From extended formulations of polytopes to the Clique-Stable Set Separation

Aurélie Lagoutte

LIP, ENS Lyon

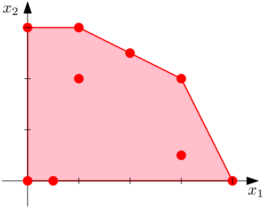
Joint work with N. Bousquet, S. Thomassé et T. Trunck

Thursday, December 4, 2014
G-SCOP Seminar

A polytope P in $\mathbb{R}^2$:

P can be defined :

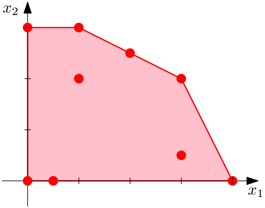
A polytope P in $\mathbb{R}^2$:



P can be defined :

- As the convex hull of a set of points : $P = \mathrm{conv}(p_1, \ldots, p_k)$

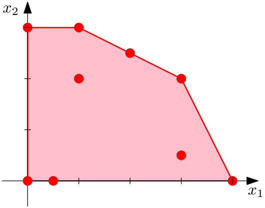
A polytope P in $\mathbb{R}^2$:

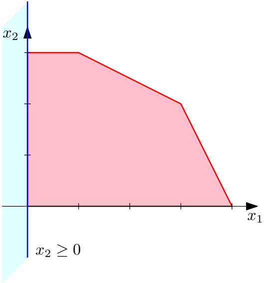


P can be defined :

- As the convex hull of a set of points : $P = \mathrm{conv}(p_1, \ldots, p_k)$
- As a bounded polyhedron, by a set of constraints (=inequalities) :
  $x = (x_1, x_2) \in \mathbb{R}^2$
  $Ax \leq b$
  $x \geq 0$

A polytope P in $\mathbb{R}^2$:

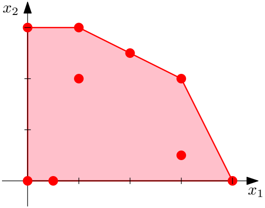


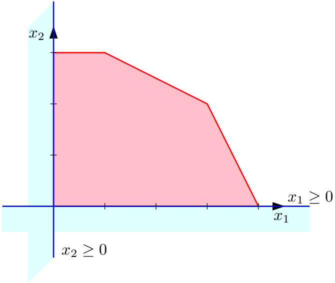


P can be defined :

- As the convex hull of a set of points : $P = \mathrm{conv}(p_1, \ldots, p_k)$
- As a bounded polyhedron, by a set of constraints (=inequalities) :
  $x = (x_1, x_2) \in \mathbb{R}^2$
  $Ax \leq b$
  $x \geq 0$

A polytope P in $\mathbb{R}^2$:



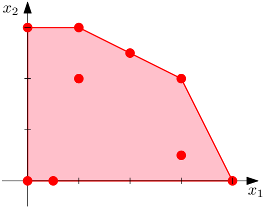


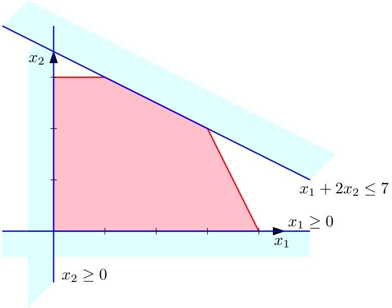
P can be defined :

- As the convex hull of a set of points : $P = \mathrm{conv}(p_1, \ldots, p_k)$

- As a bounded polyhedron, by a set of constraints (=inequalities) :
  $x = (x_1, x_2) \in \mathbb{R}^2$
  $Ax \leq b$
  $x \geq 0$

A polytope P in $\mathbb{R}^2$:

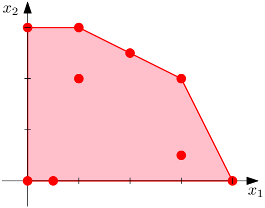


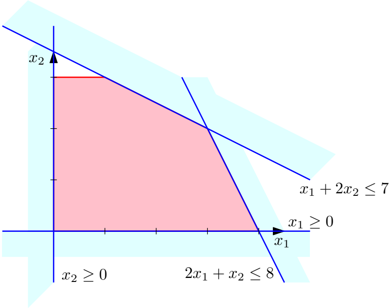


P can be defined :

- As the convex hull of a set of points : $P = \mathrm{conv}(p_1, \ldots, p_k)$
- As a bounded polyhedron, by a set of constraints (=inequalities) :
  $x = (x_1, x_2) \in \mathbb{R}^2$
  $Ax \leq b$
  $x \geq 0$

A polytope P in $\mathbb{R}^2$:

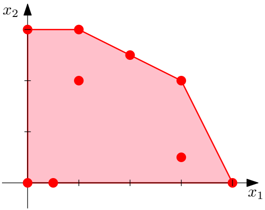


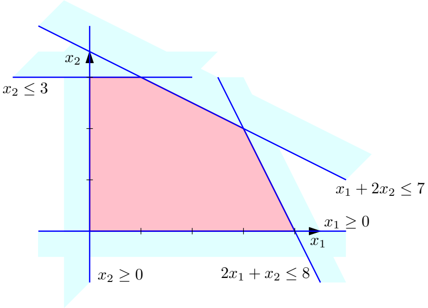


P can be defined :

- As the convex hull of a set of points : $P = \text{conv}(p_1, \ldots, p_k)$
- As a bounded polyhedron, by a set of constraints (=inequalities) : $x = (x_1, x_2) \in \mathbb{R}^2$ $Ax \leq b$ $x \geq 0$

A polytope P in $\mathbb{R}^2$:





P can be defined :

- As the convex hull of a set of points : $P = \mathrm{conv}(p_1, \dots, p_k)$
- As a bounded polyhedron, by a set of constraints (=inequalities) :
  $x = (x_1, x_2) \in \mathbb{R}^2$
  $Ax \leq b$
  $x \geq 0$

Well-studied polytopes :

Well-studied polytopes :

Stable Set polytope

$STAB(G) = \mathrm{conv}(\chi^S \in \mathbb{R}^n | S \subseteq V$ is a stable set of $G)$
where χ^S denotes the characteristic vector of $S \subseteq V$

Well-studied polytopes :

Stable Set polytope

$STAB(G) = \text{conv}(\chi^S \in \mathbb{R}^n | S \subseteq V$ is a stable set of $G)$
where χ^S denotes the characteristic vector of $S \subseteq V$

Traveling Salesman polytope (tours on $K_n = (V_n, E_n)$)

$TSP(n) = \text{conv}(\chi^F \in \mathbb{R}^{|E_n|} | F \subseteq E_n$ is a tour of $K_n)$

Well-studied polytopes :

Stable Set polytope

$STAB(G) = \mathrm{conv}(\chi^S \in \mathbb{R}^n | S \subseteq V$ is a stable set of $G)$
where χ^S denotes the characteristic vector of $S \subseteq V$

Traveling Salesman polytope (tours on $K_n = (V_n, E_n)$)

$TSP(n) = \mathrm{conv}(\chi^F \in \mathbb{R}^{|E_n|} | F \subseteq E_n$ is a tour of $K_n)$

Matching polytope

$MATCH(G) = \mathrm{conv}(\chi^M \in \mathbb{R}^{|E|} | M \subseteq E$ is a matching of $G)$

Well-studied polytopes :

Stable Set polytope

$STAB(G) = \mathrm{conv}(\chi^S \in \mathbb{R}^n | S \subseteq V$ is a stable set of $G)$
where χ^S denotes the characteristic vector of $S \subseteq V$

Traveling Salesman polytope (tours on $K_n = (V_n, E_n)$)

$TSP(n) = \mathrm{conv}(\chi^F \in \mathbb{R}^{|E_n|} | F \subseteq E_n$ is a tour of $K_n)$

Matching polytope

$MATCH(G) = \mathrm{conv}(\chi^M \in \mathbb{R}^{|E|} | M \subseteq E$ is a matching of $G)$

Parity polytope

$PAR(n) = \mathrm{conv}(x \in \{0,1\}^n | x$ has an odd number of 1. $)$

Well-studied polytopes :

Stable Set polytope

$STAB(G) = \text{conv}(\chi^S \in \mathbb{R}^n | S \subseteq V$ is a stable set of $G)$
where χ^S denotes the characteristic vector of $S \subseteq V$

Traveling Salesman polytope (tours on $K_n = (V_n, E_n)$)

$TSP(n) = \text{conv}(\chi^F \in \mathbb{R}^{|E_n|} | F \subseteq E_n$ is a tour of $K_n)$

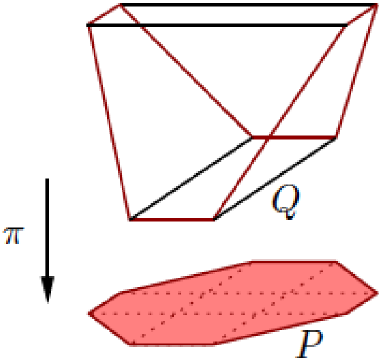
Matching polytope

$MATCH(G) = \text{conv}(\chi^M \in \mathbb{R}^{|E|} | M \subseteq E$ is a matching of $G)$

Parity polytope

$PAR(n) = \text{conv}(x \in \{0,1\}^n | x$ has an odd number of 1. $)$

These polytopes have many facets. In order to solve optimization problems with Linear Programming, we need polytopes with a small number of facets.



P : polytope in $\mathbb{R}^2$ we want to optimize on (8 facets)
Q : polytope in $\mathbb{R}^3$ which projects to P (6 facets)
$\Rightarrow$ Easier to optimize on Q and project the solution !

Extended formulation

P : a polytope in $\mathbb{R}^d$.
Q : a polytope in higher dimension $\mathbb{R}^r$.

Q is an *extension* of P if there exists a linear map π such that $\pi(Q) = P$. The *size* of Q is the number of facets of Q.

Extended formulation

P : a polytope in $\mathbb{R}^d$.
Q : a polytope in higher dimension $\mathbb{R}^r$.

Q is an *extension* of P if there exists a linear map π such that $\pi(Q) = P$. The *size* of Q is the number of facets of Q.

Extension complexity

$xc(P) = min\{\text{size of } Q \mid Q \text{ is an extension of } P\}$.

Extended formulation

P : a polytope in $\mathbb{R}^d$.
Q : a polytope in higher dimension $\mathbb{R}^r$.

Q is an *extension* of P if there exists a linear map π such that $\pi(Q) = P$. The *size* of Q is the number of facets of Q.

Extension complexity

$xc(P) = min\{\text{size of } Q \mid Q \text{ is an extension of } P\}$.

Equivalently, an *extended formulation* of P of *size* r is a linear system

$$Ex + Fy = g, \quad y \geq 0$$

in variables $(x, y) \in \mathbb{R}^{d+r}$
(E, F, g matrices/vector of suitable size).

Poly-time solvable :

- Matching polytope (Edmond's algorithm)
- Spanning Tree Polytope (Prim's and Kruskal's algorithms)
- Parity Polytope

NP-hard problems :

- Traveling Salesman Polytope
- Stable Set polytope
- Cut polytope
- Knapsack polytope

Poly-time solvable :

- Matching polytope (Edmond's algorithm) [1]
- Spanning Tree Polytope (Prim's and Kruskal's algorithms) [4]
- Parity Polytope [4]

NP-hard problems :

- Traveling Salesman Polytope [2]
- Stable Set polytope [2]
- Cut polytope [2]
- Knapsack polytope [3]

Exponential lower bound on the extension complexity
Polynomial upper bound for the extension complexity

[1] : Rothvoss 13
[2] : Fiorini, Massar, Pokutta, Tiwary, deWolf 13
[3] : Pokuta, Van Vyve 13
[4] : Conforti, Cornuéjols, Zambelli (Survey) 10

Maximum Weighted Stable set

Variables : x_v for every vertex v

Objective function : max $\Sigma_{v \in V} w_v x_v$ where $w_v :=$ weight of v

Subject to : $x_u + x_v \leq 1$ for every edge uv
$x_v \in \{0, 1\}$ for every vertex v

Maximum Weighted Stable set

Variables : x_v for every vertex v

Objective function : max $\Sigma_{v \in V} w_v x_v$ where $w_v :=$ weight of v

Subject to : $x_u + x_v \leq 1$ for every edge uv

~~$x_v \in \{0, 1\}$ for every vertex v~~

$0 \leq x_v \leq 1$ for every vertex v

Maximum Weighted Stable set

Variables : x_v for every vertex v

Objective function : max $\Sigma_{v \in V} w_v x_v$ where $w_v :=$ weight of v

Subject to : $x_u + x_v \leq 1$ for every edge uv

~~$x_v \in \{0, 1\}$ for every vertex v~~

$0 \leq x_v \leq 1$ for every vertex v

$\Rightarrow$ On the complete graph K_n with constant weight $w_v = 1$:

Optimal relaxation solution : $n/2$ (1/2 for every vertex).

Optimal Integer Linear Program solution : 1 (1 for one vertex, 0 for the others).

Maximum Weighted Stable set

Variables : x_v for every vertex v

Objective function : max $\Sigma_{v \in V} w_v x_v$ where $w_v :=$ weight of v

Subject to : $x_u + x_v \leq 1$ for every edge uv

~~$x_v \in \{0, 1\}$ for every vertex v~~

$0 \leq x_v \leq 1$ for every vertex v

$\Rightarrow$ On the complete graph K_n with constant weight $w_v = 1$:

Optimal relaxation solution : $n/2$ ($1/2$ for every vertex).

Optimal Integer Linear Program solution : 1 (1 for one vertex, 0 for the others).

$\Rightarrow$ Bad solution !

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)

$x_u + x_v \leq 1$ for every $uv \in E$ (2)

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)

$x_u + x_v \leq 1$ for every $uv \in E$ (2)

$\Sigma_{v \in K} x_v \leq 1$ for every clique K (3)

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)

$x_u + x_v \leq 1$ for every $uv \in E$ (2)

$\Sigma_{v \in K} x_v \leq 1$ for every clique K (3)

$\Sigma_{c \in C} x_v \leq (|C| - 1)/2$ for every odd cycle C (4)

...

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)
$x_u + x_v \leq 1$ for every $uv \in E$ (2)
$\Sigma_{v \in K} x_v \leq 1$ for every clique K (3)
$\Sigma_{c \in C} x_v \leq (|C| - 1)/2$ for every odd cycle C (4)
...

(1) and (2) : enough for bipartite graphs

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)
$x_u + x_v \leq 1$ for every $uv \in E$ (2)
$\Sigma_{v \in K} x_v \leq 1$ for every clique K (3)
$\Sigma_{c \in C} x_v \leq (|C| - 1)/2$ for every odd cycle C (4)
...

(1) and (2) : enough for bipartite graphs
(1) and (3) : enough for perfect graphs

Stable set polytope : valid inequalities

Stable set polytope

STAB(G)=conv($\chi^S | S$ is a stable set of G)

Valid inequalities :

$0 \leq x_v \leq 1$ for every $v \in V$ (1)
$x_u + x_v \leq 1$ for every $uv \in E$ (2)
$\Sigma_{v \in K} x_v \leq 1$ for every clique K (3)
$\Sigma_{c \in C} x_v \leq (|C| - 1)/2$ for every odd cycle C (4)
...

(1) and (2) : enough for bipartite graphs
(1) and (3) : enough for perfect graphs
(1) and (4) : enough for t-perfect graphs

How to obtain lower bounds ?

Three comparable measures on polytope :

- Rectangle covering of the slack matrix rc(M_{slack})
- Non-negative rank of the slack matrix $\text{rk}_+(M_{slack})$
- The extension complexity of the polytope xc(P)

$$rc(P) \leq rk_+(P) = xc(P)$$

Slack matrix :

$$
\begin{array}{l}
\textbf{Constraint 1 : } A_1 x \leq b_1 \\
\textbf{Constraint 2 : } A_2 x \leq b_2 \\
\vdots \\
\textbf{Constraint i : } A_i x \leq b_i \\
\vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccc} p_1 & p_2 & \dots & p_j & \dots \end{array} \\
\begin{pmatrix}
0 & 2 & & & \\
2 & 5 & & & \\
 & & & & \\
0 & 0 & & b_i - A_i p_j & \\
 & & & &
\end{pmatrix}
\end{array}
$$

$p_1, \ldots, p_j, \ldots$ are vertices of the polytope.

Slack matrix of the Stable set polytope :

$$
\begin{array}{l}
 \\
\textbf{Constraint } K_1 \textbf{ : } \Sigma_{v \in K_1} x_v \leq 1 \\
\textbf{Constraint } K_2 \textbf{ : } \Sigma_{v \in K_2} x_v \leq 1 \\
\vdots \\
\textbf{Constraint } K_i \textbf{ : } \Sigma_{v \in K_i} x_v \leq 1 \\
\vdots \\
\textbf{Other constraints}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} S_1 & S_2 & \dots & S_j & \dots \end{array} \\
\left(\begin{array}{ccccc}
0 & 1 & & & \\
1 & 1 & & & \\
 & & & & \\
0 & 0 & & 1 - |K_i \cap S_j| & \\
 & & & & \\
 & & & &
\end{array}\right)
\end{array}
$$

$S_1, \dots, S_j, \dots$ are stables sets of G.

Non-negative rank of a matrix :

$$\begin{array}{cccccccc} - & x_1 & - & - & - & - & - & - \\ - & x_2 & - & - & - & - & - & - \\ - & x_3 & - & - & - & - & - & - \end{array}$$

$$\overbrace{\begin{array}{ccc} - & - & - \\ y_1 & y_2 & y_3 \\ - & - & - \\ - & - & - \\ - & - & - \\ - & - & - \\ - & - & - \end{array}}^{r \text{ columns}} \quad \begin{pmatrix} - & - & - & - & - & - & - & - \\ - & \Sigma_{i=1}^{3} x_i y_i & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \end{pmatrix}$$

with $\forall i \quad x_i, y_i \geq 0$.

Equivalently : $rk_+(M)$ is the smallest integer such that $M = \Sigma_{i=1}^{r} R_i$ with R_i rank-1 matrices with non-negative entries.

Factorization theorem :

Theorem [Yannakakis 91]

For any polytope P and any of its slack matrix M, the following equality holds :

$$xc(P) = rk_+(M)$$

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)=$ minimum number of combinatorial rectangles needed to cover the support of M [1]

1. From now on, I will consider only 0/1 slack matrix, so $\text{supp}(M)=M$.

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & \color{red}{1} & \color{red}{1} & - & - & - & - & - \\ - & \color{red}{1} & \color{red}{1} & - & - & - & - & - \\ - & \color{red}{1} & \color{red}{1} & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)$= minimum number of combinatorial rectangles needed to cover the support of M[1]

1. From now on, I will consider only 0/1 slack matrix, so supp(M)=M.

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & \color{red}{1} & \color{red}{1} & - & - & - & - & - \\ - & \color{red}{1} & \color{red}{1} & - & - & - & - & - \\ - & \color{red}{1} & \color{red}{1} & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ \color{blue}{1} & \color{blue}{1} & 1 & 1 & - & - & 1 & - \\ \color{blue}{1} & \color{blue}{1} & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)$= minimum number of combinatorial rectangles needed to cover the support of M [1]

1. From now on, I will consider only 0/1 slack matrix, so $\text{supp}(M)=M$.

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)$= minimum number of combinatorial rectangles needed to cover the support of M [1]

1. From now on, I will consider only 0/1 slack matrix, so supp(M)=M.

Another hidden tool in the slack matrix : Rectangle covering

$$\begin{pmatrix} - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & - & - & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 1 & 1 & 1 & 1 & - & - & 1 & - \\ 1 & 1 & - & - & - & - & - & - \end{pmatrix}$$

$rc(M)$= minimum number of combinatorial rectangles needed to cover the support of M [1]

Here : $rc(M) = 3$

1. From now on, I will consider only 0/1 slack matrix, so supp(M)=M.

$$rc(M) \leq rk_+(M)$$

1	0	1	0	0	1	0	0
0	0	0	1	0	1	0	0
1	1	0	0	0	0	0	0

r columns

0	1	0
0	1	0
0	0	1
1	0	0
1	0	0
1	0	0
0	0	1

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$rc(M) \leq rk_+(M)$$

$$\begin{array}{cc} & \begin{array}{|cccccccc|} \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \\ \overbrace{\begin{array}{|ccc|} \hline 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline \end{array}}^{r \text{ columns}} & \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$rc(M) \leq rk_+(M)$$

1	0	1	0	0	1	0	0
0	0	0	1	0	1	0	0
1	1	0	0	0	0	0	0

r columns

0	1	0
0	1	0
0	0	1
1	0	0
1	0	0
1	0	0
0	0	1

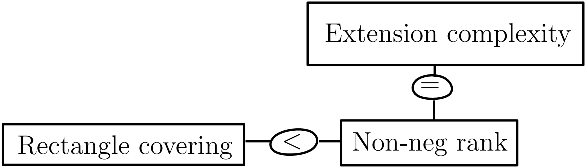
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let us sum up :

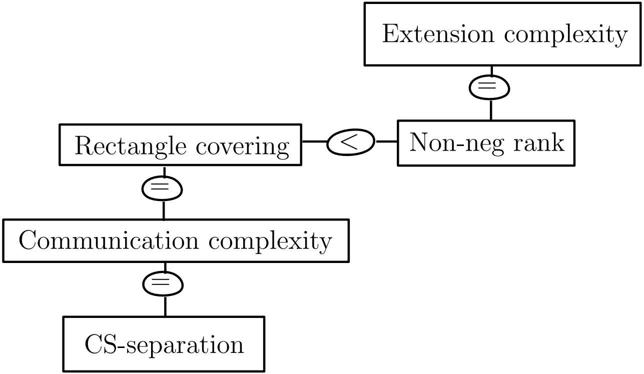
Extension complexity

Rectangle covering

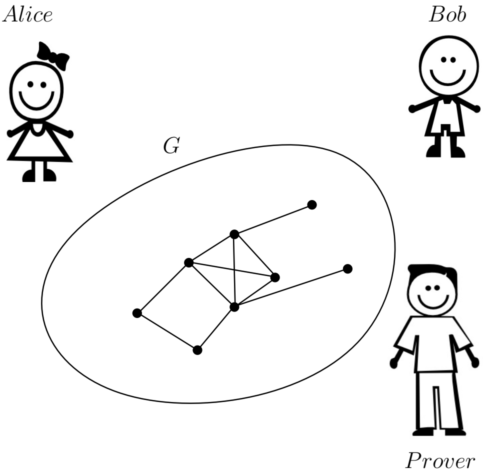
Let us sum up :



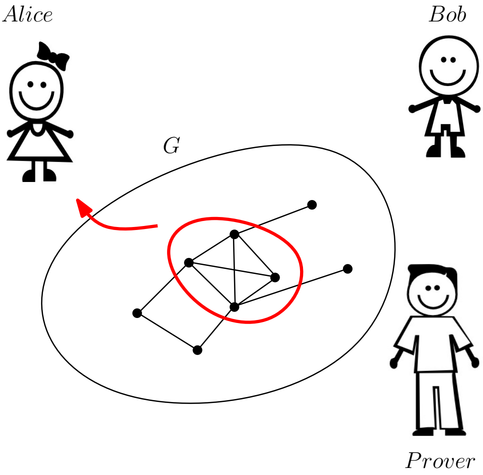
Let us sum up :
Stable set polytope for perfect graphs :



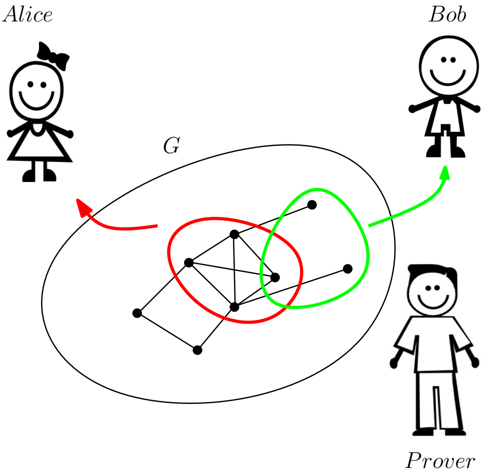
Clique vs Independent Set Problem



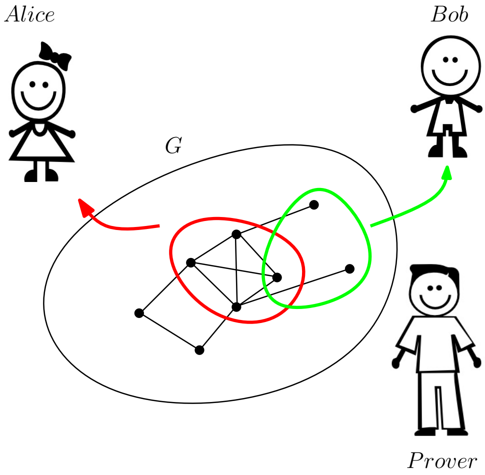
Clique vs Independent Set Problem



Clique vs Independent Set Problem

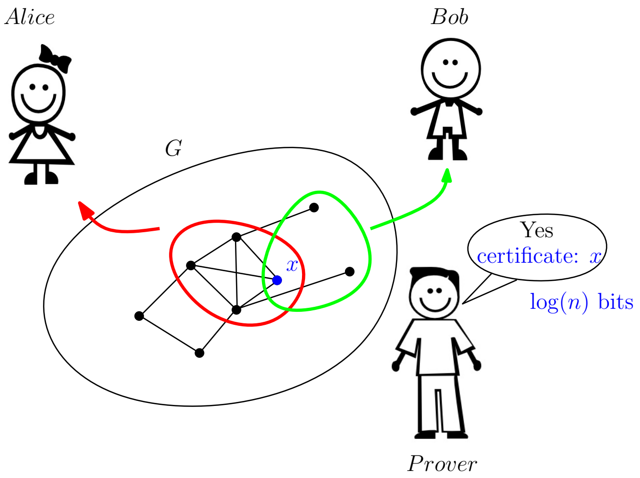


Clique vs Independent Set Problem



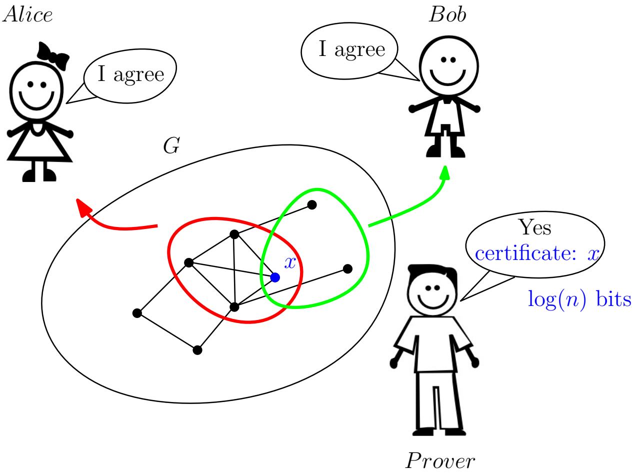
Do the clique and the stable set intersect?

Clique vs Independent Set Problem



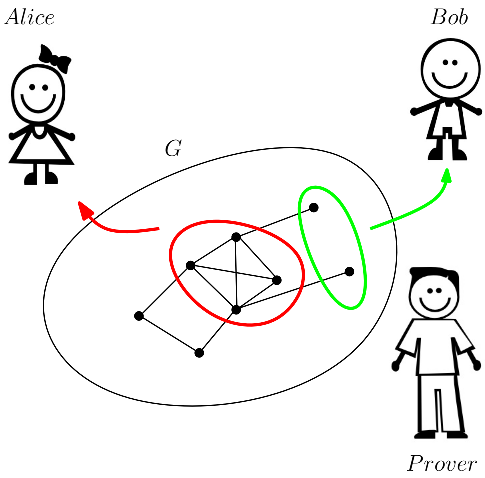
Do the clique and the stable set intersect?

Clique vs Independent Set Problem



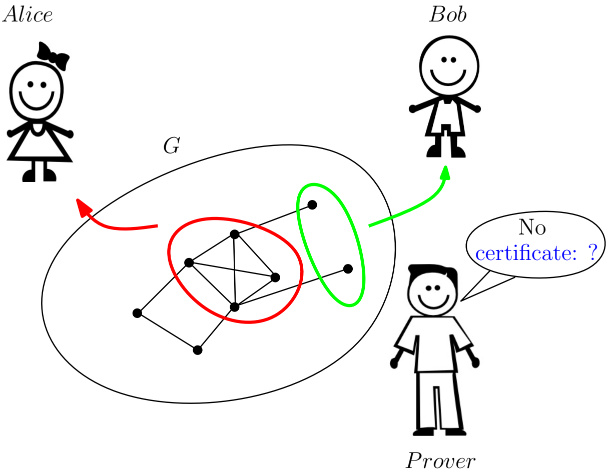
Do the clique and the stable set intersect?

Clique vs Independent Set Problem



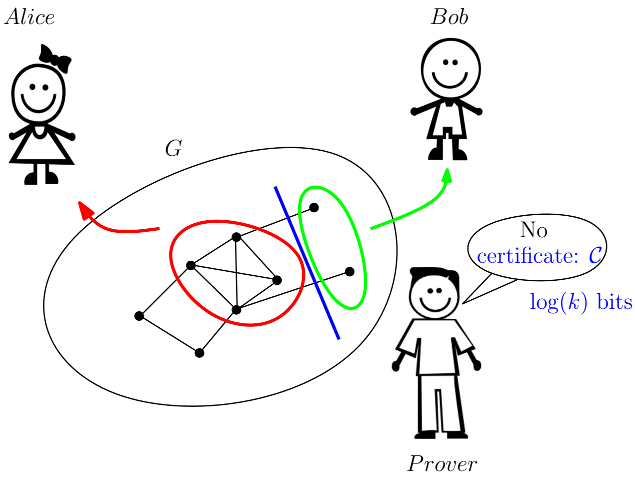
Do the clique and the stable set intersect?

Clique vs Independent Set Problem



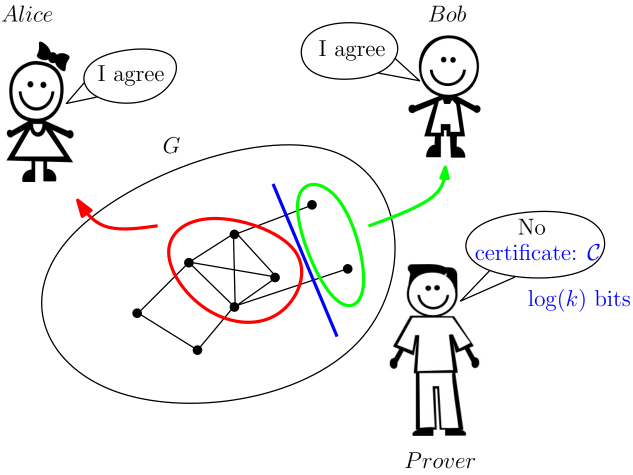
Do the clique and the stable set intersect?

Clique vs Independent Set Problem



Do the clique and the stable set intersect?

Clique vs Independent Set Problem



Do the clique and the stable set intersect?

$\log(rc(M))$ = Non-det. communication complexity for this pb

	S_1	S_2	S_3	S_4	S_5
Constr.K_1	1	1	0	0	1
Constr.K_2	1	1	1	1	0
Constr.K_3	0	1	1	1	0
Constr.K_4	1	0	0	0	1
~~Other constraints~~					

$$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$$

$\log(rc(M))$ = Non-det. communication complexity for this pb

	S_1	S_2	S_3	S_4	S_5
Constr. K_1	1	1	0	0	1
Constr. K_2	1	1	1	1	0
Alice → Constr. K_3	0	1	1	1	0
Constr. K_4	1	0	0	0	1
~~Other constraints~~					

$$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$$

$\log(rc(M)) =$ Non-det. communication complexity for this pb

$$
\begin{array}{r}
\\
\text{Constr.}K_1 \\
\text{Constr.}K_2 \\
\textit{Alice} \rightarrow \text{Constr.}K_3 \\
\text{Constr.}K_4 \\
\text{~~Other constraints~~}
\end{array}
\overset{\text{Bob} \downarrow}{\begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
 & & & &
\end{pmatrix}}
$$

~~Other constraints~~

$$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$$

$\log(rc(M))$ = Non-det. communication complexity for this pb

$$
\begin{array}{r}
\\
\\
\text{Constr.}K_1 \\
\text{Constr.}K_2 \\
Alice \rightarrow \text{Constr.}K_3 \\
\text{Constr.}K_4 \\
\text{\sout{Other constraints}}
\end{array}
\begin{array}{c}
\text{Bob} \downarrow \\
\begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
 & & & &
\end{pmatrix}
\end{array}
$$

$$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$$

$\log(rc(M))$ = Non-det. communication complexity for this pb

$$
\begin{array}{r}
\\
\text{Constr.}K_1 \\
\text{Constr.}K_2 \\
Alice \rightarrow \text{Constr.}K_3 \\
\text{Constr.}K_4 \\
\sout{\text{Other constraints}}
\end{array}
\overset{\text{Bob} \downarrow}{\begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
 & & & &
\end{pmatrix}}
$$

$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$

$\log(rc(M)) =$ Non-det. communication complexity for this pb

Bob $\downarrow$

	S_1	S_2	S_3	S_4	S_5
Constr. K_1	1	1	0	0	1
Constr. K_2	1	1	1	1	0
Alice $\rightarrow$ Constr. K_3	0	1	1	1	0
Constr. K_4	1	0	0	0	1
~~Other constraints~~					

$QSTAB(G) : M_{i,j} = 1 - |K_i \cap S_j|$

Clique vs Independent Set Problem

Goal [Yannakakis 1991]

Find a *CS-separator* : a family of cuts that can separate all the pairs Clique-Stable set.

Clique vs Independent Set Problem

Goal [Yannakakis 1991]

Find a *CS-separator* : a family of cuts that can separate all the pairs Clique-Stable set.

Upper Bound : there exists a CS-separator of size $\mathcal{O}(n^{\log n})$.

Lower Bound [Amano, Shigeta 2013] : there exists an infinite family of graphs such that any CS-separator has size $\Omega(n^{2-\varepsilon})$

Clique vs Independent Set Problem

Goal [Yannakakis 1991]

Find a *CS-separator* : a family of cuts that can separate all the pairs Clique-Stable set.

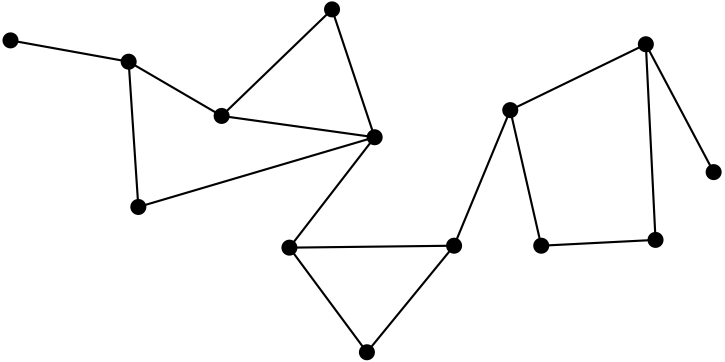
Upper Bound : there exists a CS-separator of size $\mathcal{O}(n^{\log n})$.

Lower Bound [Amano, Shigeta 2013] : there exists an infinite family of graphs such that any CS-separator has size $\Omega(n^{2-\varepsilon})$

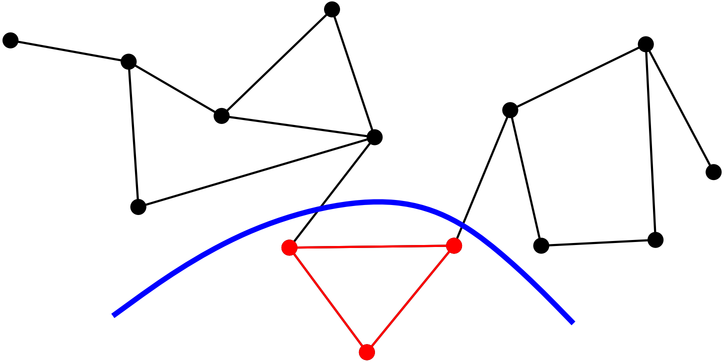
Does there exist for all graph G on n vertices a CS-separator of size poly(n) ? Or for which classes of graphs does it exist ?

In which classes of graphs do we have a polynomial CS-separator ?

In which classes of graphs do we have a polynomial CS-separator ?
An easy example : if the clique number ω is bounded, say by 3 :

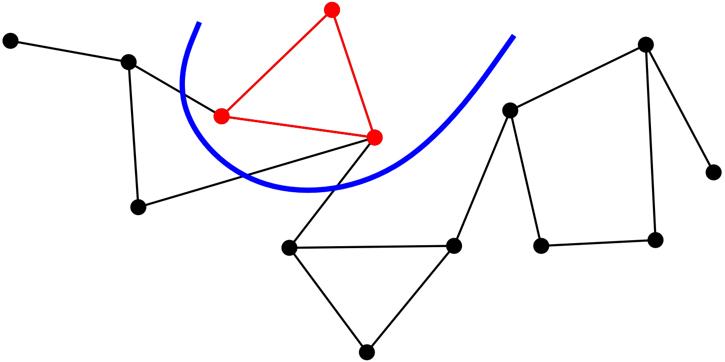


In which classes of graphs do we have a polynomial CS-separator ?
An easy example : if the clique number ω is bounded, say by 3 :



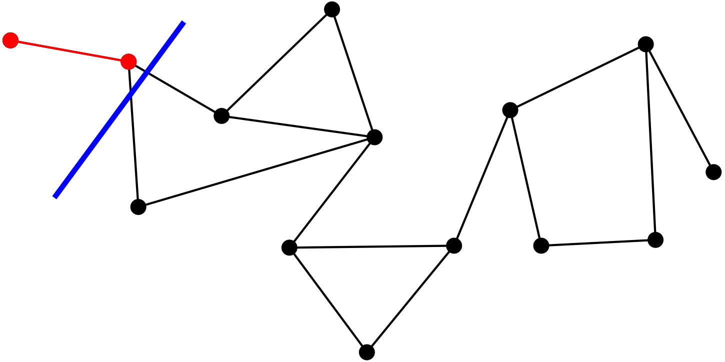
For every subset T of size ≤ 3, take the cut $(T, V \setminus T)$
$\Rightarrow$ CS-separator of size $\mathcal{O}(n^3)$.

In which classes of graphs do we have a polynomial CS-separator ?
An easy example : if the clique number ω is bounded, say by 3 :



For every subset T of size ≤ 3, take the cut $(T, V \setminus T)$
$\Rightarrow$ CS-separator of size $\mathcal{O}(n^3)$.

In which classes of graphs do we have a polynomial CS-separator ?
An easy example : if the clique number ω is bounded, say by 3 :



For every subset T of size ≤ 3, take the cut $(T, V \setminus T)$
$\Rightarrow$ CS-separator of size $\mathcal{O}(n^3)$.

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Split-free

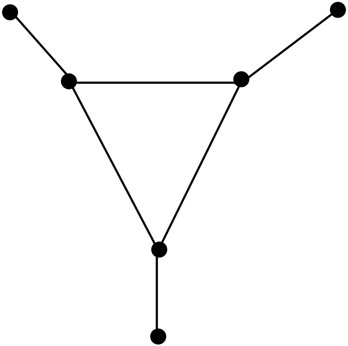
Comparability graphs [Yannakakis 1991]

Comparability graphs have a CS-separator of size $\mathcal{O}(n^2)$.

Split-free

Comparability graphs [Yannakakis 1991]

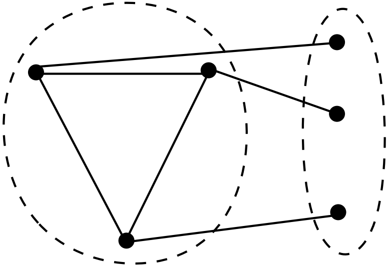
Comparability graphs have a CS-separator of size $\mathcal{O}(n^2)$.



Split-free

Comparability graphs [Yannakakis 1991]

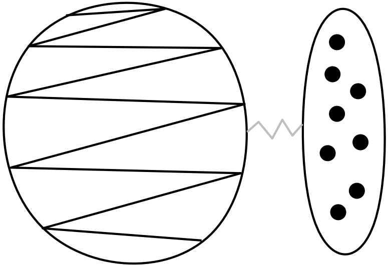
Comparability graphs have a CS-separator of size $\mathcal{O}(n^2)$.



Split-free

Comparability graphs [Yannakakis 1991]

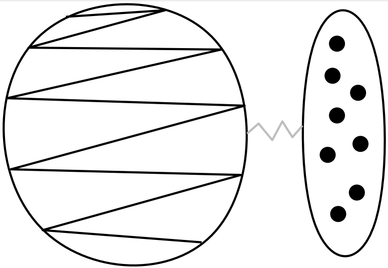
Comparability graphs have a CS-separator of size $\mathcal{O}(n^2)$.



Split-free

Split graph

A graph (V, E) is *split* if V can be partitioned into a clique and a stable set.

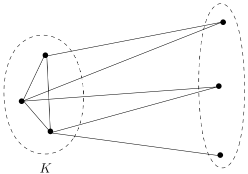


Split-free [Bousquet, L., Thomassé 2012]

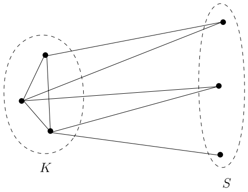
Let H be a split graph. Then every H-free graphs have a CS-separator of size $\mathcal{O}(n^{c_H})$.

Let H be a split graph.

Let H be a split graph.



Let H be a split graph.

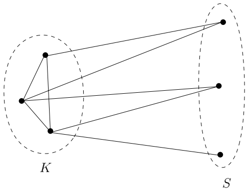


Key Lemma (using VC-dimension)

$\exists$ a constant t s. t. $\forall$ clique K and stable set S in a H-free :

- $\exists S' \subseteq S$ s. t. $|S'| = t$ and S' dominates K
- or, $\exists K' \subseteq K$ s. t. $|K'| = t$ and K' antidominates S

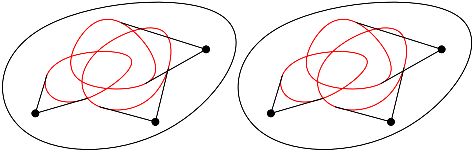
Let H be a split graph.



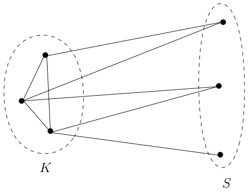
Key Lemma (using VC-dimension)

$\exists$ a constant t s. t. $\forall$ clique K and stable set S in a H-free :

- $\exists S' \subseteq S$ s. t. $|S'| = t$ and S' dominates K
- or, $\exists K' \subseteq K$ s. t. $|K'| = t$ and K' antidominates S



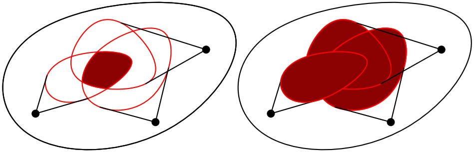
Let H be a split graph.



Key Lemma (using VC-dimension)

$\exists$ a constant t s. t. $\forall$ clique K and stable set S in a H-free :

- $\exists S' \subseteq S$ s. t. $|S'| = t$ and S' dominates K
- or, $\exists K' \subseteq K$ s. t. $|K'| = t$ and K' antidominates S



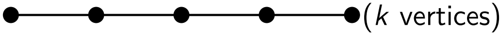
Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	?
All graphs	?	(?)	(?)
P_k-free	?	?	?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	?
All graphs	?	(?)	(?)
P_k-free	?	?	?


(k vertices)

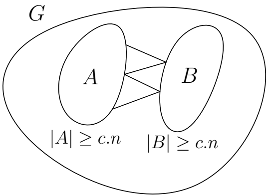
Strong Erdős-Hajnal prop. - $(P_k, \overline{P_k})$-free [Bousquet, L., Thomassé]

For every k, there exists a constant $c > 0$ such that every graph G with no P_k nor $\overline{P_k}$ has two subsets of vertices A and B of size $\geq c.n$, with A complete to B or anticomplete to B.

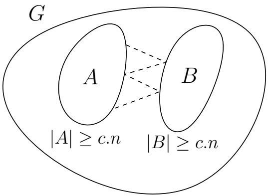


Strong Erdős-Hajnal prop. - $(P_k, \overline{P_k})$-free [Bousquet, L., Thomassé]

For every k, there exists a constant $c > 0$ such that every graph G with no P_k nor $\overline{P_k}$ has two subsets of vertices A and B of size $\geq c.n$, with A complete to B or anticomplete to B.

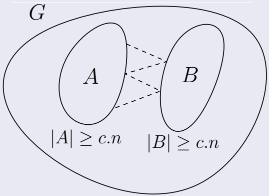


or

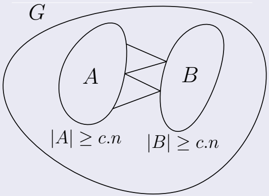


Strong Erdős-Hajnal prop. - $(P_k, \overline{P_k})$-free

$\exists c > 0$

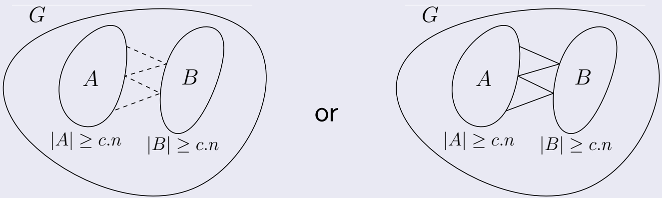


or



Strong Erdős-Hajnal prop. - $(P_k, \overline{P_k})$-free

$\exists c > 0$

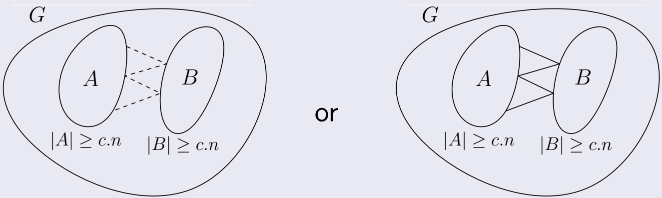


Erdős-Hajnal - $(P_k, \overline{P_k})$-free [Bousquet, L., Thomassé 2013]

There exists $\varepsilon > 0$ such that every $(P_k, \overline{P_k})$-free graph G has a clique or a stable set of size n^{ε}.

Strong Erdős-Hajnal prop. - $(P_k, \overline{P_k})$-free

$\exists c > 0$



Erdős-Hajnal - $(P_k, \overline{P_k})$-free [Bousquet, L., Thomassé 2013]

There exists $\varepsilon > 0$ such that every $(P_k, \overline{P_k})$-free graph G has a clique or a stable set of size n^{ε}.

CS-separation - $(P_k, \overline{P_k})$-free [Bousquet, L., Thomassé 2013]

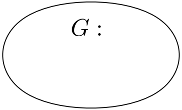
There exists a CS-separator of size $\mathcal{O}(n^{c_k})$ for every $(P_k, \overline{P_k})$-free graph .

Strong EH ⇒ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.

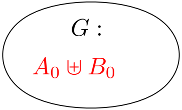
Strong EH $\Rightarrow$ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



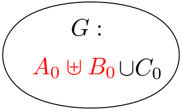
Strong EH ⇒ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



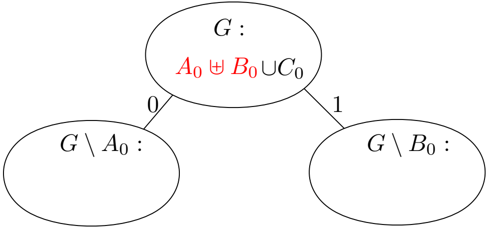
Strong EH ⇒ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



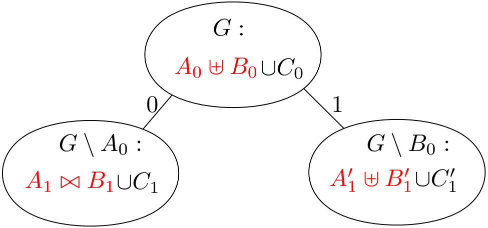
Strong EH ⇒ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



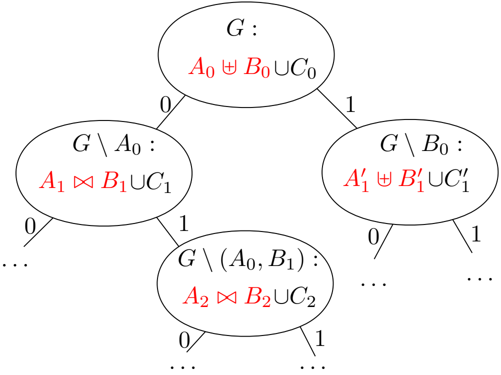
Strong EH ⇒ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



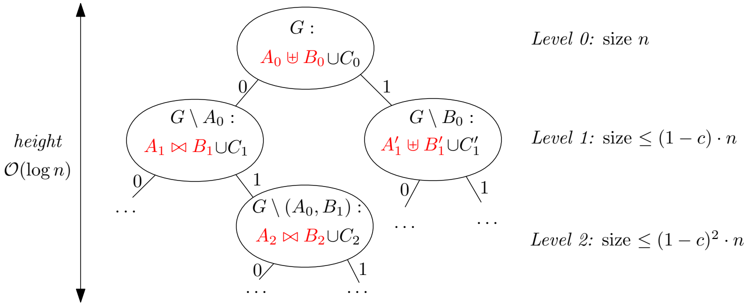
Strong EH ⇒ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



Strong EH ⇒ Deterministic protocol

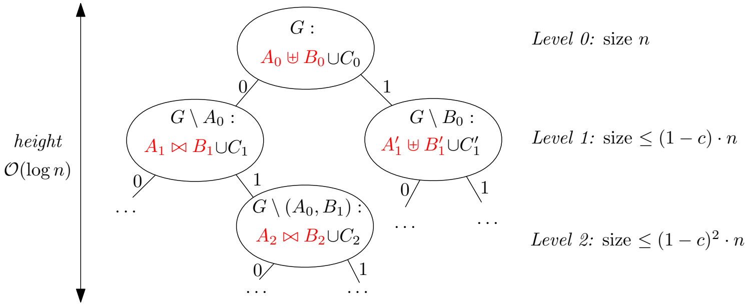
Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



At each step : Alice (for $\uplus$ nodes) or Bob (for $\bowtie$ nodes) sends 1 bit.
Number of steps : Height of the tree $\mathcal{O}(\log n)$.

Strong EH $\Rightarrow$ Deterministic protocol

Let $\mathcal{C}$ be a hereditary class of graphs satisfying the Strong Erdős-Hajnal prop. Then there exists a *deterministic* protocol for Alice and Bob to decide whether $K \cap S = \emptyset$ or not.



At each step : Alice (for $\uplus$ nodes) or Bob (for $\bowtie$ nodes) sends 1 bit.
Number of steps : Height of the tree $\mathcal{O}(\log n)$.
Excluding only P_k and not $\overline{P_k}$?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

P_5-free graphs [Loksthanov, Vatshelle, Villanger 2013]

Max. Weighted Stable Set is polytime solvable in P_5-free graphs.
(They actually proved a stronger statement.)

P_5-free graphs [Loksthanov, Vatshelle, Villanger 2013]

Max. Weighted Stable Set is polytime solvable in P_5-free graphs. (They actually proved a stronger statement.)

Consequences from the stronger statement :

P_5-free graphs [Bousquet, L., Thomassé 2013]

Every P_5-free graph has a CS-separator of size $\mathcal{O}(n^8)$.

P_5-free graphs [Lokshtanov, Vatshelle, Villanger 2013]

Max. Weighted Stable Set is polytime solvable in P_5-free graphs. (They actually proved a stronger statement.)

Consequences from the stronger statement :

P_5-free graphs [Bousquet, L., Thomassé 2013]

Every P_5-free graph has a CS-separator of size $\mathcal{O}(n^8)$.

Extended formulation for P_5-free graphs [Conforti, Di Summa, Faenza, Fiorini, Pashkovich]

For every P_5-free graph G, $STAB(G)$ has an extended formulation of polynomial size.

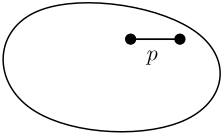
Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Random graphs [Bousquet, L., Thomassé 2012]

For every $n \in \mathbb{N}$, $p \in [0,1]$, there exists a set $\mathcal{F}$ of $\mathcal{O}(n^7)$ cuts such that

$$\forall G \in G(n,p) \qquad \Pr(\ \mathcal{F} \text{ is a CS-sep for } G) \underset{n \to +\infty}{\longrightarrow} 1$$

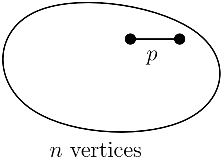


n vertices

Random graphs [Bousquet, L., Thomassé 2012]

For every $n \in \mathbb{N}$, $p \in [0,1]$, there exists a set $\mathcal{F}$ of $\mathcal{O}(n^7)$ cuts such that

$$\forall G \in G(n,p) \qquad \Pr(\ \mathcal{F} \text{ is a CS-sep for } G) \underset{n \to +\infty}{\longrightarrow} 1$$



Idea : since the edges are all drawn with the same probability p, cliques and stables sets can not both be too big.

Example for $p = 1/2$: $\alpha \approx \omega \approx 2 \log n$.

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?

Class of graphs	Poly CS-sep	Poly $rk_+(M_{QSTAB})$	Poly $rk_+(M_{STAB})$
H-free, H split	Yes	?	?
H-free, H : P_4-free split	Yes	Yes (det)	?
P_4-free	Yes	Yes	
$(P_k, \overline{P_k})$-free (Strong EH)	Yes	Yes (det)	?
P_5-free	Yes	Yes	Yes
Random	Yes	(?)	(?)
Perfect with no bal. skew part.	Yes	Not hereditary	
Perfect	?	?	
All graphs	?	?	No
P_k-free	?	?	?