Coloring perfect graphs with bounded clique number

JGA 2016

Aurélie Lagoutte

Joint work with M. Chudnovsky, P. Seymour and S. Spirkl

G-SCOP, Univ. Grenoble Alpes

November 17, 2016
Paris Dauphine

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Interactions between Cliques and Stable sets in a graph

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Partition into stable sets; relate nb of parts to size of max. clique

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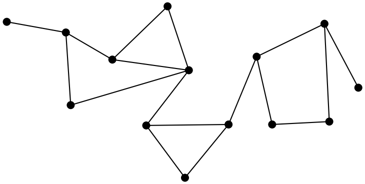
- **Clique-Stable set Separation**
  (see next slide)
- **Pbs around CS-Separation**
  PL for Max. Weighted Stable Set, Constraint Satisfaction Pbs (CSP), Alon-Saks-Seymour conjecture

Clique-Stable set Separation

Game on a graph G:



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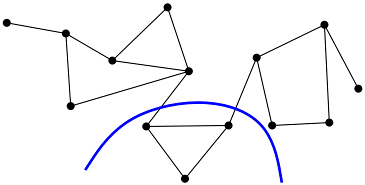

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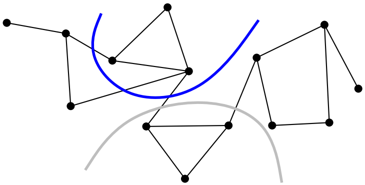

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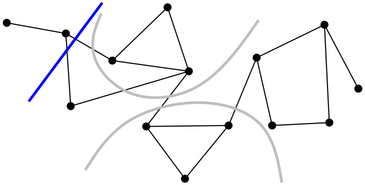

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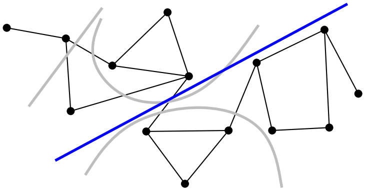

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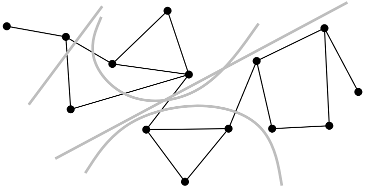

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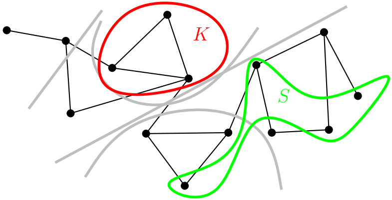

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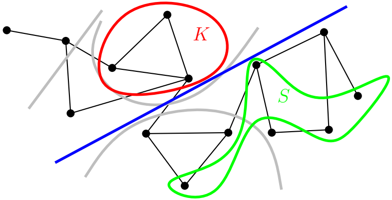

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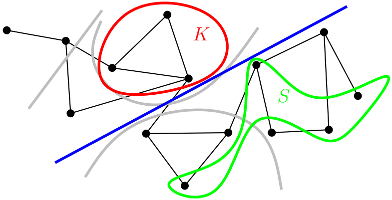

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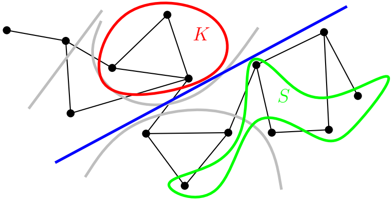

A **CS-Separator** is a family of cuts that ensures me to always win.
→ I am allowed to select only **polynomially many** cuts.

Bounds

Yannakakis (1991)

Upper Bound: $\forall G$ there exists a CS-separator of size $\mathcal{O}(n^{\log n})$.

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Does there exist for all graph G on n vertices a CS-separator of size poly(n)? Or for which classes of graphs does it exist?

It is known that the following classes of graphs admit poly-size CS-Separator:

- If ω or α is bounded (trivial)
- chordal graphs (linear number of max. cliques)
- comparability graphs (Yannakakis 1991)
- C_4-free graphs (Conseq. of Alekseev 1991)
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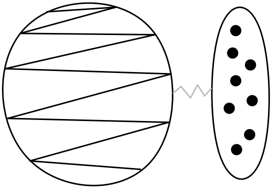
- H-free graphs where H is a split graph.



Split graph

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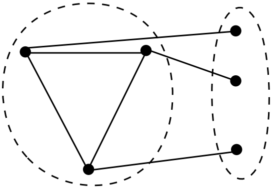

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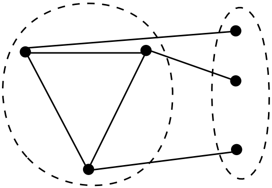

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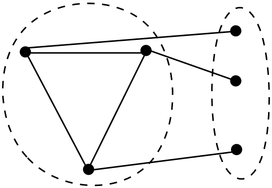
Joint work with Bousquet, Thomassé ; and Trunck

- Random graphs
- H-free graphs where H is a split graph.
- $(P_k, \overline{P_k})$-free graphs
- Perfect graphs with no BSP



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~~**Does there exist for all graph G on n vertices a CS-separator of size poly(n)?**~~ **No!**

Lower Bound: (Göös 2015): we need $n^{\Omega(\log^{0.128} n)}$ cuts for some graphs.

What now?

$\rightarrow$ Want to learn more about perfect graphs and try to close the CS-Separation question on them.

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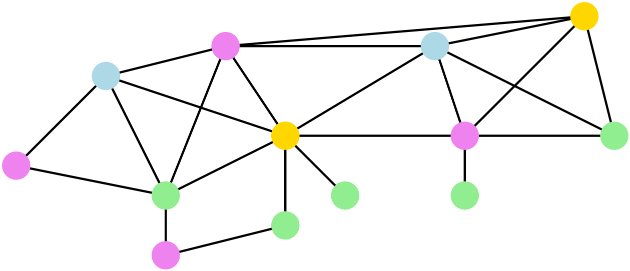

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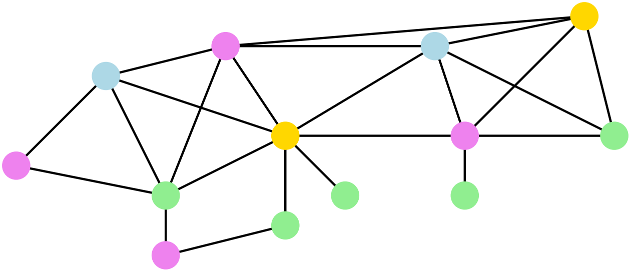

Graph Coloring

Input: A graph G and an integer k.
Output: Does G admits a proper k-coloring?

Graph Coloring is NP-complete.

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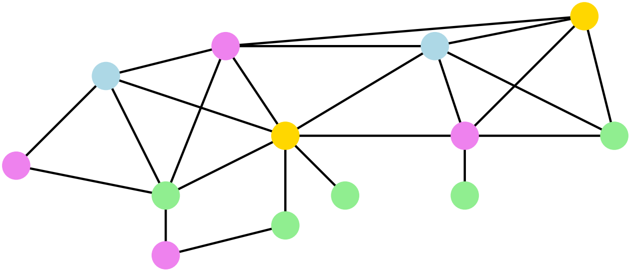

~~Graph Coloring~~ 3-Coloring

Input: A graph G ~~and an integer k.~~

Output: Does G admits a proper ~~k-coloring?~~ 3-coloring?

Graph Coloring is NP-complete. Even 3-Coloring is!

- $\chi(G)$: **chromatic number** of G, i.e. minimum number of color in a proper coloring.
- $\omega(G)$: **clique number**, i.e. size of the largest clique.

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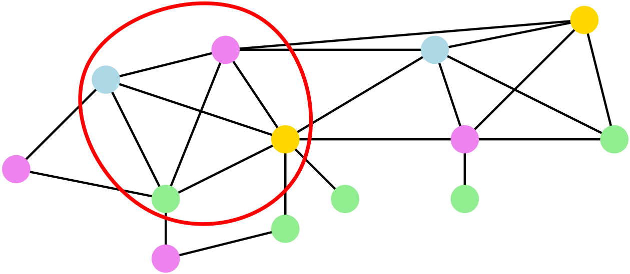
$$\chi(G) \geq \omega(G)$$



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G is **perfect** if and only if $\chi(G) = \omega(G)$ and the equality holds for every induced subgraph H of G.

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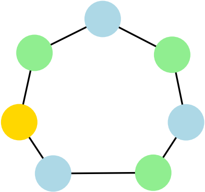
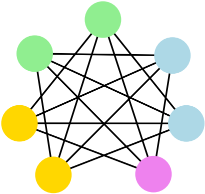


Odd hole C_7



Odd antihole $\overline{C_7}$

Berge's Conjecture (1960's)

A graph G is perfect if and only if G contains no odd hole and no odd antihole as induced subgraph.

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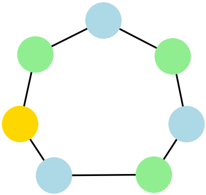

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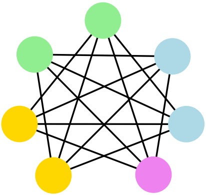

Odd antihole $\overline{C_7}$

Berge's Conjecture (1960's) ⇒ Strong Perfect Graph Theorem

A graph G is perfect if and only if G contains no odd hole and no odd antihole as induced subgraph.

Proved in 2002 by Chudnovsky, Robertson, Seymour and Thomas.

What about coloring perfect graphs?

Theorem [Grötschel, Lovász, Schrijver 1981]

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Not satisfying! We know so much on perfect graphs that we want a *combinatorial* algorithm.

We know *so much*?

Decomposition theorem from [CRST'02]

If G is Berge, then

- either G is *basic* (bipartite, line graph of bipartite, ...),
- or G can be *decomposed* in a given way.

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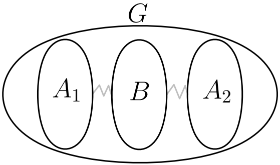

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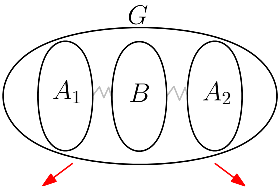

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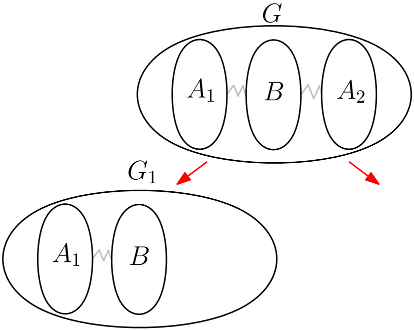

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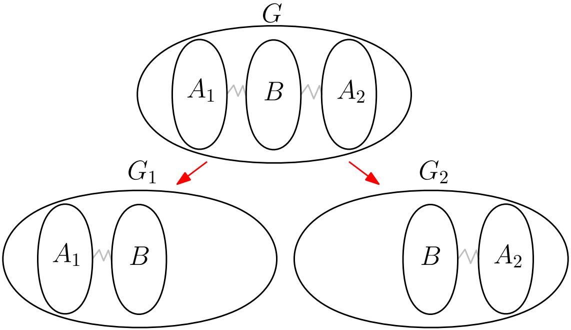

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$\Rightarrow$ Contradiction!

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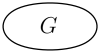

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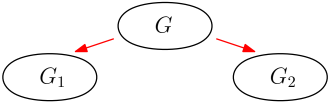

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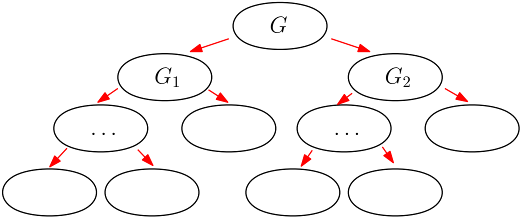

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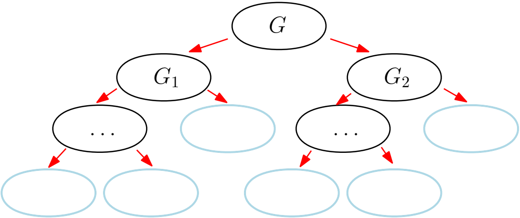

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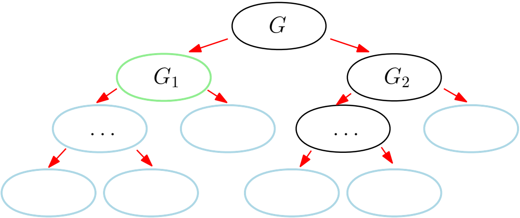

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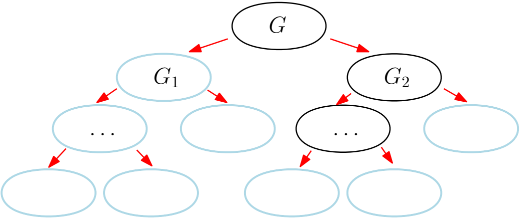
3. From bottom to top: **combine solutions for children** to get a solution for the father.

How to use it for algorithmic purposes?

Want to compute something (coloring, largest stable set, ...).

Meta-algorithm:

1. Construct the decomposition tree:



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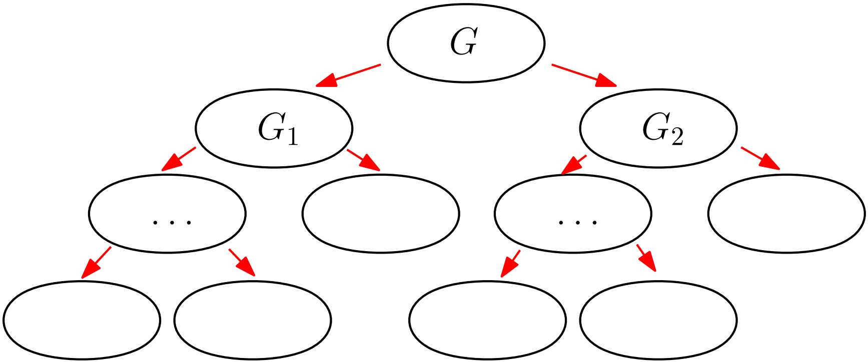
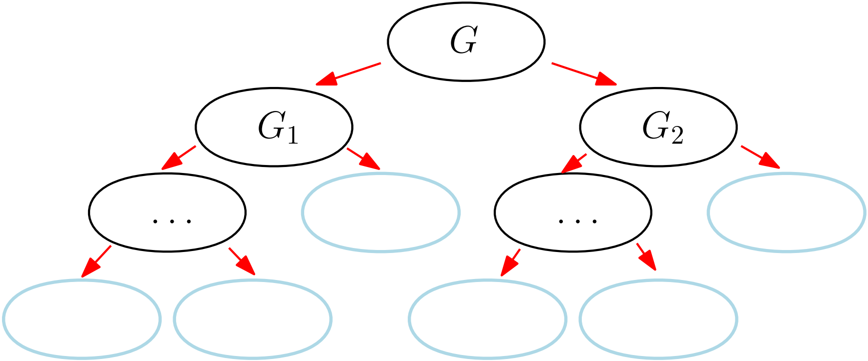
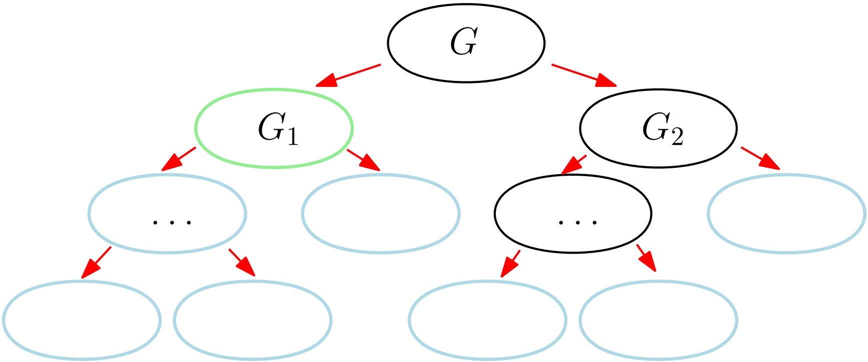


Hence four intermediate steps to reach:



Hence four intermediate steps to reach:

- Know how to directly solve the problem on leaves.



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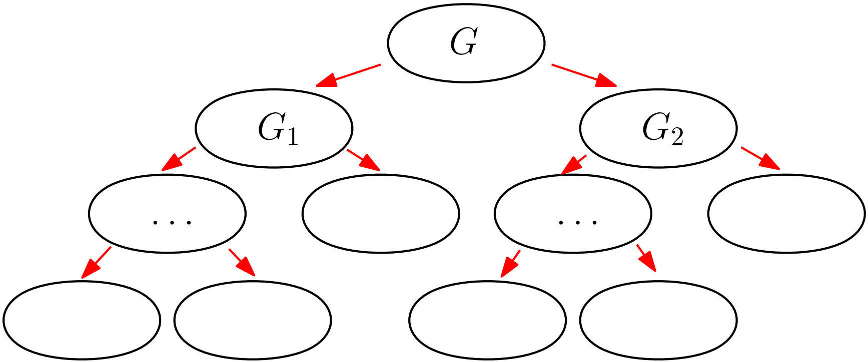

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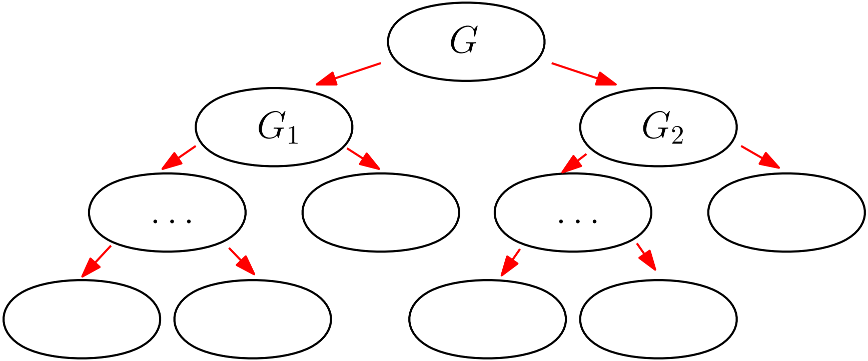
- Bound the size of the tree by a polynomial in n.



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- Know how to algorithmically construct the tree.

Our result

Theorem [Chudnovsky, L., Seymour, Spirkl]

We design an algorithm with the following specification:

Algorithm $\mathcal{A}_k$:

Input: A perfect graph G with $\omega(G) \leq k$.

Output: A proper coloring of G with $\chi(G) = \omega(G)$ colors.

Running time: $\mathcal{O}(n^{(\omega(G)+1)^2})$

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We proceed by induction on $k \rightarrow$ we can call $\mathcal{A}_{k-1}$ when needed.

Previous results in this direction:

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A combinatorial algorithm that optimally colors:

- *any Berge graph with no BSP*

  [Chudnovsky, Trotignon, Trunkc, Vušković 2015]

- *any C_4-free Berge graph*

  [Chudnovsky, Lo, Maffray, Trotignon, Vušković 2015^+]

Outline

Five intermediate steps to reach:

- Describe the decomposition tree that is used
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Decomposing perfect graphs

Decomposition theorem

If G is perfect, then at least one of the following holds:

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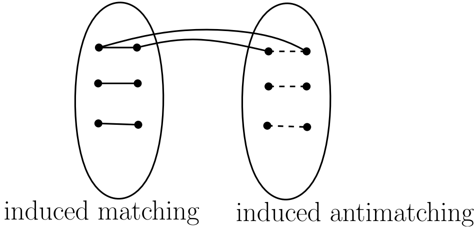
- G or $\overline{G}$ lies in one of the following classes: bipartite graphs, line graphs of a bipartite graph, double split.



Double-Split

Decomposing perfect graphs

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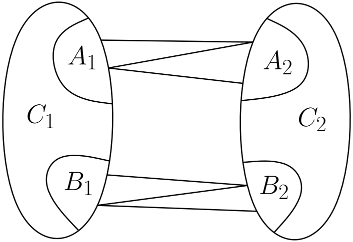

2-join

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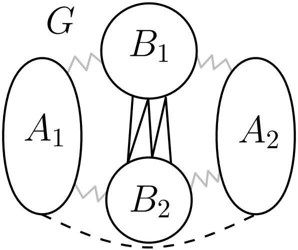
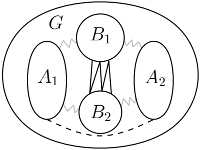
- G or $\overline{G}$ lies in one of the following classes: bipartite graphs, line graphs of a bipartite graph, double split.
- G or $\overline{G}$ admits a decomposition by 2-join,
- G admits a decomposition by balanced skew partition (BSP).



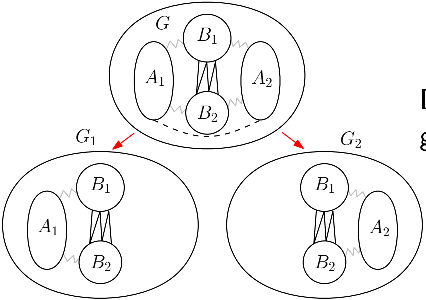
Skew partition

Our decomposition tree



Decompose along BSP until the graph:

Our decomposition tree



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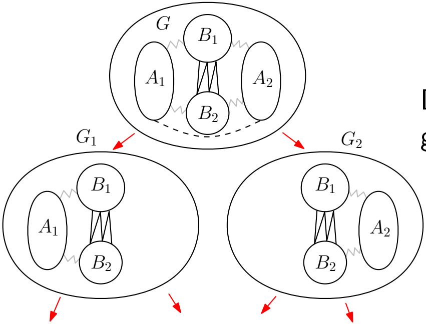

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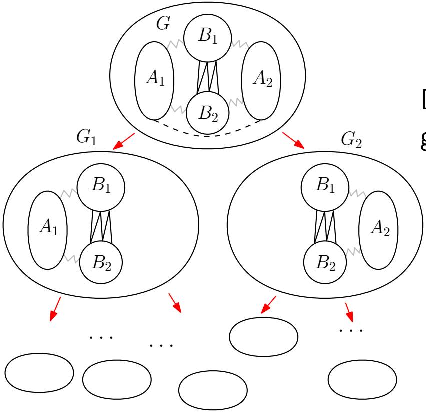

Decompose along BSP until the graph:

- admits no BSP,
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There is an algorithm that, given as input a perfect graph G, outputs a BSP of G or asserts that there is none.

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There is an algorithm that, given as input a perfect graph G, outputs a BSP of G or asserts that there is none.

Previous results:

- Deciding if a graph has a BSP is NP-complete. [Trotignon 08]
- A poly-time algorithm that decides if a **perfect** graph has a BSP can be done in polynomial-time (but, if yes, the algo does not output such a partition). [Trotignon 08]
- A poly-time algo that decides if a graph has a skew partition and, if yes, outputs such a partition. [Kennedy & Reed 08]

Outline

Five intermediate steps to reach:

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  $\rightarrow$ time $\mathcal{O}(n^{\max(7,\omega(G)^2)})$
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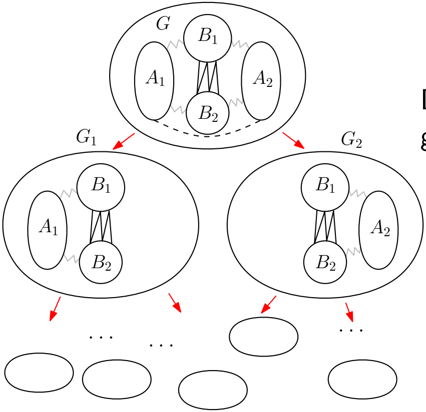
Our decomposition tree



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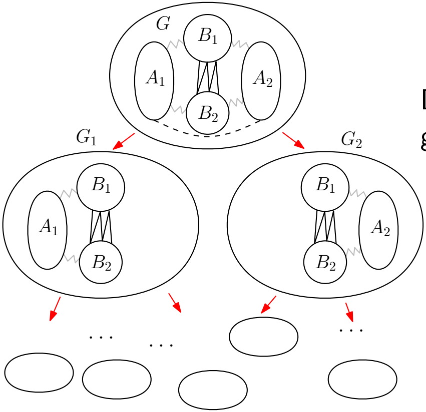

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Color with CTTV algo $\rightarrow \mathcal{O}(n^7)$

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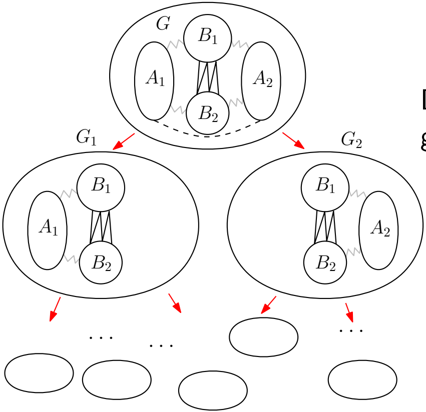

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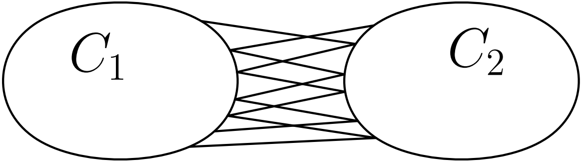
If G is not anticonnected:



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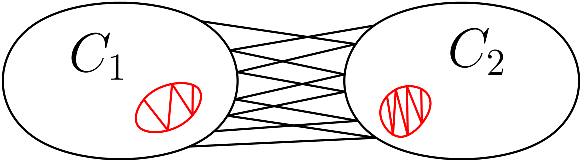

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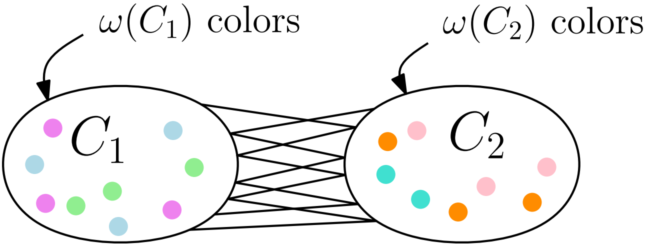

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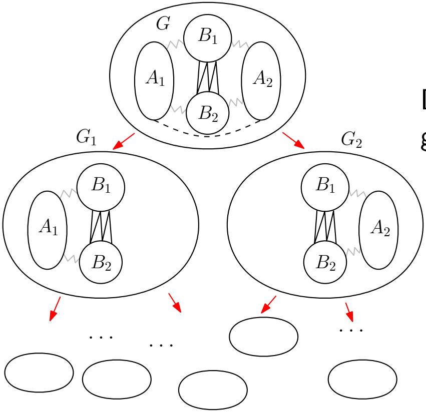
Color each side with $\mathcal{A}_{k-1}$

Our decomposition tree



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Color with CTTV algo $\to \mathcal{O}(n^7)$
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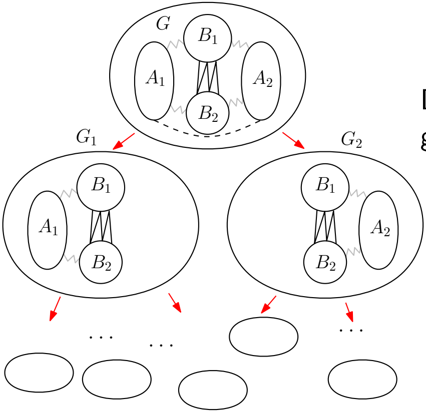
Our decomposition tree



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Color with CTTV algo $\rightarrow \mathcal{O}(n^7)$
Color with $\mathcal{A}_{k-1} \rightarrow \mathcal{O}(n^{\omega(G)^2})$
Easy to color in $f(k)$

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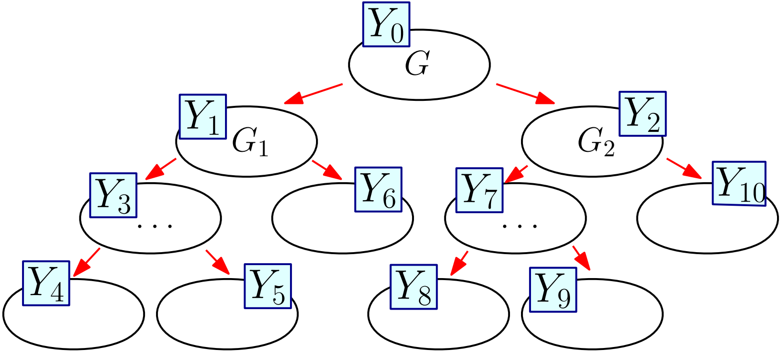
How to bound the size of the tree?

Label each node of the tree with some well-chosen $Y \subseteq V(G)$:



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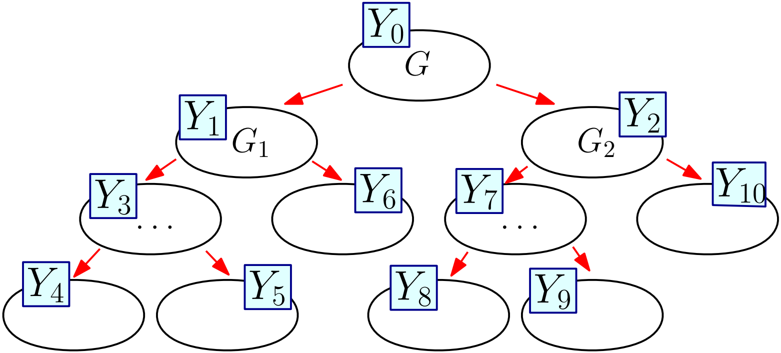

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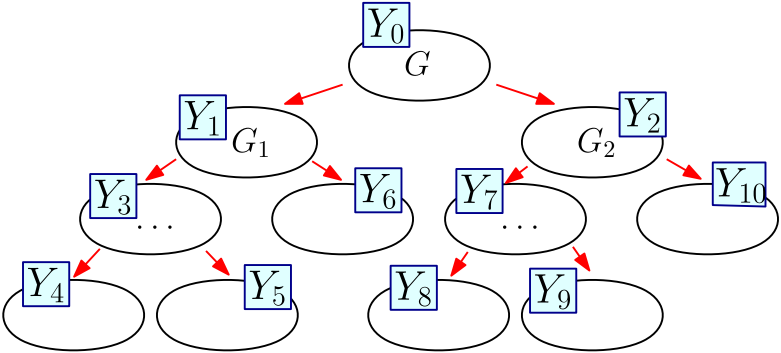

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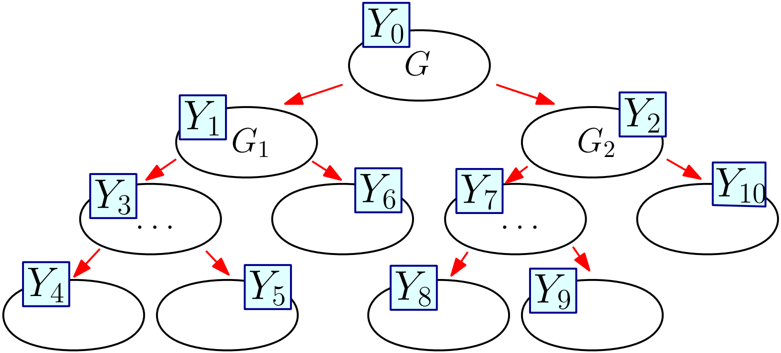

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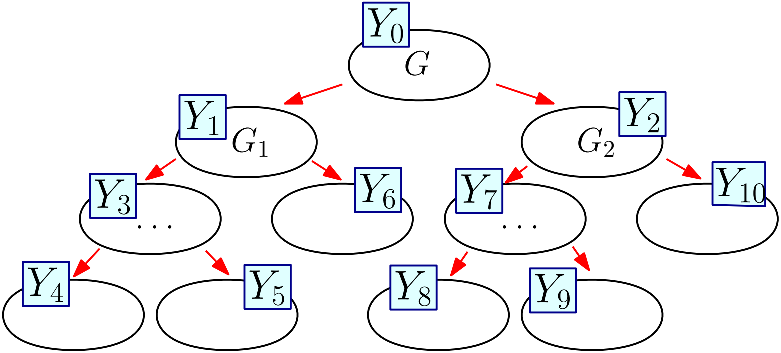

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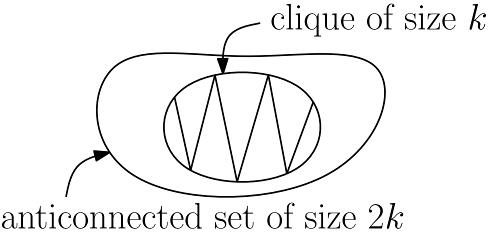
$\Rightarrow$ Bounds the number of nodes by a polynomial.

Key ingredient: k-pellet

Definition: k-pellet

A subset $Y \subseteq V(G)$ is a k-pellet if

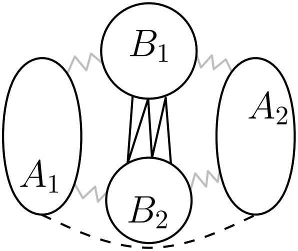
- Y contains a clique of size k,
- Y is anticonnected,
- and $|Y| = 2k$.



Number of $\omega(G)$-pellets: at most $\mathcal{O}(n^{2\omega(G)})$!!

Good property of $\omega(G)$-pellet

An $\omega(G)$-pellet can not lie in the middle part $B_1 \cup B_2$ of a BSP.



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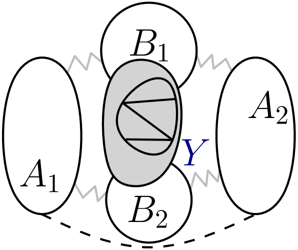

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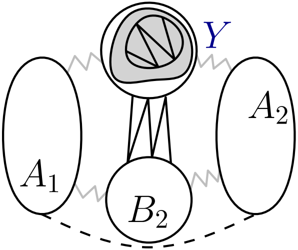

Y is anticonnected.

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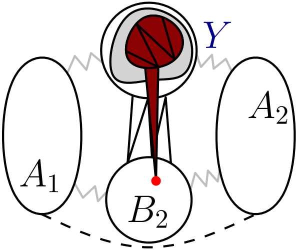

Y is anticonnected.

Y contains a clique of size $\omega(G)$ and any $v \in B_2$ is complete to it.

$\Rightarrow$ Contradiction!

Unique labeling

Two nodes getting the same label Y?

Unique labeling

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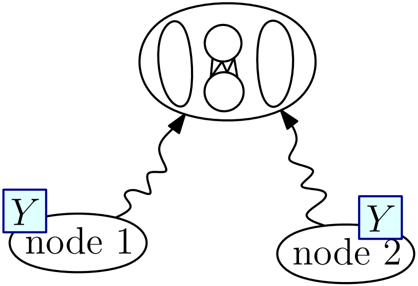
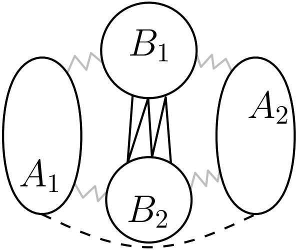


Where is Y?



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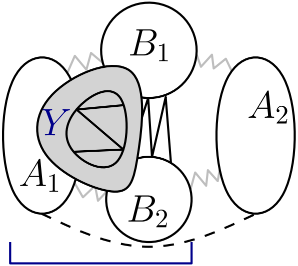
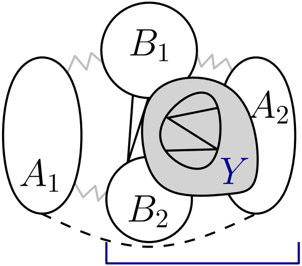


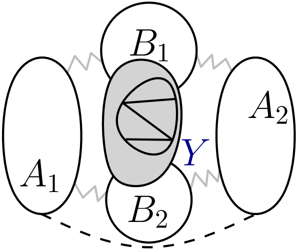
Y appears in left descendants.

Where is Y?



Y appears in left descendants.
Y appears in right descendants.

Where is Y?



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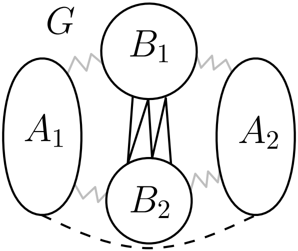
How to combine solutions?

Problem when gluing solutions



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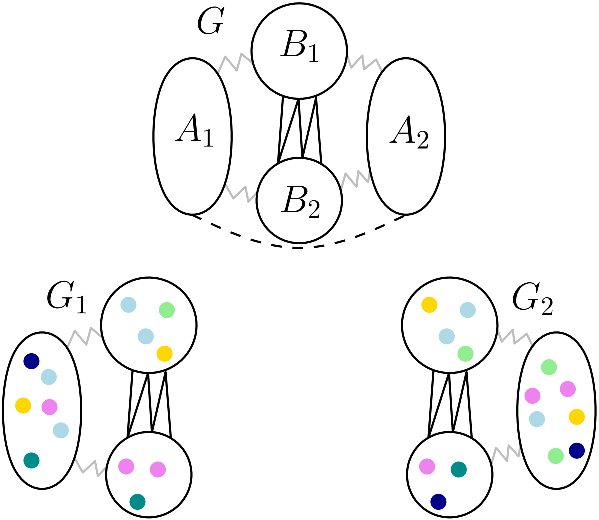

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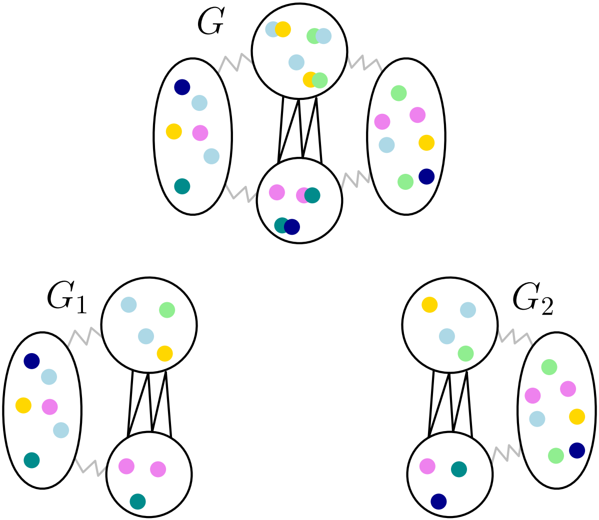

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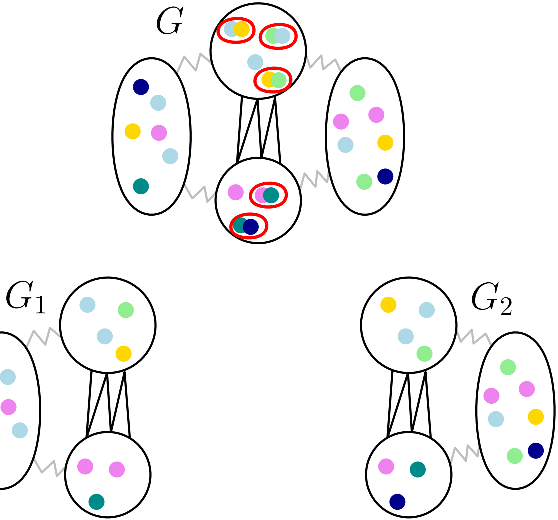

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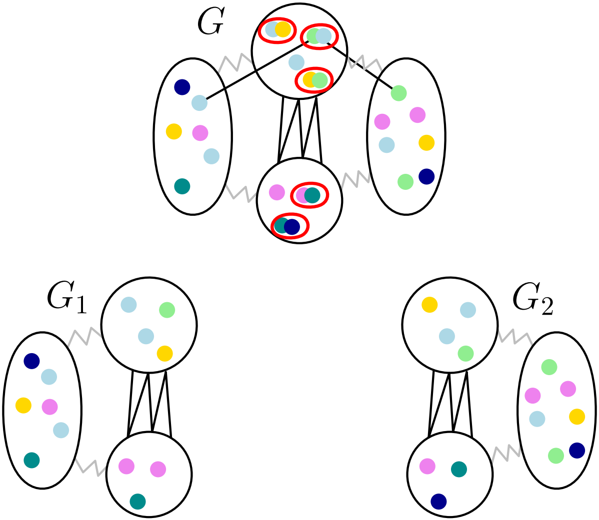
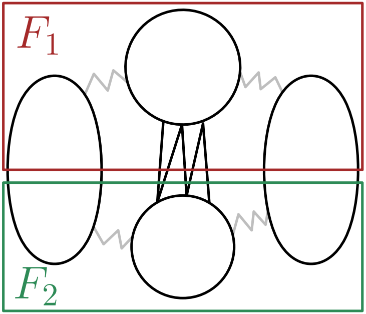


Goal: Find a partition in two sets (F_1, F_2):

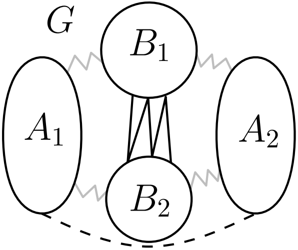
- $\omega(F_1) = k_1 < \omega(G)$;
- $\omega(F_2) = k_2 < \omega(G)$;
- $k_1 + k_2 = \omega(G)$.



Then we will call $\mathcal{A}_{k-1}$ on F_1 and F_2.

How to find such a partition (F_1, F_2)?

1. Compute $k_1 = \omega(B_1) < k$ (test every X s.t. $|X| < k$).



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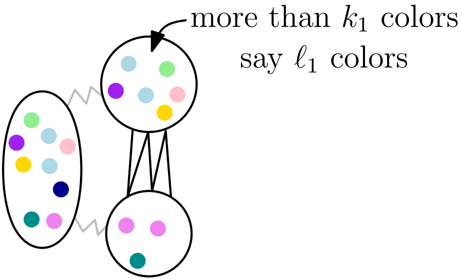

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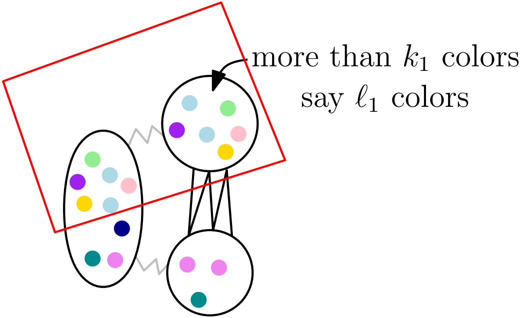

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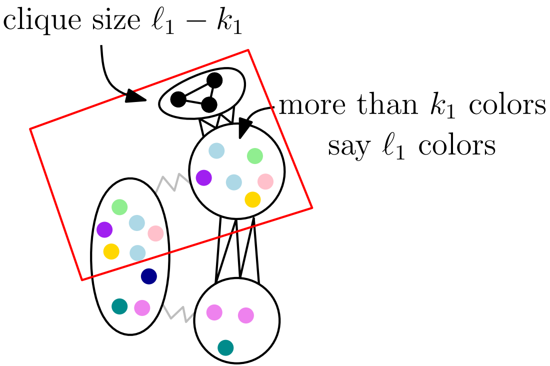

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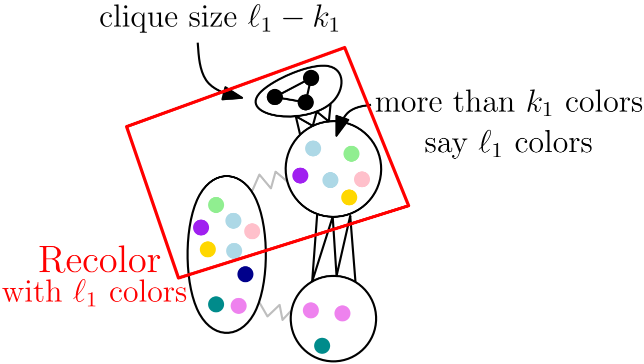

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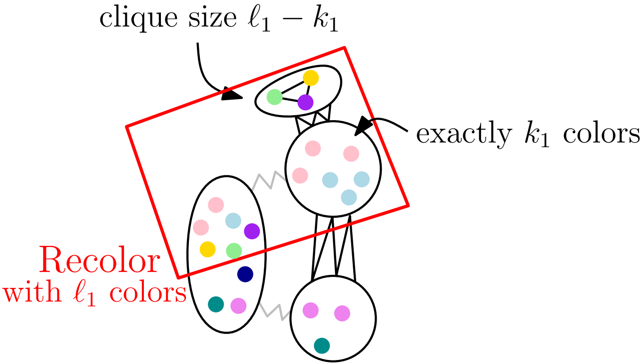

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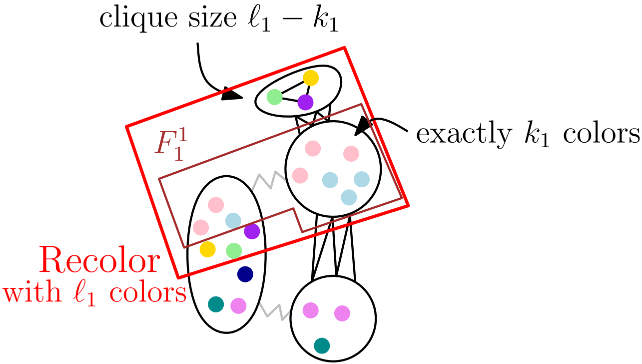

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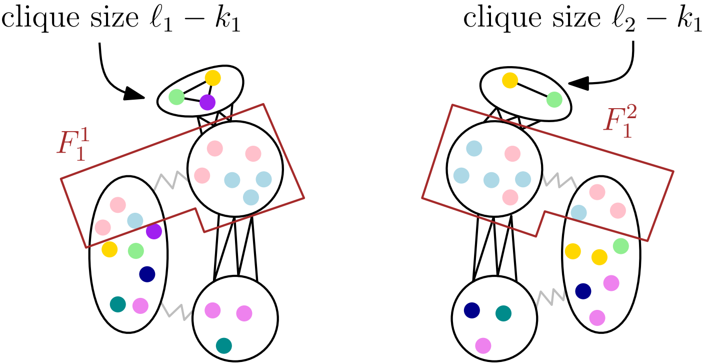

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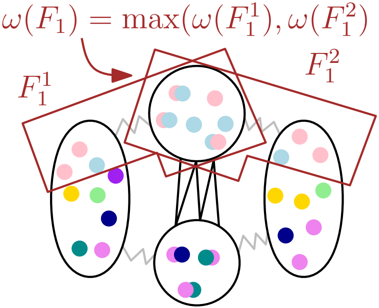

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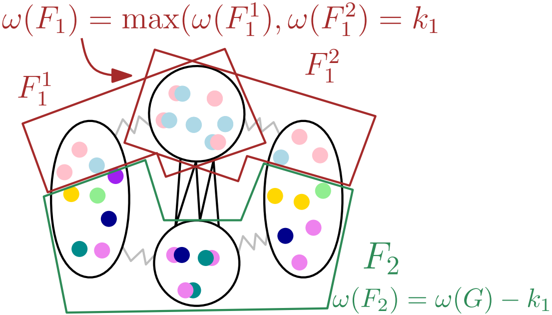

Goal: Find a partition in two sets (F_1, F_2):

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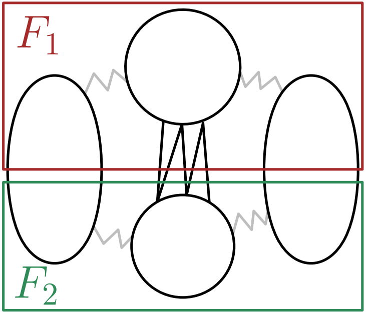

Call $\mathcal{A}_{k-1}$ on F_1 and F_2. This gives a proper $\omega(G)$-coloring of G.

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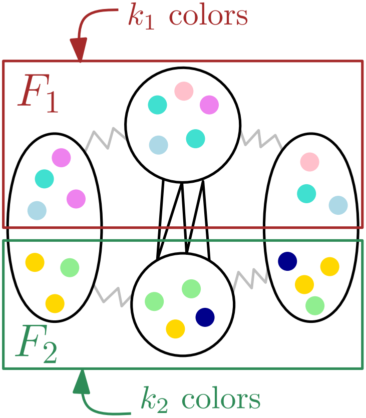

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Outline

Five intermediate steps to reach:

- ✓ Describe the decomposition tree that is used
- ✓ Know how to algorithmically construct the tree
- ✓ Know how to directly solve the problem on leaves
  $\rightarrow$ time $\mathcal{O}(n^{\max(7,\omega(G)^2)})$
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$\rightarrow$ Algorithm $\mathcal{A}_k$ is well-defined and runs in time $\mathcal{O}(n^{(\omega(G)+1)^2})$

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Thank you for your attention!