On minimally 2-T-connected digraphs

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Joint work with :

Olivier Durand de Gevigney

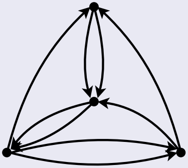
Outline

- Definitions on connectivity
- Motivation
- Result
- Definitions on bi-sets
- Proof

k-arc-connected (k-ac) digraphs

Definition

1. k-ac : $\forall(u,v) \in V^2$, $\exists$ k arc disjoint (u,v)-paths,



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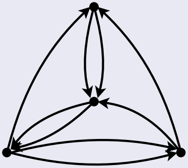

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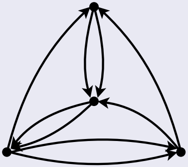

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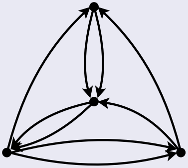

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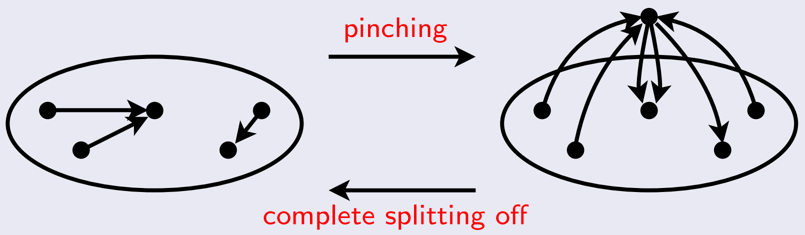

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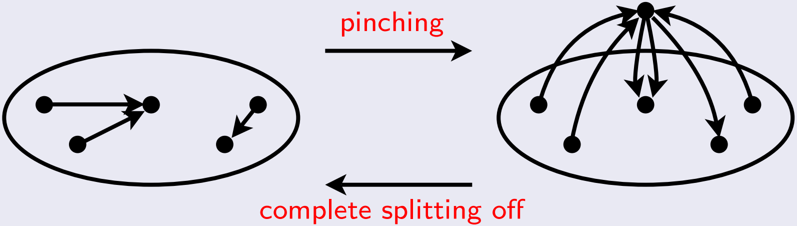

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Theorem 1 (Mader 1978)

1. A digraph is k-ac $\Longleftrightarrow$ it can be constructed from a vertex by repeated applications of operations 2 and 3.

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Theorem 1 (Mader 1978)

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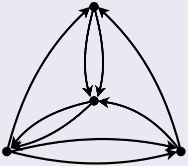
3. In a k-ac digraph for $|\partial^-(s)| = |\partial^+(s)|$, $\exists$ a complete splitting off at s resulting in a k-ac digraph.

Construction of a 2-ac digraph

Example



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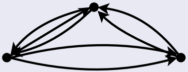

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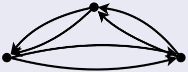

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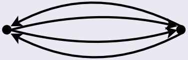

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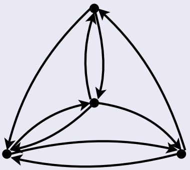

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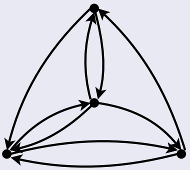

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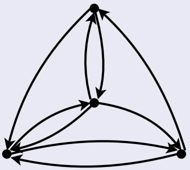

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In a minimally k-vc digraph $\exists$ a vertex v : $|\partial^-(v)| = |\partial^+(v)| = k$.

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Open problem

Find a constructive characterization of 2-vc digraphs.

2-T-connected (2-T-c) digraphs

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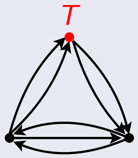

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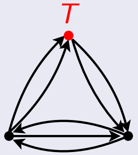

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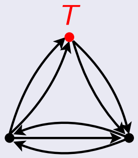

Remark

1. 2-$\emptyset$-c $=$ 2-ac and 2-V-c $=$ 2-vc.

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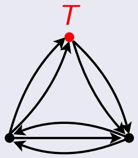

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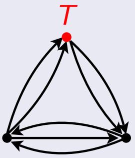

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Theorem 3 (Durand de Gevigney, Szigeti)

In a minimally 2-T-c digraph with $(\star)$ no parallel arc leaving a vertex in T, $\exists$ a vertex v : $|\partial^-(v)| = |\partial^+(v)| = 2$.

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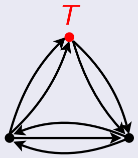

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Remark

Theorem 3 implies Theorem 1 (2) for $k = 2$ and Theorem 2.

Bi-sets

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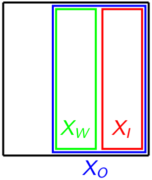

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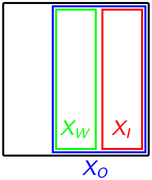

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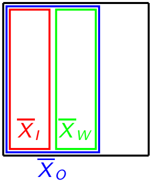

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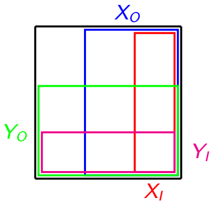

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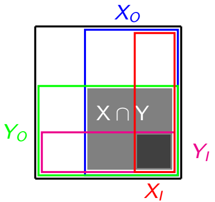

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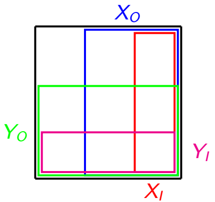

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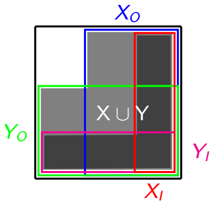

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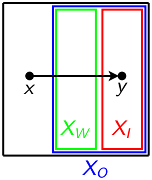

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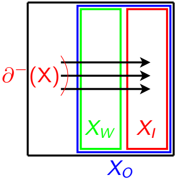
7. in-degree of bi-set X : $|\partial^-(\mathsf{X})|$ = number of arcs entering X.



Bi-sets and 2-T-c

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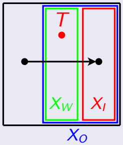

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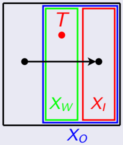

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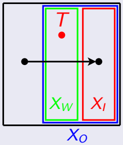

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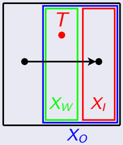

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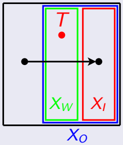

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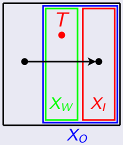

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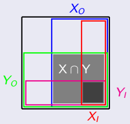

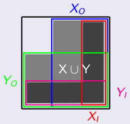

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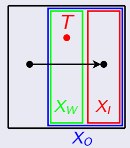

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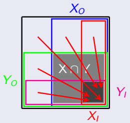

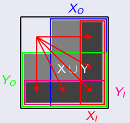

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Theorem 3 (Durand de Gevigney, Szigeti)

In a minimally 2-T-c digraph with $(\star)$ no parallel arc leaving a vertex in T, $\exists$ a vertex v : $|\partial^-(v)| = |\partial^+(v)| = 2$.

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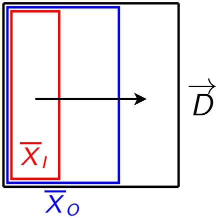

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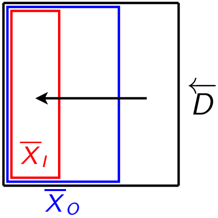

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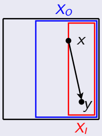

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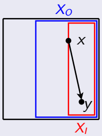

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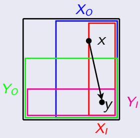

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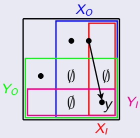

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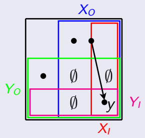

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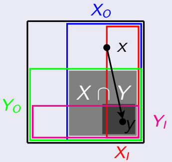

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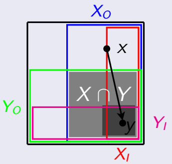

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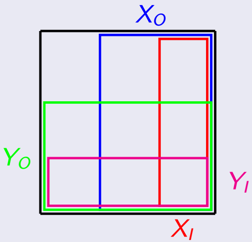

Proof of Theorem 3

Claim : $X_I \cap Y_I = y$, $(\mathrm{X} \sqcap \mathrm{Y})_W = \emptyset$ and $|\mathrm{X}_W| = |\mathrm{Y}_W| = 1$.

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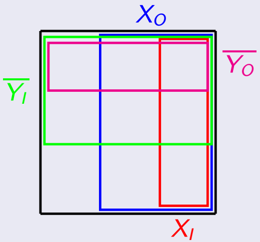

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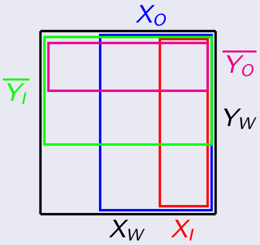

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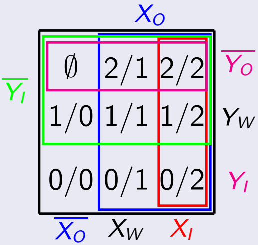

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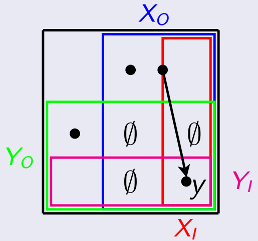
4. Thus we have equality everywhere and the claim follows.



Proof of Theorem 3

Lemma 3 : $D[X_l]$ is 1-ac.

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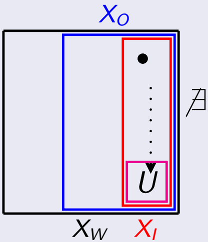

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Lemma 3 : $D[X_I]$ is 1-ac.

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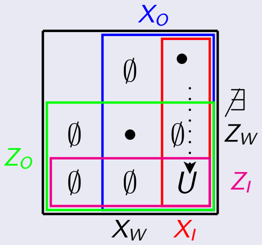

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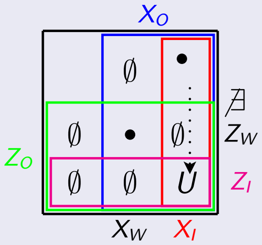

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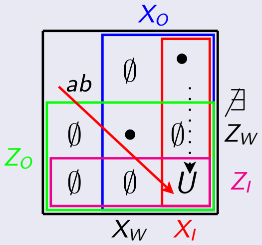
3. $2 \leq f_D^T(Z) = |\partial^-_D(Z)| + g^T(Z_W) \leq |\partial^-_D(X)| + g^T(X_W) = f_D^T(X) = 2$.
4. Z is tight and $\partial^-_D(Z) = \partial^-_D(X)$, so ab enters Z, thus $Z \in \mathcal{T}$.



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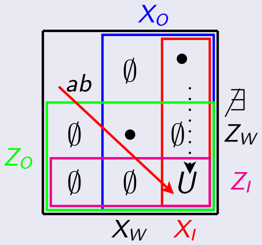
5. By $|Z_O| + |Z_I| < |X_O| + |X_I|$, contradiction.



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1. If $|\partial_D^-(v)| > 2$ and $uv \in A \setminus A_0$, then $u \in V_+$.
   - By condition, $|\partial_D^+(u)| = 2$, and then, since D is a counterexample, $|\partial_D^-(u)| > 2$ and hence $u \in V_+$.

Proof of Theorem 3

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1. If $|\partial_D^-(v)| > 2$ and $uv \in A \setminus A_0$, then $u \in V_+$.
2. $X_I \subseteq V_+$.

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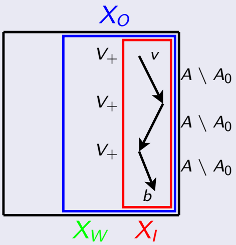
1. If $|\partial^-_D(v)| > 2$ and $uv \in A \setminus A_0$, then $u \in V_+$.
2. $X_I \subseteq V_+$.
   - By Lemmas 2, 3, and (1) :
   - $X_I \subseteq \{v : \exists$ nontrivial (v, b)-path in $D - A_0\} \subseteq V_+$.



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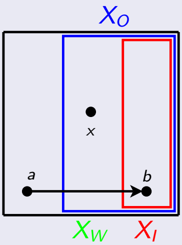

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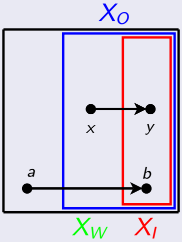
   - By $x = X_W, \partial_D^-(X) = ab$ and hence $\exists xy \in \partial_D(X_W, X_I)$,



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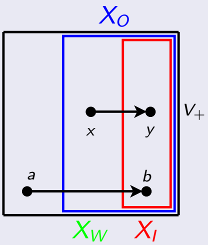

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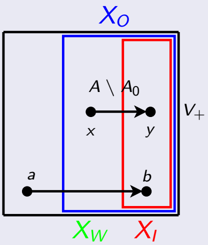

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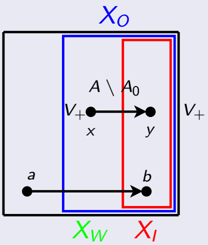

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1. If $X_l \neq b$: by Lemma 4 (3) and (2), we have a contradiction :
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$$3-2 \geq |\partial_D^-(\mathrm{X})| + 2|X_W| - 2 \geq |\partial_D^-(\mathrm{X})| + |\partial_D(X_W, X_I)| - |\partial^+(X_I)|$$
$$= |\partial_D^-(X_I)| - |\partial_D^+(X_I)| = \sum_{v \in X_I} (|\partial_D^-(v)| - |\partial_D^+(v)|) \geq |X_I| \geq 2.$$
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