$ are stable sets, so there is a separator $(U, W) \in \mathcal{F}_2$ such that $A_4 \subseteq U$ and $A_1 \cup A_2 \subseteq W$ (see Figure 6.6), and there is a corresponding 2-list assignment $\mathcal{L}' \in \mathcal{F}'$. Consequently, the 2-constraint $\mathcal{L}'(v)$ built from rules 1 and 3 are compatible with $\mathcal{S}$. Finally, as $\mathcal{S}$ is maximal, there is no $v \in A_1$ such that $A_3 \in \mathcal{L}(v)$: the 2-constraints built from rule 2 are also compatible with $\mathcal{S}$. □

*Proof of Theorem 6.5.* Lemmas 6.6, 6.8 and 6.9 conclude the proof of Theorem 6.5. □

# Conclusion

THIS thesis was concerned with different types of interactions between cliques and stable sets. Let us sum up the content and the results of this thesis, and give some directions for further work.

In Chapter 1, we first surveyed the classical notions of perfect graphs and of $\chi$-bounded classes. We then proved the following coloring bound: *there exists a constant $c$ such that every triangle-free graph with no even hole of length at least 6 has chromatic number at most $c$*. The motivation for it was threefold: first, because of a famous series of conjectures by Gyárfás [113] of type *the class of graphs with no odd (resp. long, resp. long odd) hole is $\chi$-bounded*; second, because of recent work by Scott and Seymour [185] who proved the odd-hole-free case; finally, because of another recent work of Bonamy, Charbit and Thomassé who proved that forbidding induced cycles of length divisible by three induces a class of graphs with bounded chromatic number [17]. In addition to that, Scott and Seymour have just announced the following result: *for every $k$, there exists $c$ such that every triangle-free graph $G$ with $\chi(G) \geqslant c$ admits holes of $k$ consecutive lengths*. This breakthrough basically smashes any work of type *forbidding triangles and some hole lengths leads to a $\chi$-bounded class of graphs*. The next giant step would be to adapt their result to remove the triangle-free hypothesis.

Chapter 2 was devoted to the Erdős-Hajnal conjecture. We proved that *the class of $(P_k, \overline{P_k})$-free graphs has the Erdős-Hajnal property*, by showing that they fit in an even stronger framework, called the *Strong* Erdős-Hajnal property and introduced by Fox and Pach: instead of requiring a polynomial-size clique or stable set, it requires a linear-size biclique or complement biclique. We then developed new tools that may help to solve the conjecture for more classes of graphs; in the first place, we generalized the Strong Erdős-Hajnal property, and in the second place we generalized the degeneracy.

In Chapter 3, we introduced the Clique-Stable set separation and proved that *the following classes of graphs admit polynomial Clique-Stable set separators*: *random graphs*, *$H$-free graphs when $H$ is split*, *$(P_k, \overline{P_k})$-free graphs* and *perfect graphs with no balanced skew partition*. We also discussed similarities between tools for the CS-Separation and tools for the Erdős-Hajnal property (in particular the Strong Erdős-Hajnal property).

Despite these visible similarities, the behavior of the Erdős-Hajnal property and the Clique-Stable Set Separation differ for random graphs: they have a polynomial CS-Separator but they admit only logarithmic-size cliques and stable sets. We can also mention the case of $P_5$-free graphs and net-free graphs, for which the CS-Separation is proved but the Erdős-Hajnal property is still open. On the contrary, the Erdős-Hajnal property is trivially true for perfect graphs, whereas the CS-Separation in perfect graphs is a 20-year-old open problem from Yannakakis, at the very origin of the CS-Separation [211].

Similarly, we can focus on the relationships between $\chi$-boundedness on the one hand and both the CS-Separation and the Erdős-Hajnal property on the other hand. In this respect, the first example that comes to mind is triangle-free graphs: both the Erdős-Hajnal property and the CS-Separation are obviously true, but $\chi$ is known to be unbounded. In the other direction, as already pointed out in Observation 2.5, a polynomial $\chi$-bounding function trivially implies the Erdős-Hajnal property. However, there is no known implication on the CS-Separation. Moreover, a superpolynomial $\chi$-bounding function gives no further clue: for instance, Gyárfás' classical $\chi$-boundedness result on $P_k$-free graphs (Theorem 1.10) could not be extended so far to prove the Erdős-Hajnal property for $k \geqslant 5$ nor a polynomial CS-Separation for $k \geqslant 6$. Nonetheless, the proof method still led us to the aforementioned result on $(P_k, \overline{P_k})$-free graphs (Theorem 2.17), which is a valuable partial result.

Chapters 2 and 3 open several directions for further work since both the Erdős-Hajnal conjecture and the CS-Separation still have many open cases worthy of consideration. Proving the Erdős-Hajnal property for $P_5$-free graph is a natural goal to aim at, and is enforced by the positive results on this class for the Maximum Weighted Stable Set problem [149] and its consequence on the CS-Separation. As for 6-vertex graphs, it would be interesting to study the Erdős-Hajnal property for net-free graphs, especially because it is a split graph and thus the CS-Separation is known in this case. Concerning the CS-Separation, there also are a lot of open questions. The recent superpolynomial lower bound of Göös provides a breakthrough and we can wonder how far this lower bound can be improved. On the other hand, it would be interesting to find polynomial upper bounds for another class of graphs; maybe the next natural classes to study are $P_6$-free graphs or $H$-free graphs when $H$ is a subdivision of a claw. By mimicking the Erdős-Hajnal conjecture, we can even hope that $H$-free graphs admits polynomial CS-Separators for *every* $H$. However, even perfect graphs is still a big open problem. Hopefully new attempts on various classes of graphs can give rise to useful methods to solve the perfect graphs case.

In Chapter 4, we explained the connection between the CS-Separation and the extension complexity of the Stable set Polytope, which was Yannakakis' initial motivation for introducing the CS-Separation problem: indeed, a lower bound on the CS-Separation immediately gives a lower bound on the extension complexity of the Stable set Polytope of perfect graphs. However, the results we obtained only give upper bounds, so there is one question: *can we still extend some of these CS-Separation results to bound the extension complexity?* This is a new research direction, for which we started to obtain some nice and promising observations, which we plan to develop in further work.

Chapter 5 was dedicated to the link between the CS-Separation and the Alon-Saks-Seymour conjecture. More precisely, we introduced an oriented version of the biclique partition number, which we denote $\mathbf{bp}_{or}$, and we proved that *there exists a CS-Separator of size $f(|V(G)|)$ for every graph $G$ if and only if the inequality $\chi(H) \leqslant f(\mathbf{bp}_{or}(H))$ holds for every graph $H$*. In other words, the maximum gap between $|V(G)|$ and the minimum size of a CS-Separator is the same as the gap between $\chi(H)$ and $\mathbf{bp}_{or}(H)$. We then went further into this question and investigated on the $t$-biclique covering number $\mathbf{bp}_t$, where the edges are allowed to be covered at most $t$ times. Here the result on the gap is not as precise as in the oriented case, but we still showed that *there exists a polynomial CS-Separator for every graph if and only if $\chi$ is polynomially upper bounded in terms of $\mathbf{bp}_t$ for every graph*, and this holds for all $t$. In particular, since $\mathbf{bp}_1$ is the standard biclique partition number $\mathbf{bp}$, we reached our initial goal concerning the equivalence between the polynomial relaxation of the Alon-Saks-Seymour conjecture and a general polynomial upper bound for the CS-Separation. Although the Alon-Saks-Seymour attracts a little less attention since it was disproved in 2012 [122], the oriented version was already used by Amano and Shigeta [190] to give a lower bound on the CS-Separation. Moreover, the equivalence result presented here is surely useful when combined with Göös' superpolynomial lower bound for the CS-Separation, proved in the meantime.

Likewise, Chapter 6 showed the equivalence between the CS-Separation and 2-list-covering for the Stubborn Problem and 3-CCP. More precisely, we showed that *there exists a polynomial CS-Separator for every graph if and only if there exists a polynomial 2-list covering for every instance of the Stubborn Problem if and only if there exists a polynomial 2-list covering for every instance of 3-CCP*. Although the Stubborn Problem was proved to be polynomially solvable without using 2-list-covering [59], this is still a quite natural method for solving 3-SAT-like problems. Moreover, Göös' result can be combined once again with this equivalence result to give negative answers to all three questions.

# Contributions

The results presented in this thesis gave rise to the following research articles, submitted or published in specialized journals:

[I] **Coloring graphs with no even holes $\geqslant 6$: the triangle-free case**, preprint on *ArXiv*, *Computer Science > Discrete Mathematics*, abs/1503.08057, Submitted for publication.

[II] **The Erdős-Hajnal Conjecture for Paths and Antipaths**, with N. BOUSQUET and S. THOMASSÉ, *Journal of Combinatorial Theory, Series B*, 113:261–264, 2015.

[III] **Clique versus Independent Set**, with N. BOUSQUET and S. THOMASSÉ, *European Journal of Combinatorics*, 40:73–92, 2014.

[IV] **Clique- Stable set separation in perfect graphs with no balanced skew-partition**, with T. TRUNCK, preprint on *ArXiv*, *Computer Science > Discrete Mathematics*, abs/1312.2730, Submitted for publication.

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