Clique-Stable Set separation in Berge graphs with no balanced skew partition

Aurélie Lagoutte, Théophile Trunck

LIP, ENS Lyon

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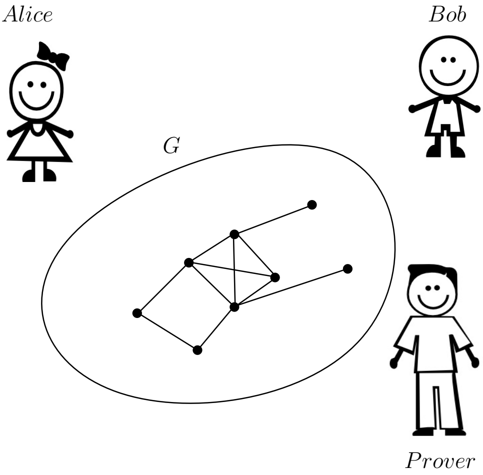
Bertinoro Workshop on Algorithms and Graphs

Clique vs Independent Set Problem



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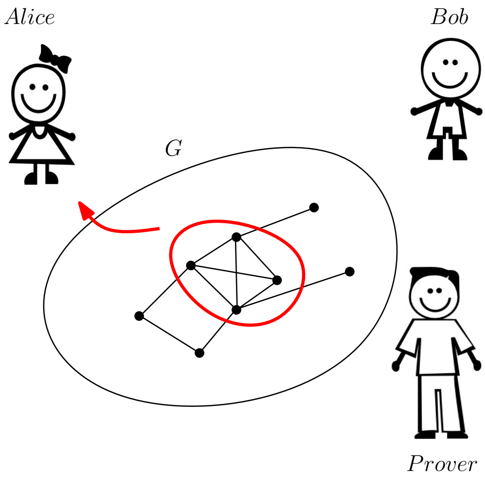

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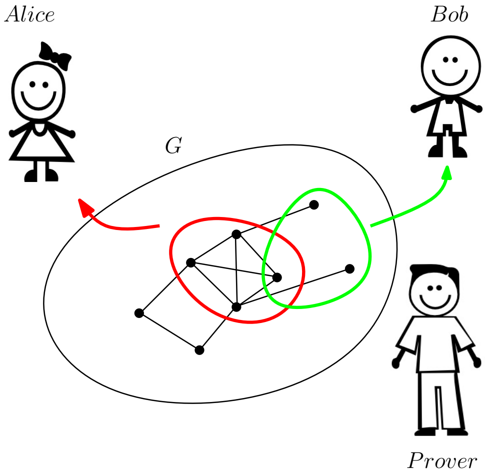

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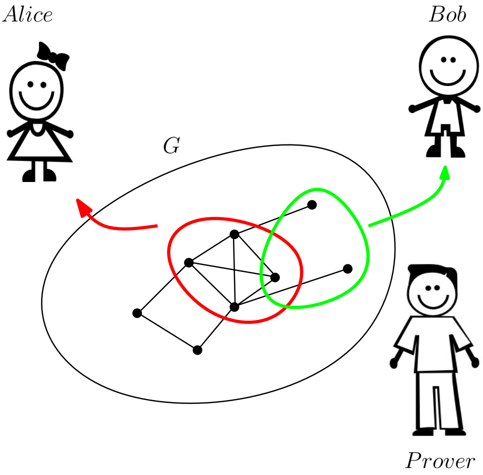

Do the clique and the stable set intersect?

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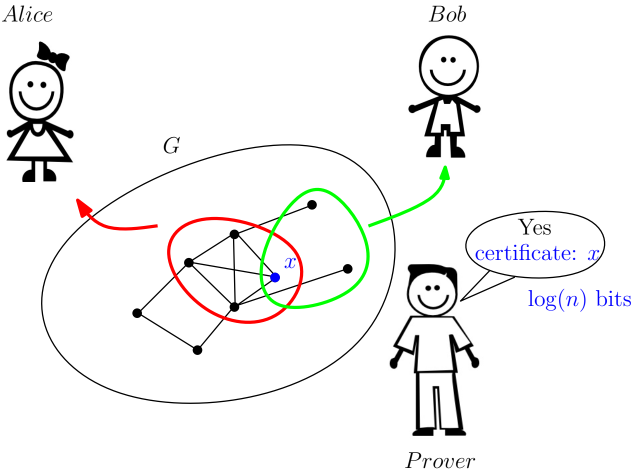

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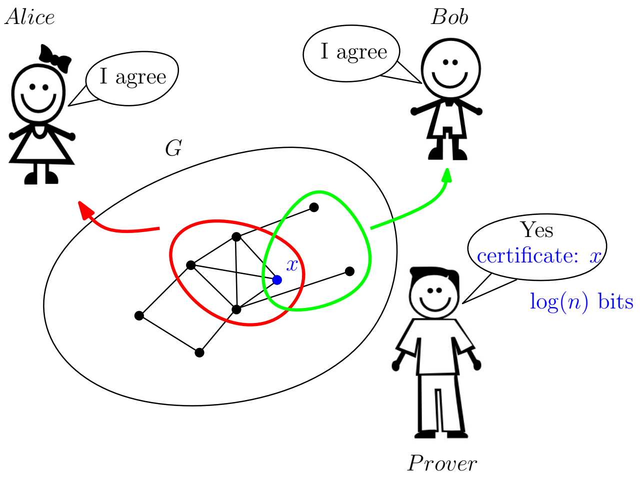

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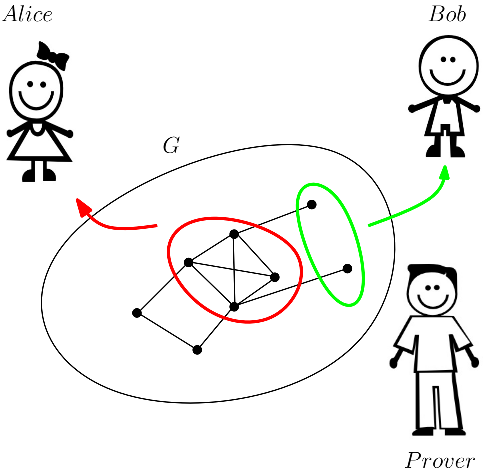

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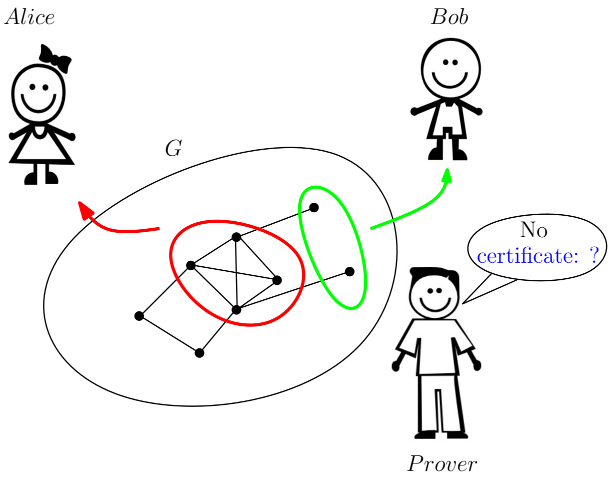

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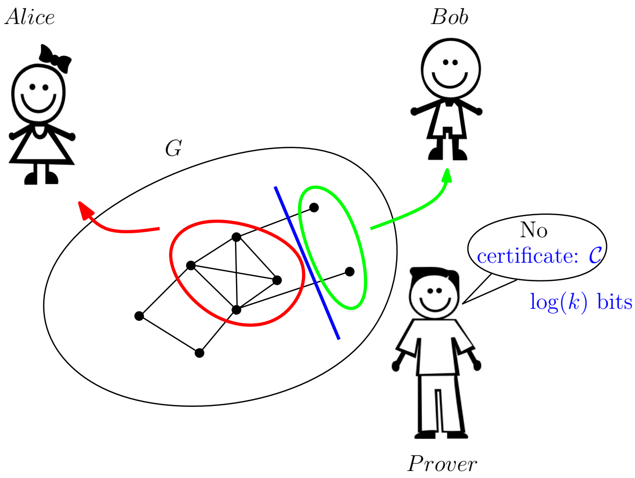

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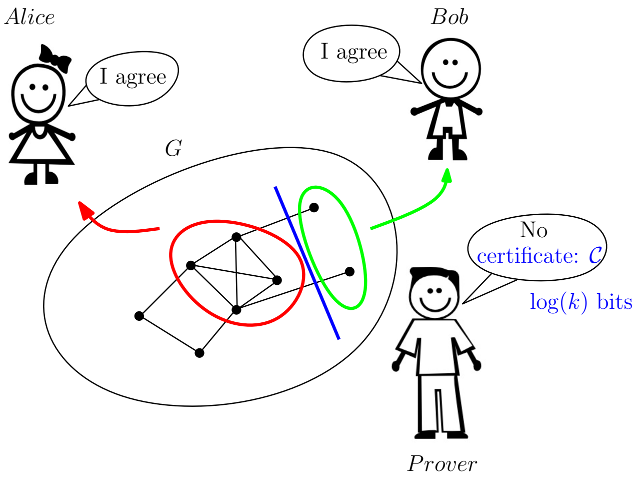

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Goal

Find a *CS-separator* : a family of cuts that can separate all the pairs Clique-Stable set.

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Non-det. communication complexity $= \log k$
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If $k = n^c$, then complexity$=\mathcal{O}(\log n)$.

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Upper Bound : there exists a CS-separator of size $\mathcal{O}(n^{\log n})$.
Lower Bound [Amano, Shigeta 2013] : there exists an infinite family of graphs such that any CS-separator has size $\Omega(n^{2-\varepsilon})$

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Does there exist for all graph G on n vertices a CS-separator of size poly(n) ? Or for which classes of graphs does it exist ?

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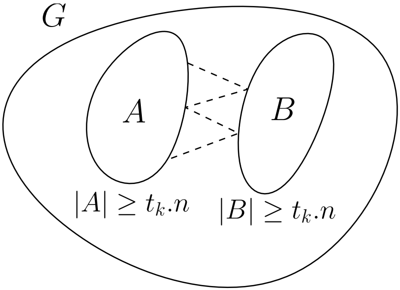

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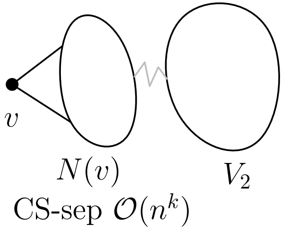
- Simple-neighbourhood property : $\exists k,\ \exists v \in V$ s.t. $N(v)$ admits a $\mathcal{O}(n^k)$ CS-separator.



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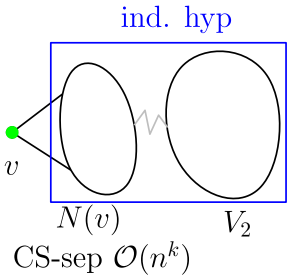

v in stable set side

Cuts every (K, S) with $v \in S$

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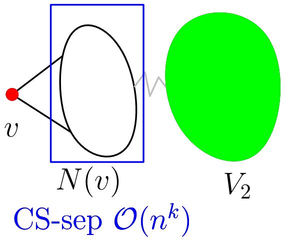

v in clique side

V_2 in stable set side

Cuts every (K, S) with $v \in K$

Perfect graph

We denote $\omega(G)$ the size of the biggest clique and $\chi(G)$ the chromatic number of G. A graph is called *perfect* if for every induced subgraph of H, we have :

$$\chi(H) = \omega(H)$$

Berge graph

A *hole* is an induced (chordless) cycle of length at least 4.
A graph G is Berge if neither G nor $\overline{G}$ has an odd hole.

Strong Perfect Graph Theorem [Chudnovsky, Roberston, Seymour, Thomas, 2006]

A graph is perfect if and only if it is Berge.

Decomposition [Chudnovsky, Roberston, Seymour, Thomas]

If a graph is Berge, then for G or $\overline{G}$, one of the following holds :

- It is a basique graph : bipartite, line graph of bipartite, or double split.
- There is a 2-join
- There is a balanced skew partition.

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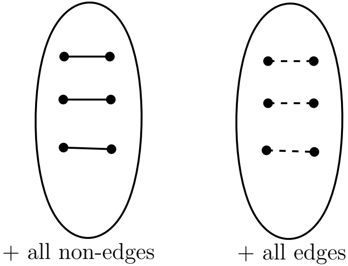

FIGURE : Double split

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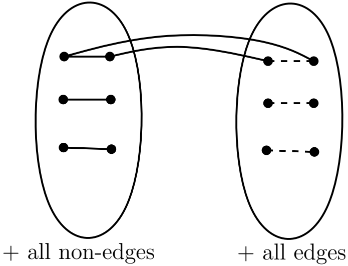

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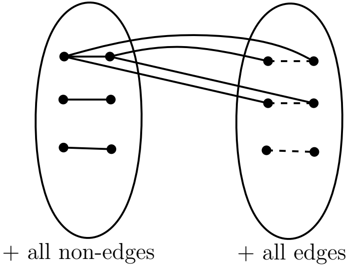

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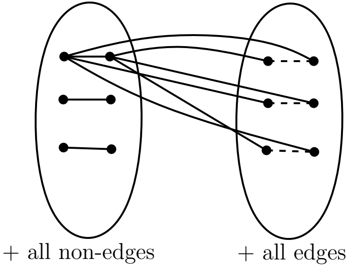

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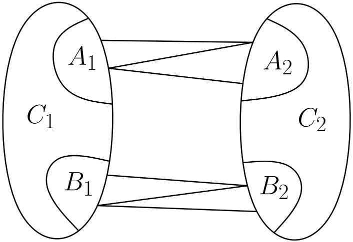

Figure : 2-join

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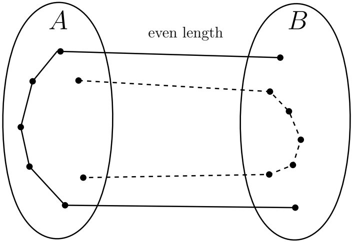

Figure : Balanced partition

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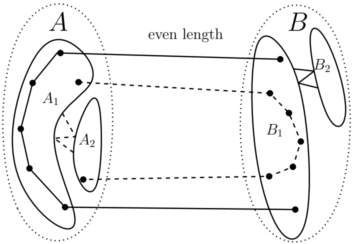
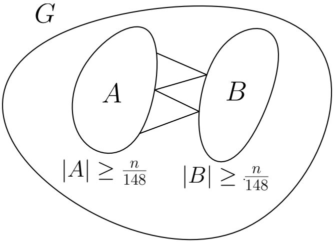


FIGURE : Balanced skew partition

Linear bipartite property [L., Trunck 2013]

Every Berge graph with no balanced skew partition admits two subsets of vertices A and B of size at least $n/148$, with A complete or anticomplete to B.



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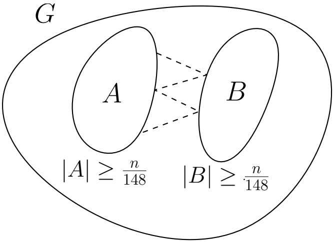

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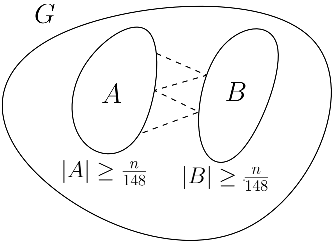

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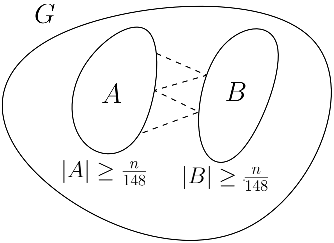

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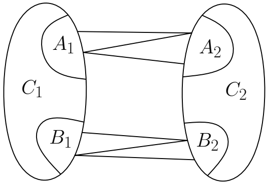
But there exist perfect graphs that do not verify the linear bipartite property [Fox 2006] $\Rightarrow$ Evidence of some structure

[L., Trunck, 2013]

Let G be a Berge graph with no balanced skew partition, then there exists a CS-separator for G of size $\mathcal{O}(n^2)$.

Proof by induction :

- For basic graphs
- For a graph G with a 2-join :
  from G, we build two Berge graphs G_1 and G_2 with no balanced skew partition, each corresponding to a side of the 2-join + a gadget.
  [Chudnovsky, Trotignon, Trunck, Vušković 2012]
  $\Rightarrow$ CS-separators for G_1 and G_2 by induction hypothesis
  $\Rightarrow$ we transform them into a CS-separator for G.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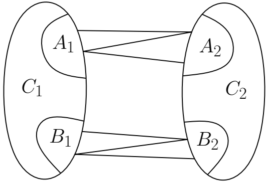
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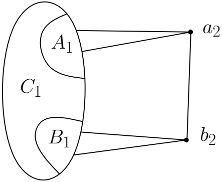
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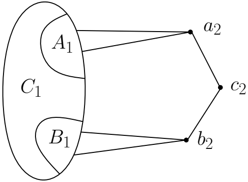
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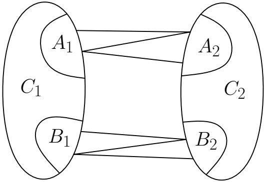
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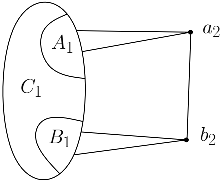
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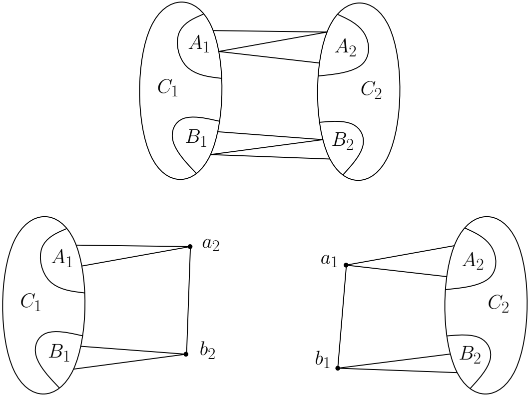
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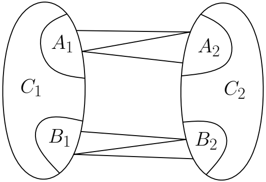
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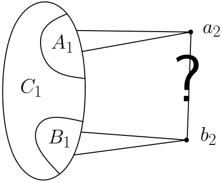
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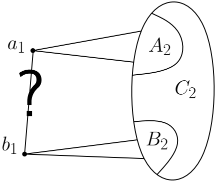
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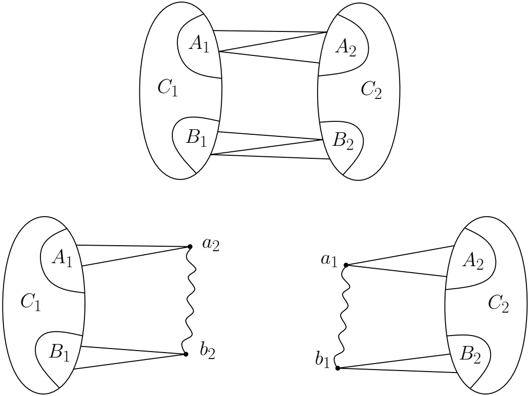
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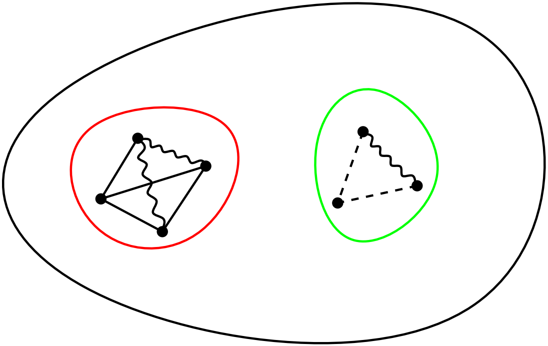
Trigraphs [Chudnovsky, 2006]

A trigraph is composed of a set of vertices V, and between each pair of vertices u and v, there can be either :

- A strong edge : $u \bullet\!\!\longrightarrow\!\!\bullet\ v$
- A strong antiedge : $u \bullet - - - - \bullet\ v$ ou $u \bullet \qquad \bullet\ v$
- A switchable pair (which can play the role both of an edge and an non-edge) : $u \bullet\!\!\sim\!\sim\!\sim\!\!\bullet\ v$

A trigraph has a hole if we can chose the switchable pair in such a way to create a hole. A trigaph T is Berge if neither T nor $\overline{T}$ has odd holes.

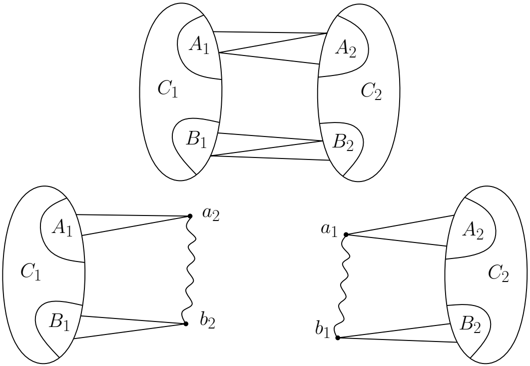
Clique-Stable set separation in trigraphs :
A clique (resp. a stable set) can contain switchable pairs.



Decomposition [Chudnovsky, 2006]

If a trigraph T is Berge, then for T or $\overline{T}$, one of the following holds :

- It is a basique trigraph : bipartite, line trigraph, or doubled.
- It admits a 2-join
- It admits a balanced skew partition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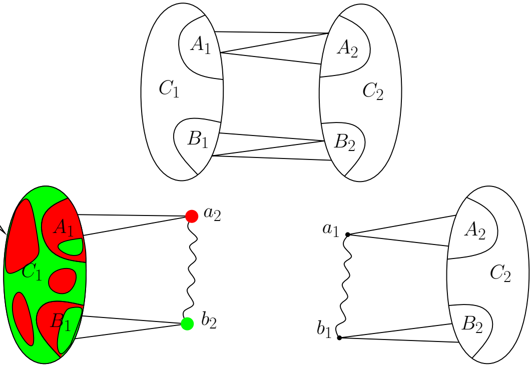
Red= What is put on the left (clique side)

Green= What is put on the right (stable set s.)



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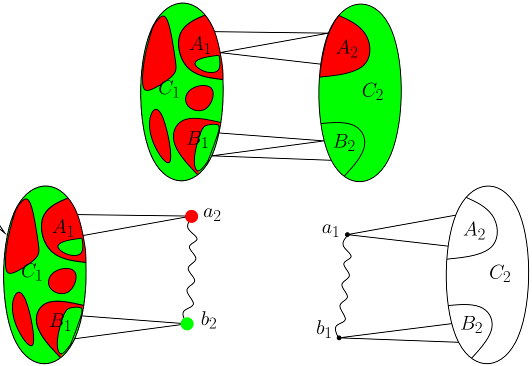

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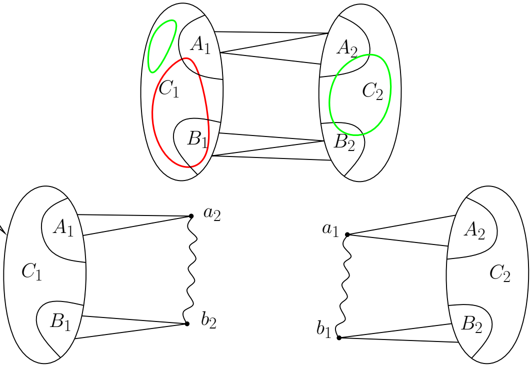

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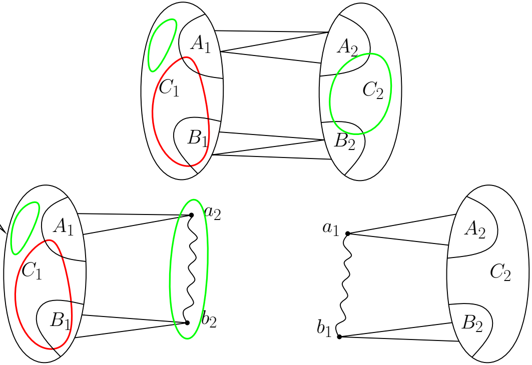

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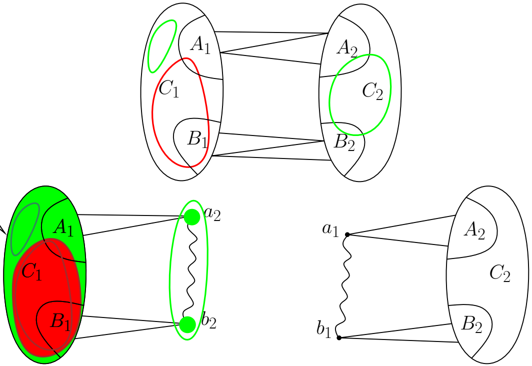
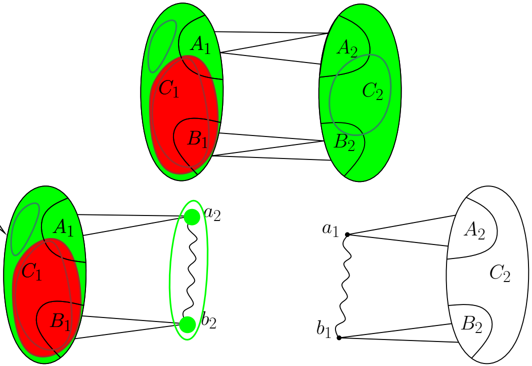
What is put on the right (stable set s.)



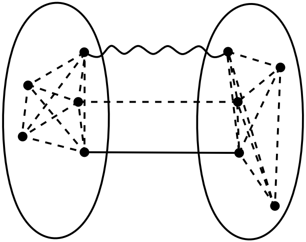
Red= What is put on the left (clique side)

Green= What is put on the right (stable set s.)



In basic trigraphs :

- In a bipartite trigraph, ω is bounded by 2.
  $\Rightarrow$ CS-separator of size $\mathcal{O}(n^2)$.



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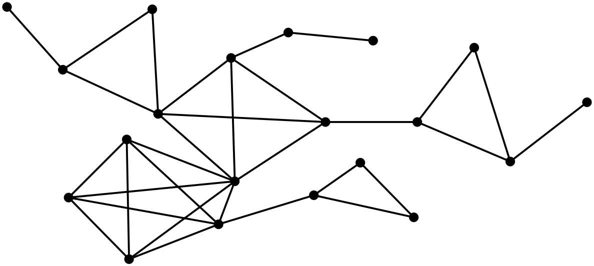

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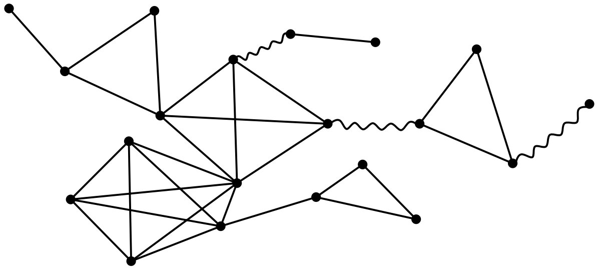

Perspectives

- Study the Clique-Stable Set separation on other classes of graphs (perfect graphs ?)
- Prove that there does not exist a polynomial CS-separator in general.
- What are the links between the CS-separation and other graph classes properties ? (Erdős-Hajnal for example).

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Thank you for your attention !