Extended formulations of polytopes and Communication complexity

Aurélie Lagoutte

LIP, ENS Lyon

Joint work with N. Bousquet, S. Thomassé et T. Trunck

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São Paulo Workshop

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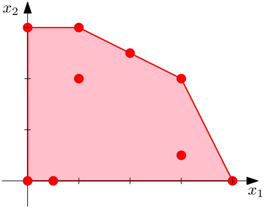

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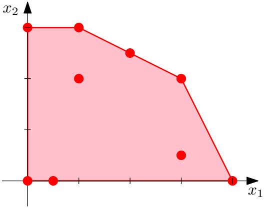

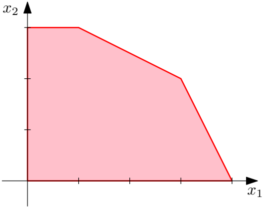

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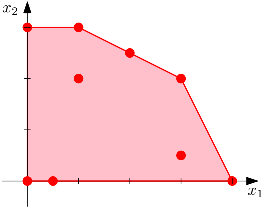

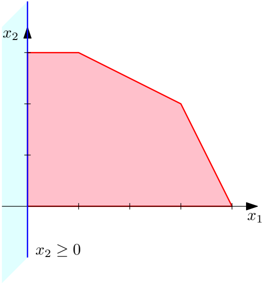

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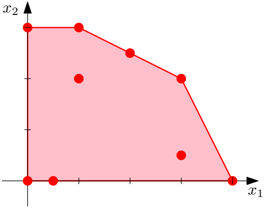

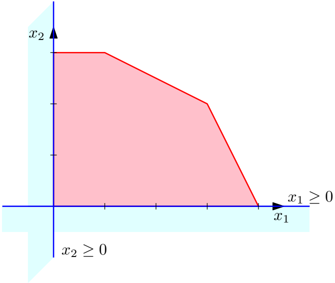

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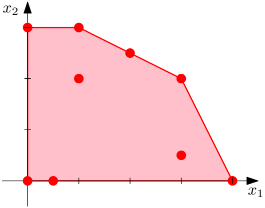

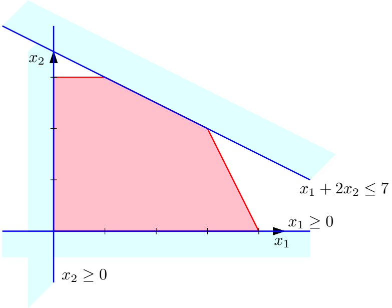

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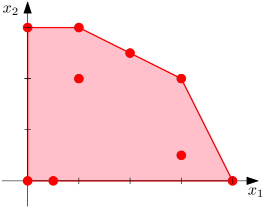

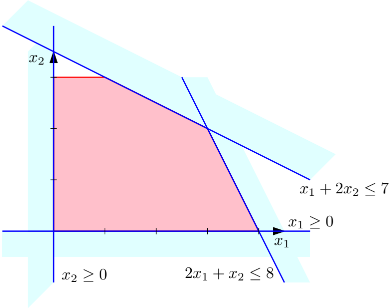

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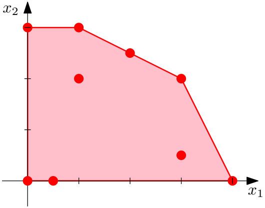

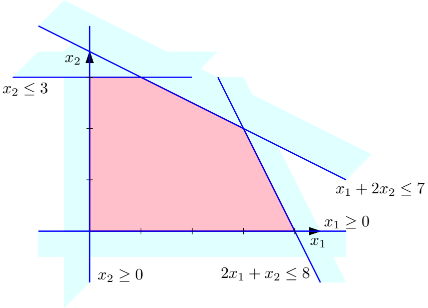

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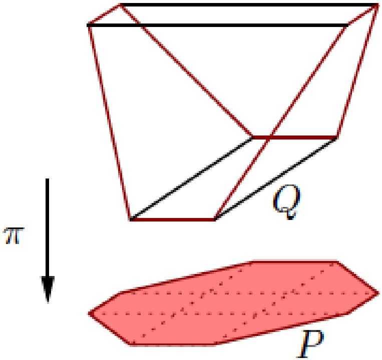
These polytopes have many facets. In order to solve optimization problems with Linear Programming, we need polytopes with a small number of facets.



P : polytope in $\mathbb{R}^2$ we want to optimize on (8 facets)
Q : polytope in $\mathbb{R}^3$ which projects to P (6 facets)
$\Rightarrow$ Easier to optimize on Q and project the solution !

Extended formulation

P : a polytope in $\mathbb{R}^d$.
Q : a polytope in higher dimension $\mathbb{R}^r$.

Q is an *extension* of P if there exists a linear map π such that $\pi(Q) = P$. The *size* of Q is the number of facets of Q.

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Equivalently, an *extended formulation* of P of *size* r is a linear system

$$Ex + Fy = g, \quad y \geq 0$$

in variables $(x, y) \in \mathbb{R}^{d+r}$
(E, F, g matrices/vector of suitable size).

Poly-time solvable :

- Matching polytope (Edmond's algorithm)
- Spanning Tree Polytope (Prim's and Kruskal's algorithms)
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NP-hard problems :

- Traveling Salesman Polytope
- Stable Set polytope
- Cut polytope
- Knapsack polytope

Poly-time solvable :

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Exponential lower bound on the extension complexity
Polynomial upper bound for the extension complexity

[1] : Rothvoss 13
[2] : Fiorini, Massar, Pokutta, Tiwary, deWolf 13
[3] : Pokuta, Van Vyve 13
[4] : Conforti, Cornuéjols, Zambelli (Survey) 10

Maximum Weighted Stable set

Variables : x_v for every vertex v

Objective function : max $\Sigma_{v \in V} w_v x_v$ where $w_v :=$ weight of v

Subject to : $x_u + x_v \leq 1$ for every edge uv
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Optimal relaxation solution : $n/2$ ($1/2$ for every vertex).

Optimal Integer Linear Program solution : 1 (1 for one vertex, 0 for the others).

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$\Rightarrow$ Bad solution !

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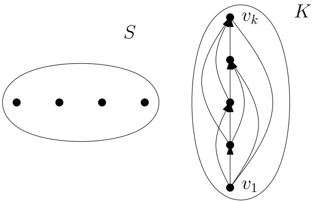

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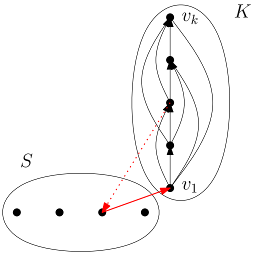

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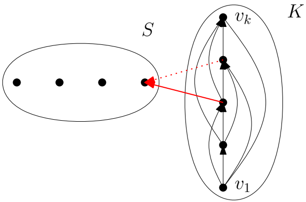

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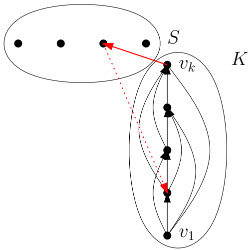

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Given an integer solution $x = \chi^S$:

- $b_v = \left| \begin{array}{ll} 1 & \text{iff } \nexists s \in S \quad v \leq s \\ 0 & \text{otherwise} \end{array} \right.$
- $t_v = \left| \begin{array}{ll} 1 & \text{iff } \exists s \in S \quad v < s \\ 0 & \text{otherwise} \end{array} \right.$
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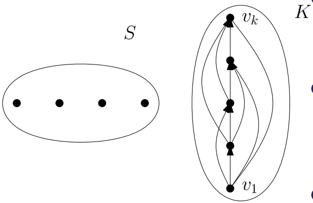

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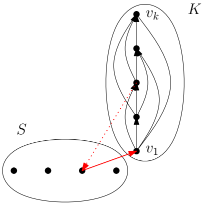
- $z_{v_i v_i+1} = \left| \begin{array}{ll} 1 & \text{iff } \exists s \in S \quad u < s \\ & \& \ \nexists s' \ v < s' \& \ v \notin S \\ 0 & \text{otherwise} \end{array} \right.$

Extended formulation for comparability graphs

Constraints :

$$\begin{array}{rl} \forall v \in V & x_v, b_v, t_v \geq 0 \\ \forall u < v \in V & z_{uv} \geq 0 \\ \forall K = v_1 < v_2 < \cdots < v_k & \Sigma_{i=1}^{k} x_{v_i} + b_{v_1} + t_{v_k} + \Sigma_{i=1}^{k-1} z_{v_i v_{i+1}} = 1 \end{array}$$

Given an integer solution $x = \chi^S$:



- $b_{v_1} = \begin{vmatrix} 1 & \text{iff } \nexists s \in S \quad v \leq s \\ 0 & \text{otherwise} \end{vmatrix}$
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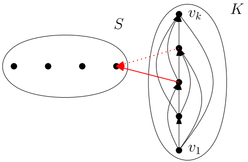

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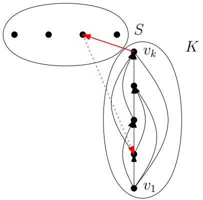

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How to obtain lower bounds ?

Three comparable measures on polytope :

- Rectangle covering of the slack matrix rc(M_{slack})
- Non-negative rank of the slack matrix $\text{rk}_+(M_{slack})$
- The extension complexity of the polytope xc(P)

$$rc(P) \leq rk_+(P) = xc(P)$$

Slack matrix :

$$
\begin{array}{l}
\textbf{Constraint 1 : } A_1 x \leq b_1 \\
\textbf{Constraint 2 : } A_2 x \leq b_2 \\
\vdots \\
\textbf{Constraint i : } A_i x \leq b_i \\
\vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccc} p_1 & p_2 & \dots & p_j & \dots \end{array} \\
\begin{pmatrix}
0 & 2 & & & \\
2 & 5 & & & \\
 & & & & \\
0 & 0 & & b_i - A_i p_j & \\
 & & & &
\end{pmatrix}
\end{array}
$$

$p_1, \dots, p_j, \dots$ are vertices of the polytope.

Slack matrix of the Stable set polytope :

$$\begin{array}{l} \\ \textbf{Constraint } K_1 : \Sigma_{v \in K_1} x_v \leq 1 \\ \textbf{Constraint } K_2 : \Sigma_{v \in K_2} x_v \leq 1 \\ \vdots \\ \textbf{Constraint } K_i : \Sigma_{v \in K_i} x_v \leq 1 \\ \vdots \\ \textbf{Other constraints} \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & \dots & S_j & \dots \end{array} \\ \begin{pmatrix} 0 & 1 & & & \\ 1 & 1 & & & \\ & & & & \\ 0 & 0 & & 1 - |K_i \cap S_j| & \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

$S_1, ..., S_j, ...$ are stables sets of G.

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)=$ minimum number of combinatorial rectangles needed to cover the support of M

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & \color{red}{1} & \color{red}{1} & - & - & - & - & - \\ - & \color{red}{1} & \color{red}{1} & - & - & - & - & - \\ - & \color{red}{1} & \color{red}{1} & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

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$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

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$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

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$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)$= minimum number of combinatorial rectangles needed to cover the support of M
Here : $rc(M) = 3$

Let us sum up :

Extension complexity

Rectangle covering

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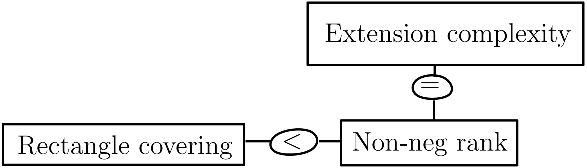
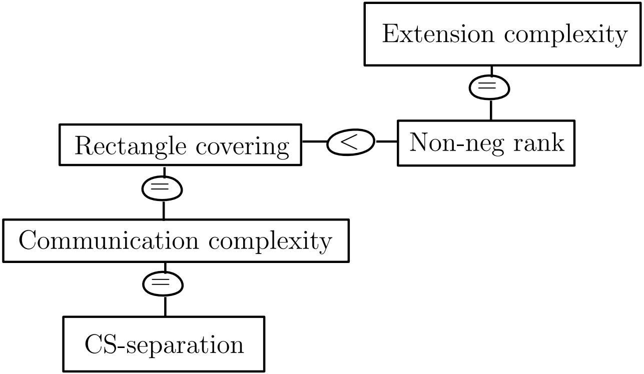
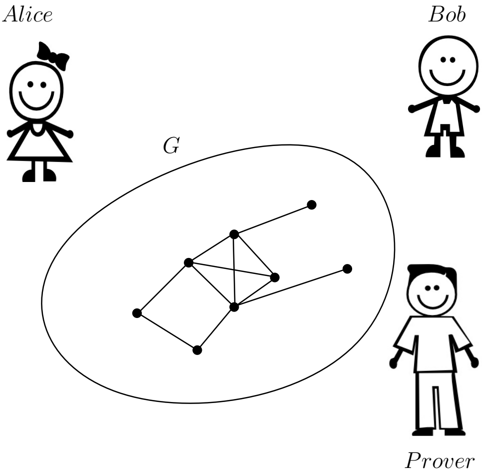


Let us sum up :

Stable set polytope for perfect graphs :



Clique vs Independent Set Problem



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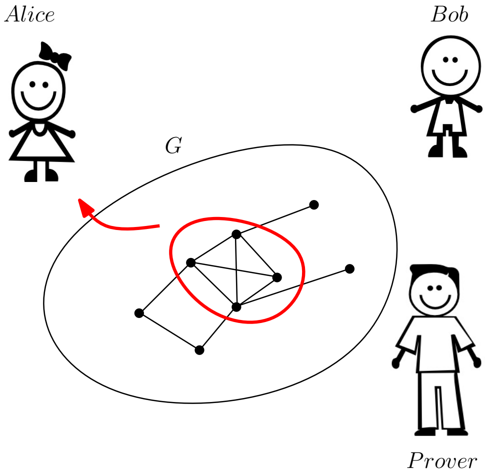

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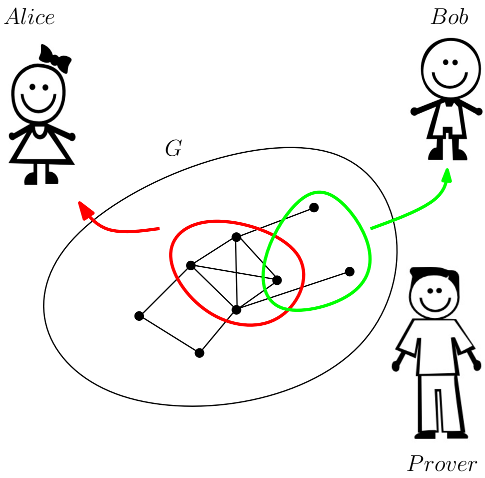

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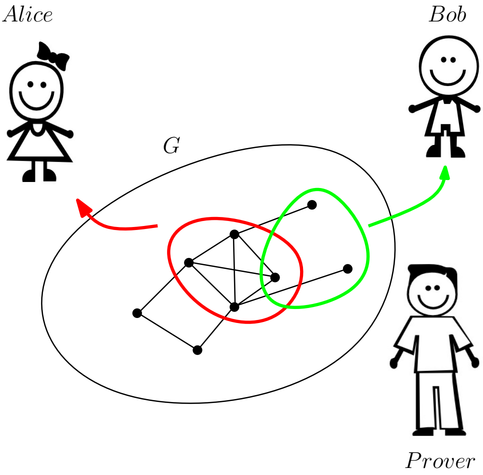
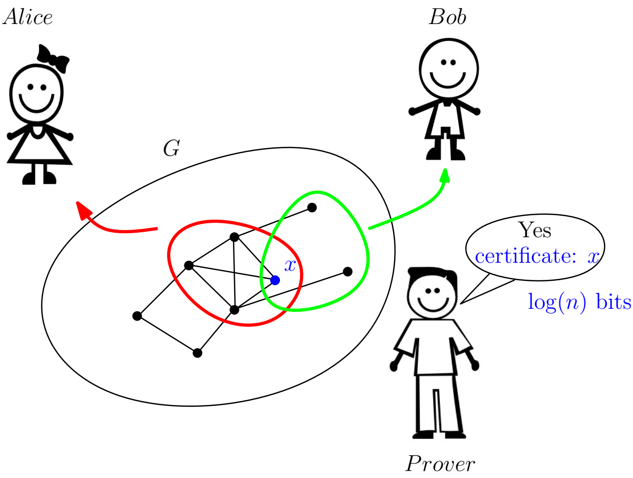


Do the clique and the stable set intersect?

Clique vs Independent Set Problem



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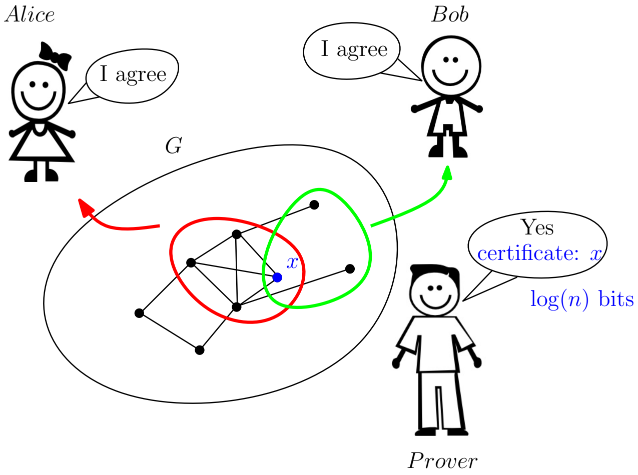

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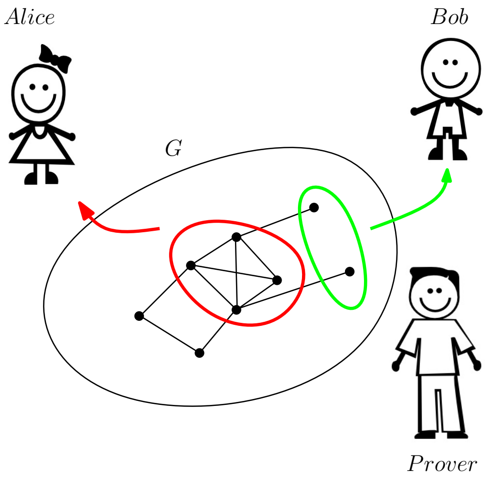

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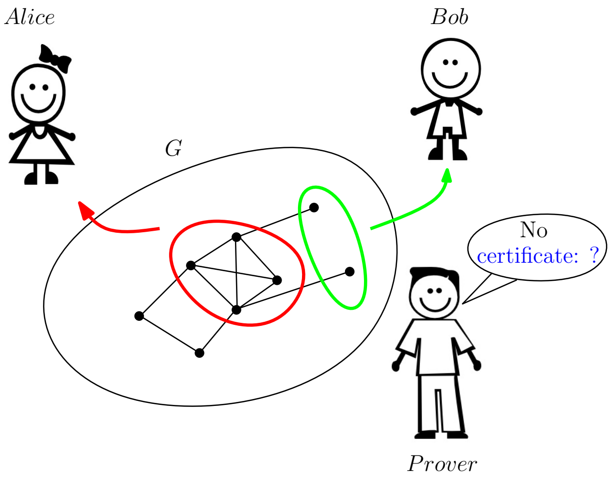

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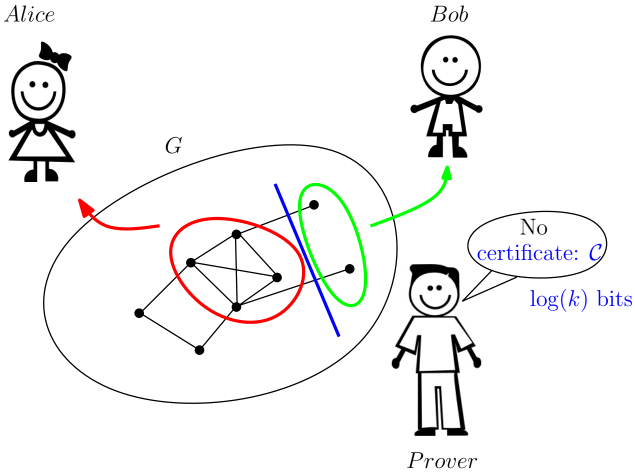

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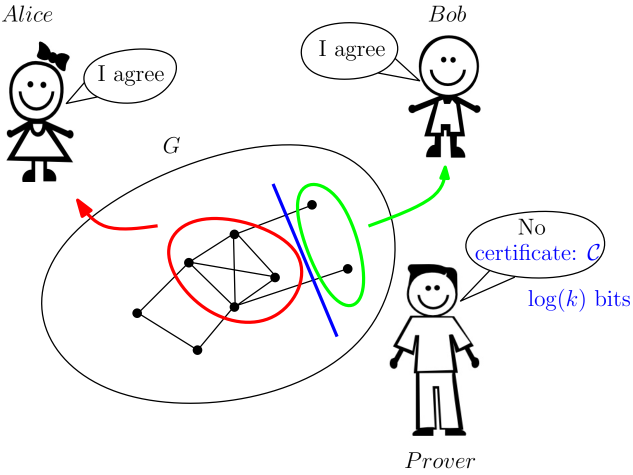

Do the clique and the stable set intersect?

$\log(rc(M)) =$ Non-det. communication complexity for this pb

$$
\begin{array}{r}
\\
\text{Constr.}K_1 \\
\text{Constr.}K_2 \\
\text{Constr.}K_3 \\
\text{Constr.}K_4 \\
\text{~~Other constraints~~}
\end{array}
\begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
 & & & &
\end{pmatrix}
$$

$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$

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$$\begin{array}{r} \\ \text{Constr.}K_1 \\ \text{Constr.}K_2 \\ Alice \rightarrow \text{Constr.}K_3 \\ \text{Constr.}K_4 \\ \text{~~Other constraints~~} \end{array} \begin{pmatrix} S_1 & S_2 & S_3 & S_4 & S_5 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ & & & & \end{pmatrix}$$

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\begin{array}{r}
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\text{Constr.}K_1 \\
\text{Constr.}K_2 \\
Alice \rightarrow \text{Constr.}K_3 \\
\text{Constr.}K_4 \\
\text{~~Other constraints~~}
\end{array}
\overset{\text{Bob} \downarrow}{\begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
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\end{pmatrix}}
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	S_1	S_2	S_3	S_4	S_5
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Goal [Yannakakis 1991]

Find a *CS-separator* : a family of cuts that can separate all the pairs Clique-Stable set.

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Does there exist for all graph G on n vertices a CS-separator of size poly(n) ? Or for which classes of graphs does it exist ?

In which classes of graphs do we have a polynomial CS-separator ?

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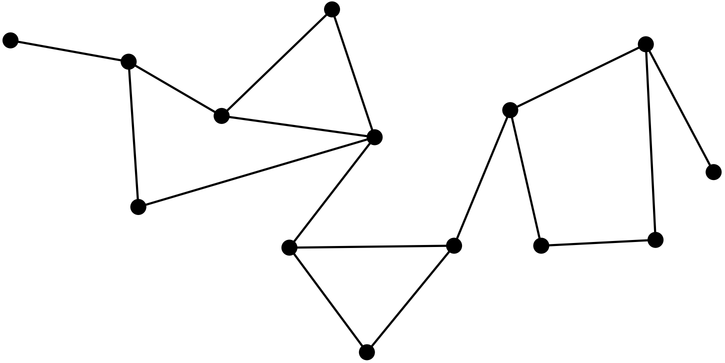
An easy example : if the clique number ω is bounded, say by 3 :



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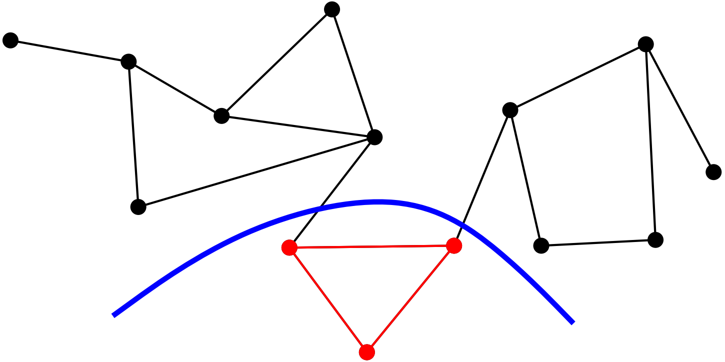

For every subset T of size ≤ 3, take the cut $(T, V \setminus T)$
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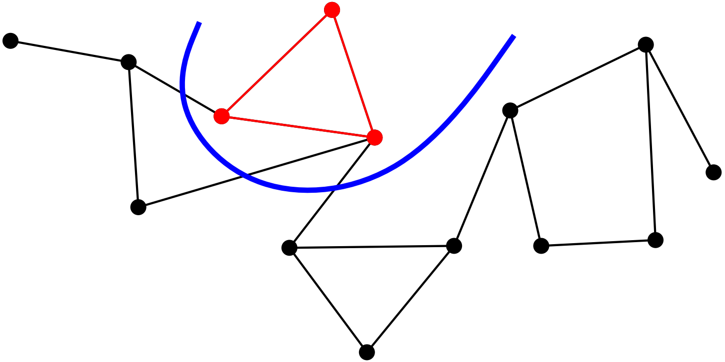

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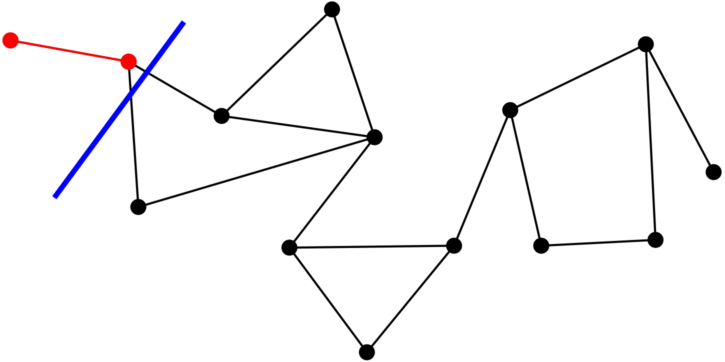

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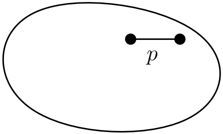
Random graphs [Bousquet, L., Thomassé 2012]

For every $n \in \mathbb{N}$, $p \in [0,1]$, there exists a set $\mathcal{F}$ of $\mathcal{O}(n^7)$ cuts such that

$$\forall G \in G(n,p) \qquad \Pr(\ \mathcal{F} \text{ is a CS-sep for } G) \underset{n \to +\infty}{\longrightarrow} 1$$



n vertices

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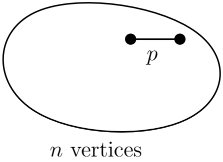

Idea : since the edges are all drawn with the same probability p, cliques and stables sets can not both be too big.

Example for $p = 1/2$: $\alpha \approx \omega \approx 2 \log n$.

Split-free

Comparability graphs [Yannakakis 1991]

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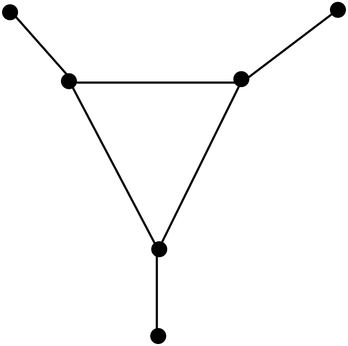

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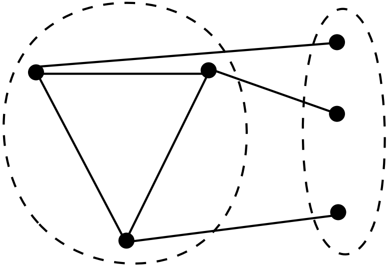

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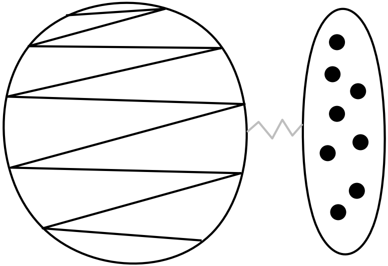
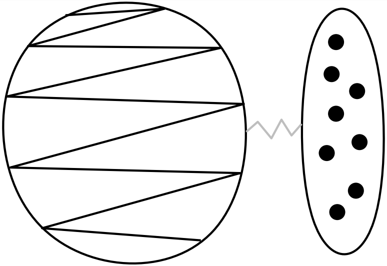


Split-free

Split graph

A graph (V, E) is *split* if V can be partitioned into a clique and a stable set.



Split-free [Bousquet, L., Thomassé 2012]

Let H be a split graph. Then every H-free graphs have a CS-separator of size $\mathcal{O}(n^{c_H})$.

Let H be a split graph.

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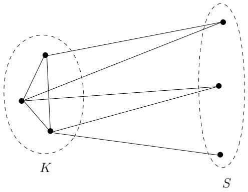

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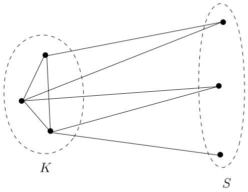

Key Lemma (using VC-dimension)

$\exists$ a constant t s. t. $\forall$ clique K and stable set S in a H-free :

- $\exists S' \subseteq S$ s. t. $|S'| = t$ and S' dominates K
- or, $\exists K' \subseteq K$ s. t. $|K'| = t$ and K' antidominates S

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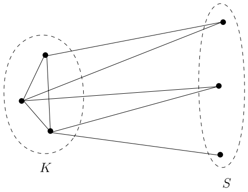

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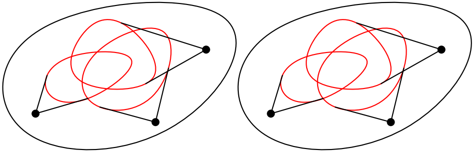

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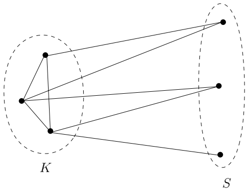

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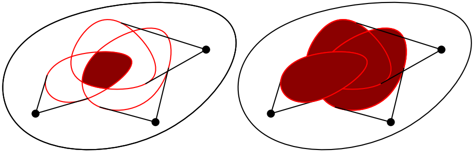

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Extended formulation for P_5-free graphs [Conforti, Di Summa, Faenza, Fiorini, Pashkovich]

For every P_5-free graph G, $STAB(G)$ has an extended formulation of polynomial size.

Back to perfect graphs :

Decomposition [Chudnovsky, Roberston, Seymour, Thomas]

If a graph is Berge, then for G or $\overline{G}$, one of the following holds :

- It is a basique graph : bipartite, line graph of bipartite, or double split.
- There is a 2-join
- There is a balanced skew partition.

[L., Trunck, 2013]

Let G be a Berge graph with no balanced skew partition, then there exists a CS-separator for G of size $\mathcal{O}(n^2)$.

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- Yannakakis question : CS-separation in perfect graphs ?
- Better lower bound for the CS-separation in general ?