Packing arborescences : a survey

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Outline

- Definitions, Applications
- Old results
- New results
- Algorithmic aspects

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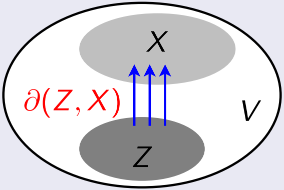

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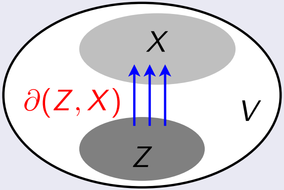

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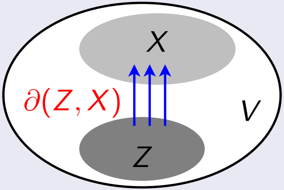

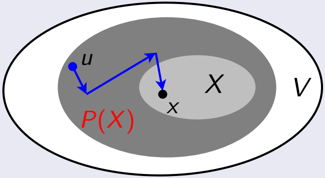

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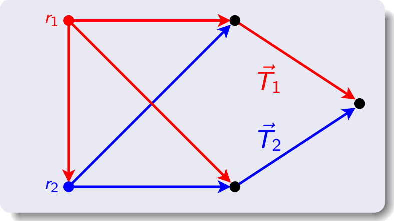

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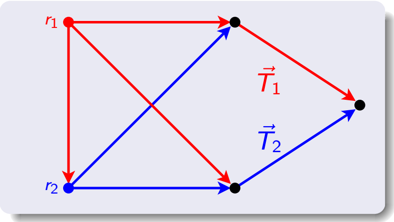

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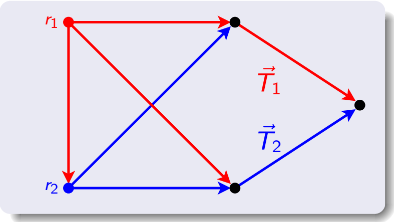

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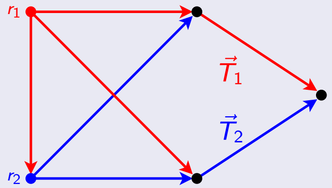

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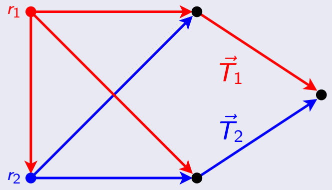

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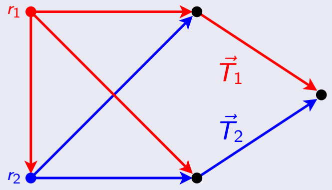

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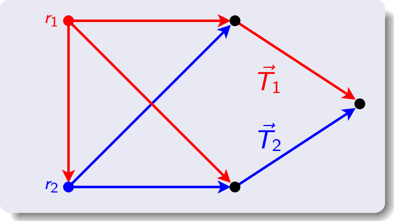

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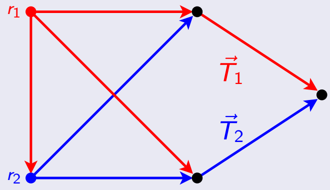
3. Packing of arborescences is a set of pairwise arc-disjoint arborescences.



Applications :

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- From each agent to any other agent some secret channels exist.
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  - each message was assigned to one agent and
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- Today the security rules changed :
  - the transmission of at most one message is allowed via any channel.
- Can now each agent receive each message ?
- and the messages that they could have received before ?

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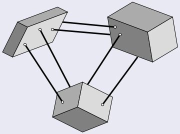
- For each channel, one must decide which message is sent (if any).
- The minimal set of channels through which the same message is sent forms an arborescence.

Applications : Rigidity

Body-Bar Framework



Theorem (Tay 1984)

"Rigidity" of a Body-Bar Framework can be characterized by the existence of a spanning tree decomposition.

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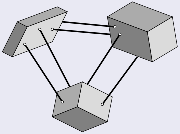

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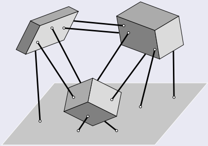
Body-Bar Framework with Bar-Boundary



Theorem (Katoh, Tanigawa 2013)

"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a matroid-based rooted-tree decomposition.

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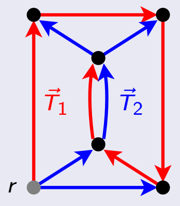

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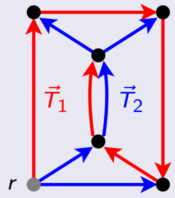

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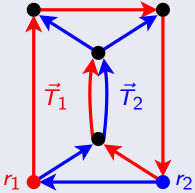

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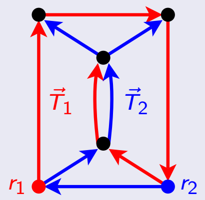

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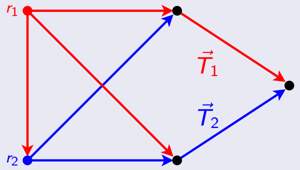

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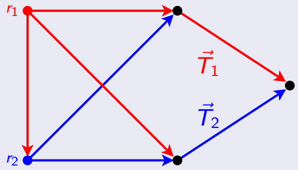

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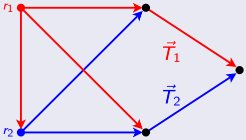

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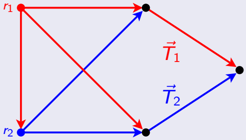

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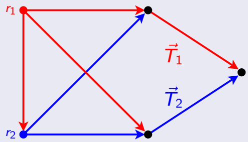

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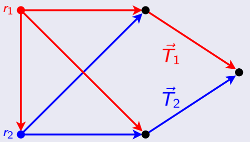

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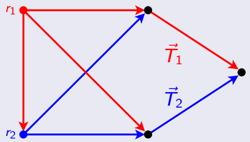

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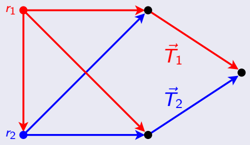

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2. Thus there exists a packing of reachability r_i-arborescences

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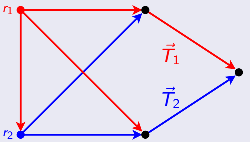

Remark

It implies Edmonds' theorem.

1. $|\partial(X)| \geq |\{r_i \in V \setminus X\}|$ for all $\emptyset \neq X \subseteq V$ implies the above condition and that each vertex is reachable from each r_i.
2. Thus there exists a packing of reachability r_i-arborescences and hence spanning r_i-arborescences.

Matroids

Definition

For $\mathcal{I} \subseteq 2^E$ (independent sets), $\mathcal{M} = (E, \mathcal{I})$ is a matroid if

1. $\mathcal{I} \neq \emptyset$,
2. If $X \subseteq Y \in \mathcal{I}$ then $X \in \mathcal{I}$,
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Examples for matroids

1. Linear : Sets of linearly independent vectors in a vector space,
2. Graphic : Edge-sets of forests of a graph,
3. Uniform : $U_{n,k} = \{X \subseteq E : |X| \leq k\}$ where $|E| = n$,
4. Free : $U_{n,n}$,
5. Transversal : end-vertices in S of matchings of bipartite graph $(S, T; E)$

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Notion

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The minimum of a submodular function can be found in polynomial time.

Theorem (Edmonds 1970,1979)

Let $\mathcal{M}_1 = (E, r_1), \mathcal{M}_2 = (E, r_2)$ be matroids on E, $k \in \mathbb{Z}_+$, $w : E \to \mathbb{R}$.

1. *$\mathcal{M}_1$ and $\mathcal{M}_2$ have a common independent set of size k $\iff$ $r_1(X) + r_2(E \setminus X) \geq k \;\; \forall\, X \subseteq E$.*
2. *A common base of $\mathcal{M}_1$ and $\mathcal{M}_2$ of minimum weight can be found in polynomial time.*

Definition

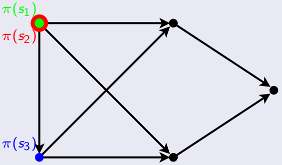
A matroid-based rooted-digraph is a quadruple $(\vec{G}, \mathcal{M}, S, \pi)$:

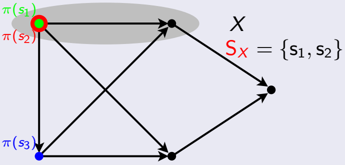
1. $\vec{G} = (V, A)$ is a digraph,
2. $\mathcal{M}$ is a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. π is a placement of the elements of S at vertices of V such that $S_v \in \mathcal{I}$ for every $v \in V$, where $S_X = \pi^{-1}(X)$, the elements of S placed at X.

$\vec{G}$

$S = \{s_1, s_2, s_3\}$

$\mathcal{M} = U_{3,2}$





Matroid-based packing of rooted-arborescences

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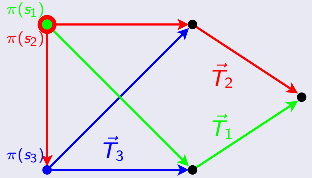

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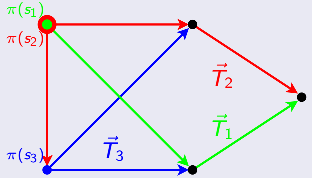

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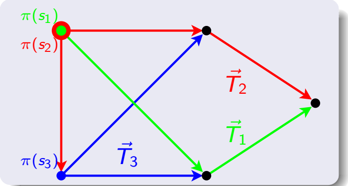

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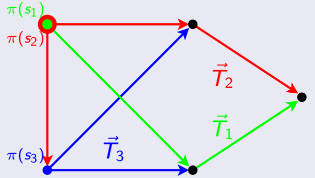

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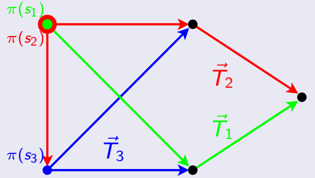

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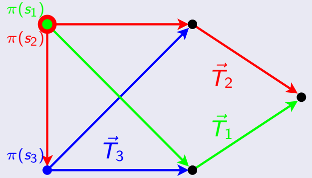

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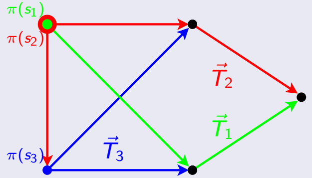

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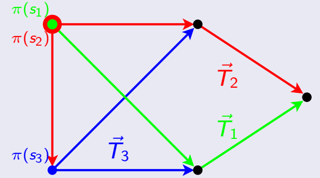

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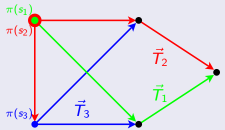

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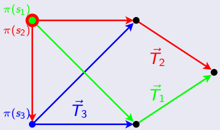

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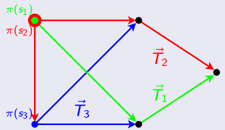

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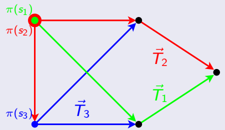

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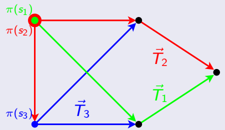

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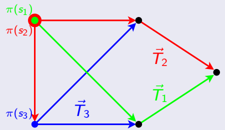

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2. Thus there exists a matroid-based packing of rooted-arborescences and, by Remark, a packing of spanning r_i-arborescences.

Reachability-based packing of rooted-arborescences

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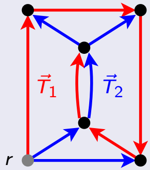

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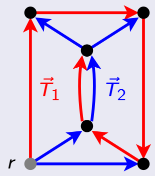

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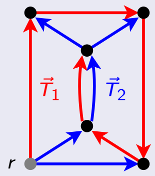

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Given a digraph $\vec{G} = (V + s, A)$, $k \in \mathbb{Z}_+$ and a matroid $\mathcal{M} = (A, r)$ which is the direct sum of the matroids $\mathcal{M}_v = (\partial(v), r_v)$ $\forall v \in V$.

- $\vec{G}$ *has an $\mathcal{M}$-restricted packing of k spanning s-arborescences* $\Longleftrightarrow$
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Matroid-restricted packing of spanning s-arborescences

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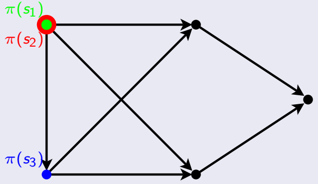
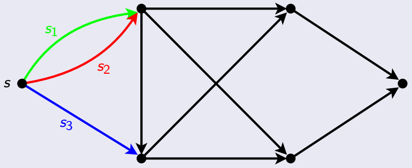
3. For general matroid $\mathcal{M}$, the problem is NP-complete, even for $k = 1$.

New model : Matroid-rooted digraphs

Transformation



Matroid on the vertices



Matroid on the arcs leaving s

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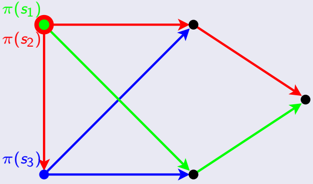
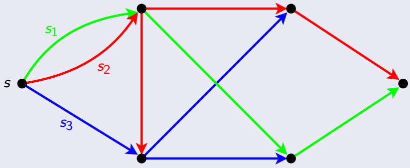


Matroid-based packing



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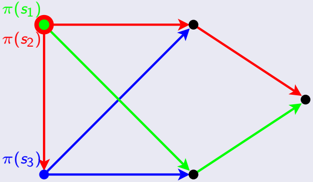

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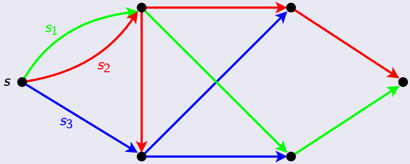

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Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)

Let $(\vec{G} = (V + s, A), \mathcal{M})$ *be a matroid-rooted digraph.*

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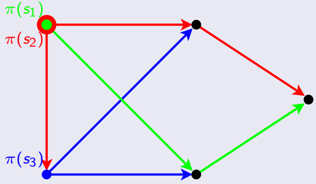

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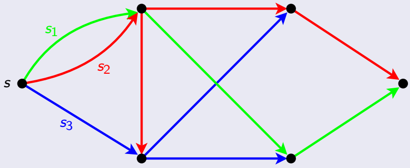

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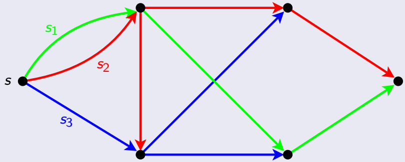
Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)

Let $(\vec{G} = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph.

1. *There is a $\mathcal{M}$-based packing of s-arborescences* $\iff$
2. $r_{\mathcal{M}}(\partial(s, X)) + |\partial(V \setminus X, X)| \geq r_{\mathcal{M}}(\partial(s, V))$ *for all* $\emptyset \neq X \subseteq V$.

Matroid-based packing of spanning s-arborescences



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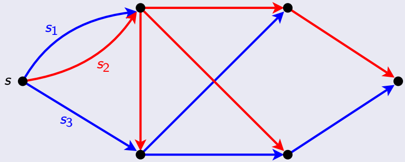

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Conjecture (Bérczi, Frank 2015)

Let $(\vec{G} = (V + s, A), \mathcal{M})$ be a matroid-rooted digraph.

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$\mathcal{M}_1$-based $\mathcal{M}_2$-restricted packing of s-arborescences

Theorem (Cs. Király, Szigeti 2016-)

Let $\vec{G} = (V + s, A)$, $\mathcal{M}_1 = (\partial(s, V), r_1)$, $\mathcal{M}_2 = (A, r_2) = \oplus_{v \in V}\mathcal{M}_v$.

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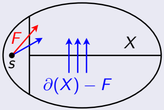

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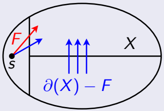

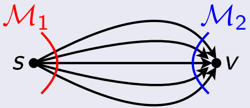

Remarks

1. It contains matroid-restricted packing of spanning s-arborescences, even matroid intersection. For matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on S, our problem on $(\vec{G} = (\{s, v\}, \{|S| \times sv\}), \mathcal{M}_1, \mathcal{M}_2)$ reduces to it.

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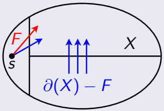

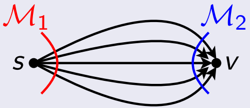

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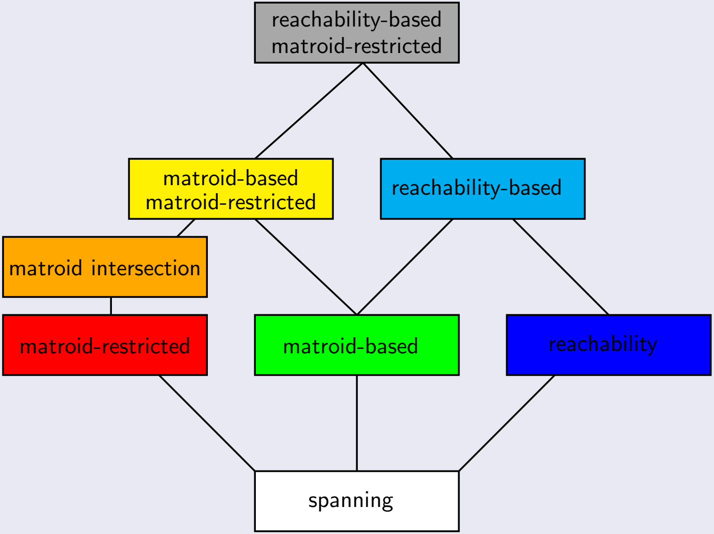
2. It implies Cs. Király's theorem, if $\mathcal{M}_2$ is free matroid.

Diagram of results



Algorithmic aspects

- (Matroid-restricted) Packing of spanning arborescences

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- Matroid-based matroid-restricted packing of arborescences
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Algorithmic aspects

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  - (even the weighted case) by (weighted) matroid intersection
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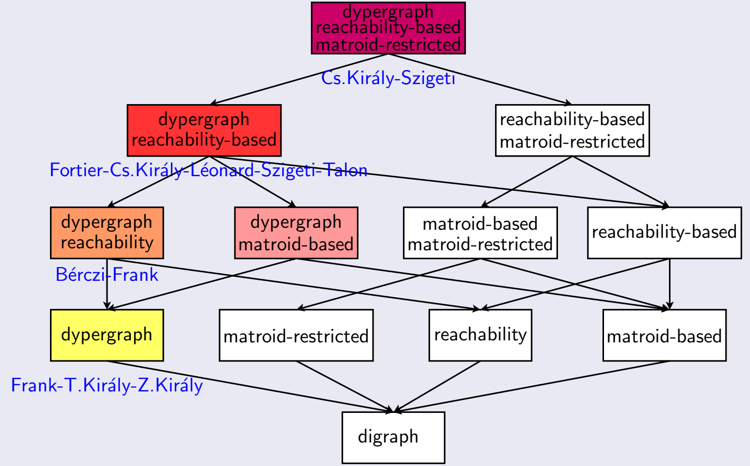
Directed hypergraphs



Thank you for your attention !