Edge-connectivity augmentations of graphs

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Edge- connectivity augmentation of graphs

(1) edge- connectivity — global / local ~~vertex- connectivity~~

(2) augmentation = min. cardinality ~~min. weight~~

(3) graph = undirected graph ~~directed graph~~

Outline of the talk

Basic problem

↓

solution

↓

splitting off

↓

detachment

↓

principal result
(Jordán- Szigeti)

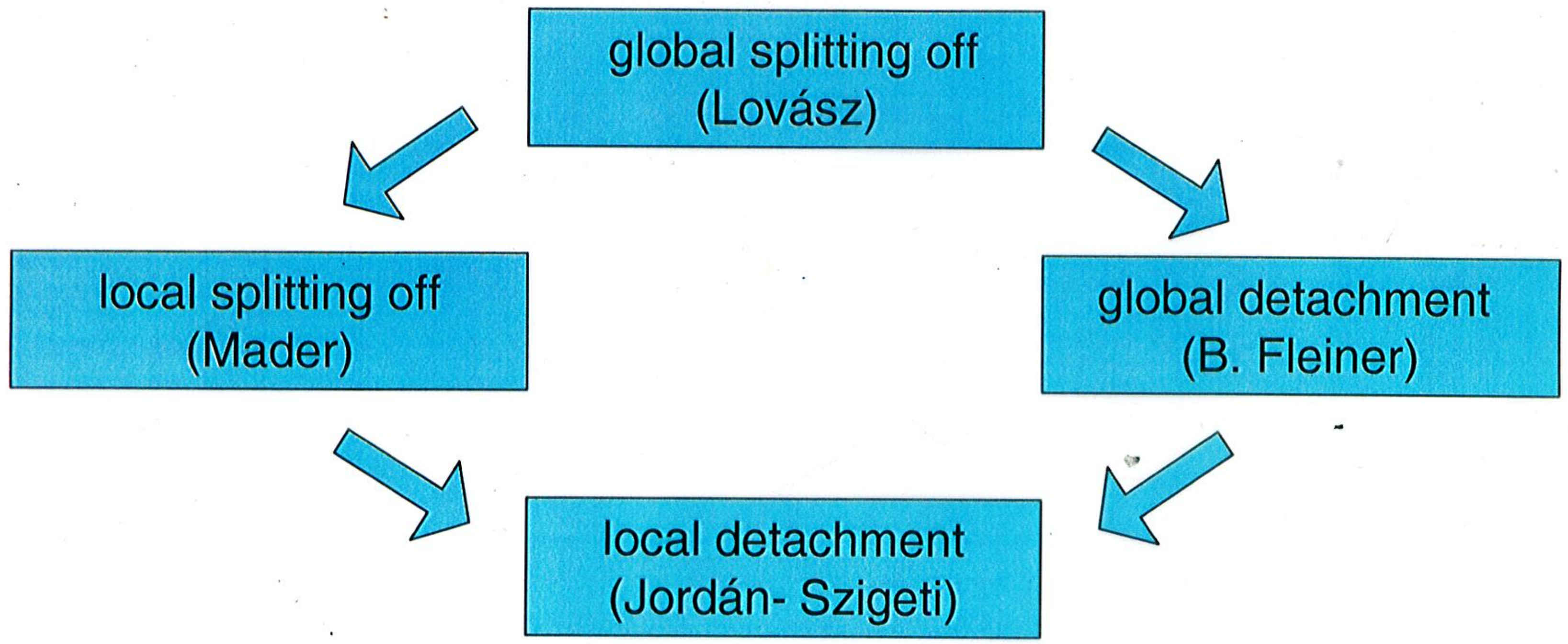
↓

application

↓

related
problems

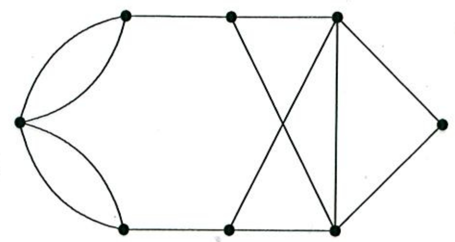
Aim


global splitting off
(Lovász)
local splitting off
(Mader)
global detachment
(B. Fleiner)
local detachment
(Jordán- Szigeti)

1 Basic problem

Telephone network:

vertex = telephone center
edge = connection
edge-connectivity = reliability
edge-connectivity augmentation
=
augmentation of reliability

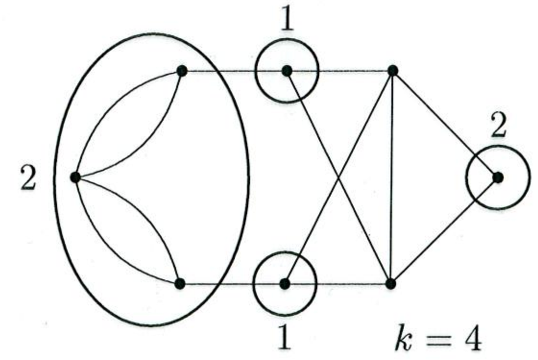


Description of the problem: *Global edge-connectivity augmentation in graphs:*

- Given graph G, requirement $k \in \mathbb{Z}_+$,
- Minimize the number of new edges whose addition results a k-edge-connected graph:

  $$\gamma := \min\{|F| : G + F \quad k\text{-edge-connected}\}.$$

Lower bound



deficient set X_i

$def_G(X_i) = k - d(X_i)$

deficient subpartition:

$\mathcal{X} = \{X_1, ..., X_l\}$

$def_G(\mathcal{X}) = \sum_1^l (k - d(X_i))$

$def_{G+e}(\mathcal{X}) \geq def_G(\mathcal{X}) - 2$

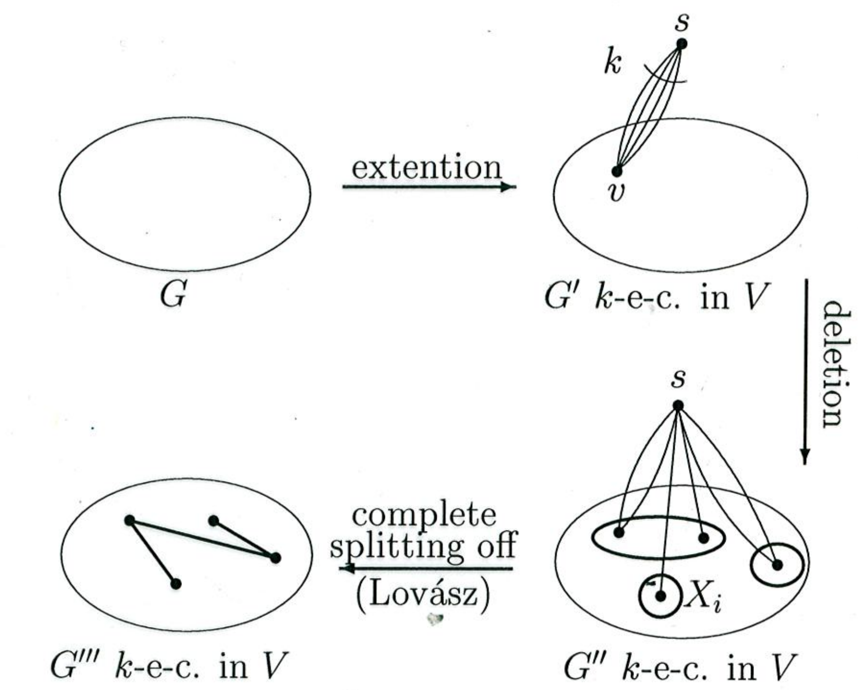
$\mathcal{S}(V) := \{$all subpartitions $\mathcal{X} = \{X_1, ..., X_l\}$ of $V\}$.

$$2\gamma \geq \sum_{X_i \in \mathcal{X}} (k - d(X_i)) \quad \forall \mathcal{X} \in \mathcal{S}(V)$$

Watanabe, Nakamura; Cai, Sun:

$$\gamma = \lceil \frac{1}{2} \max_{\mathcal{X} \in \mathcal{S}(V)} \{ \sum_{X_i \in \mathcal{X}} (k - d(X_i)) \rceil.$$

2 Frank's algorithm



Optimality:

$$k = d_{G''}(X_i) = d_G(X_i) + d_F(X_i),$$

$$\gamma = \lceil \frac{d_{G''}(s)}{2} \rceil = \lceil \frac{1}{2} \sum_1^l d_F(X_i) \rceil = \lceil \frac{1}{2} \sum_1^l (k - d_G(X_i)) \rceil.$$

Global edge-connectivity I:

Given graph $G = (U, E)$, $k \in \mathbb{Z}^+$; equivalent:

- G is k-**edge-connected**,
- $G - F$ is connected $\forall F \subseteq E, |F| \leq k - 1$,
- $d_G(X) \geq k \quad \forall \emptyset \neq X \subset U$,
- $\exists k$ edge-disjoint (u, v)-paths in G $\forall u, v \in U$.

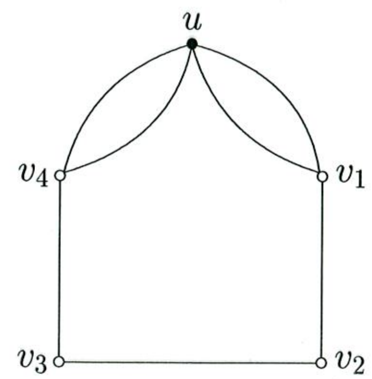
Local edge-connectivity:

Given graph $G = (U, E)$, $u, v \in U$
local edge-connectivity between u and v

$$\boldsymbol{\lambda_G(u, v)} = \text{max. number of edge-disjoint } (u, v)\text{-paths} = \text{min. card. of a cut separating } u \text{ and } v$$

Global edge-connectivity II:

Given graph $G = (U, E)$, $k \in \mathbb{Z}^+$, $V \subseteq U$,
G is k-**e-c. in** V if $\lambda_G(u, v) \geq k$ for all $u, v \in V$.



$G = (U, E)$

G is 2-e-c.

$\lambda_G(v_2, v_3) = 2$

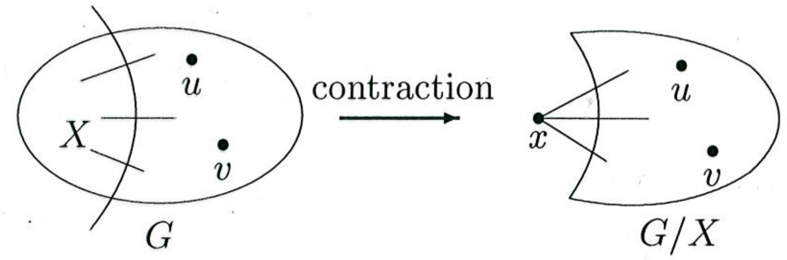
$\lambda_G(v_1, v_4) = 3$

G 2-e-c. in V

4 Remarks

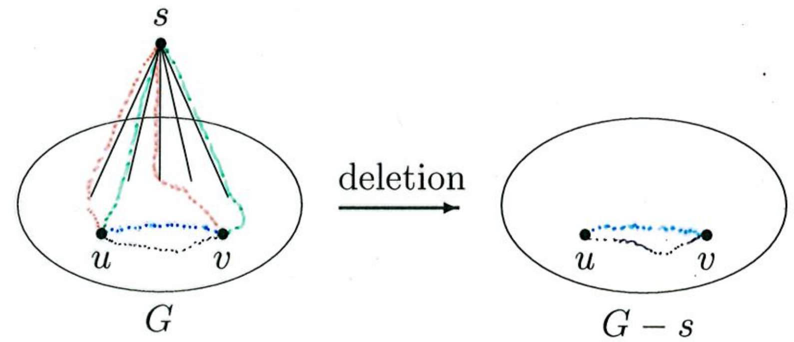
Remark 1 $G = (U, E)$, $X \subset U$, $u, v \in U - X$

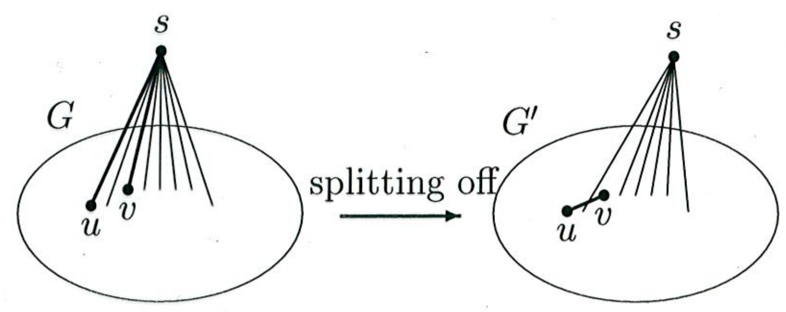
$$\lambda_{G/X}(u, v) \geq \lambda_G(u, v).$$



Remark 2 $G = (U, E)$, $s \in U$, $u, v \in U - s$

$$\lambda_{G-s}(u, v) \geq \lambda_G(u, v) - \lfloor \frac{d_G(s)}{2} \rfloor.$$





Global edge-connectivity:

Theorem 1 (Lovász) *Given*

- $G = (V + s, E)$ *graph,*
- $d(s)$ *even,*
- G *k-edge-connected in* V $(\boldsymbol{k \geq 2})$.

Then there exists a (complete) splitting off at s *that maintains* k**-edge-connectivity in** V.

Local edge-connectivity:

Theorem 2 (Mader) *Given*

- $G = (V + s, E)$ *graph,*
- $d_G(s) \neq 3$,
- *no cut edge incident to* s.

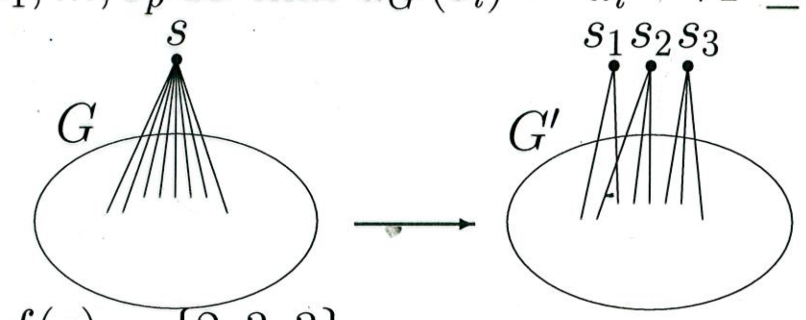
Then there exists a splitting off at s *that maintains* **local edge-connectivities in** V.

Theorem 2 implies Theorem 1.

6 Detachment

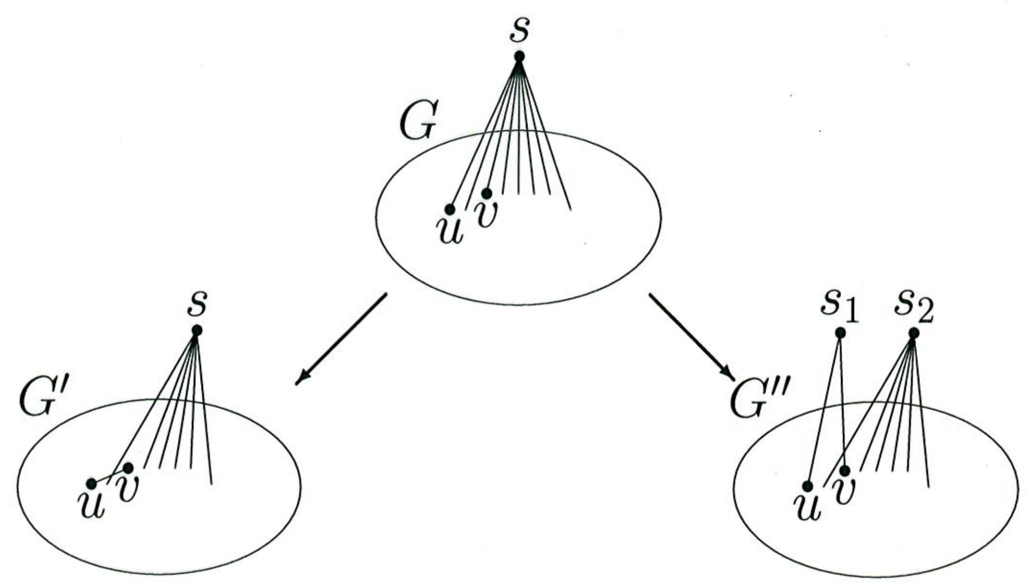
- Graph: $G = (V + s, E)$,
- Degree specification: $f(s) = \{d_1, ..., d_p\}$,
  $d_i \in \mathbb{Z}_+, \Sigma_1^p d_i = d_G(s)$.

G' is an $\boldsymbol{f(s)}$**-detachment** of G, if s is split into $s_1, ..., s_p$ so that $d_{G'}(s_i) = d_i \quad \forall 1 \leq i \leq p$.



$f(s) = \{2, 3, 3\}$

Remark 3 *splitting off* $= (2, d(s)-2)$*-detachment*



splitting off $\qquad$ $(2, d(s) - 2)$-detachment

Global edge-connectivity

Theorem 3 (B. Fleiner) *Given*

- $G = (V + s, E)$ *graph,*
- $\boldsymbol{k \geq 2}$,
- $f(s) = \{d_1, ..., d_p\}$ *degree spec.* $\mathbf{d_i \geq 2}$ $\forall i$.

Then there exists a k**-edge-connected in** V $f(s)$*-detachment of* G *iff* $\forall \emptyset \neq X \subset V$

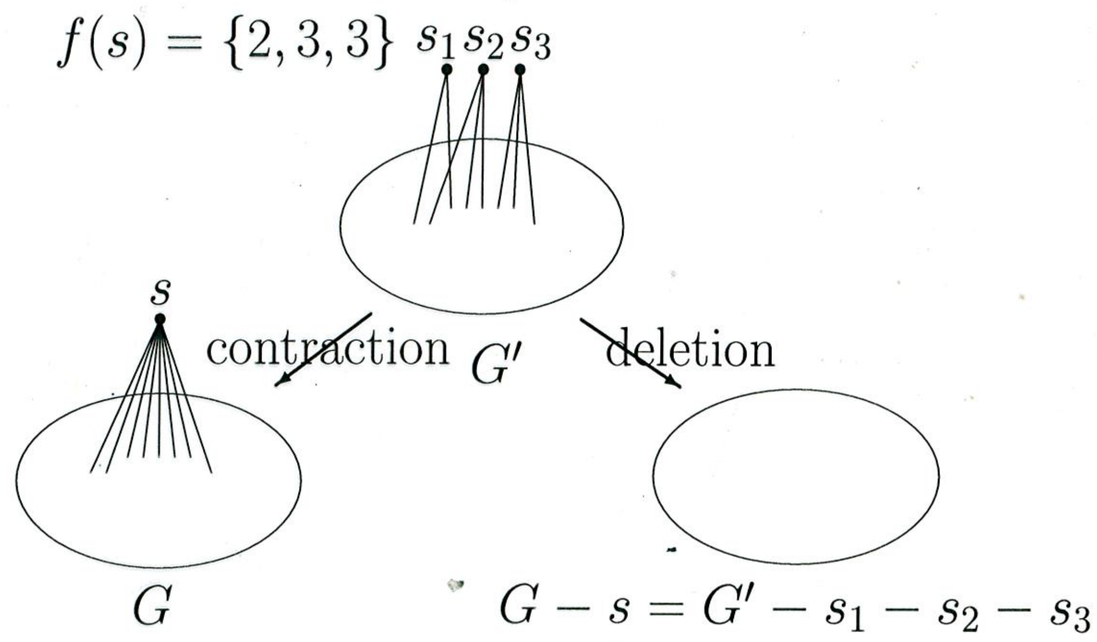
$$d_G(X) \geq k, \tag{1}$$

$$d_{G-s}(X) \geq k - \sum_{1}^{p} \lfloor \frac{d_i}{2} \rfloor. \tag{2}$$

Theorem 3 implies

- Lovász' splitting off theorem,
- Nash-Williams' simultanious detachment theorem.

Necessity:



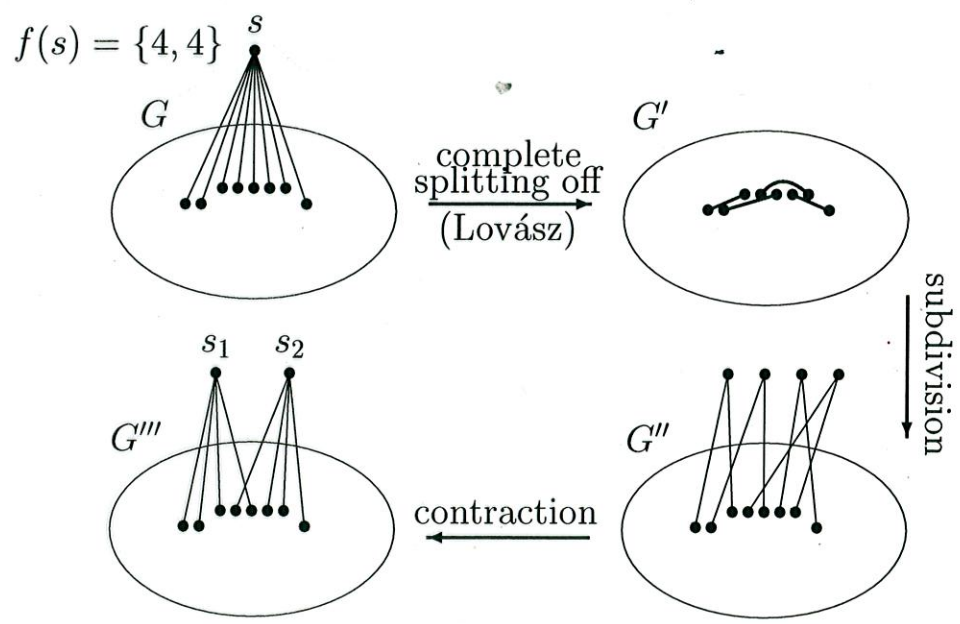
Special case: each d_i is **even** in $f(s)$.

(1) Then (1) implies (2) by ~~Remarks~~ Remark 2.

$$d_{G-s}(X) \geq d_G(X) - \lfloor \tfrac{d_G(s)}{2} \rfloor \geq k - \Sigma_1^p \lfloor \tfrac{d_i}{2} \rfloor.$$

(2) Thus this special case of Theorem 3 implies Lovász' theorem.

(3) Lovász' theorem implies this special case of Theorem 3.



Local edge-connectivity I:

Theorem 4 (Jordán, Szigeti) *Given*

- $G = (V + s, E)$ *2-edge-connected graph,*
- $f(s) = \{d_1, ..., d_p\}$ *degree spec.* $\boldsymbol{d_i \geq 2}$ $\forall i$.

Then there is an $f(s)$*-detachment maintaining* **local edge-connectivities in** V *iff* $\forall u, v \in V$,

$$\lambda_{G-s}(u, v) \geq \lambda_G(u, v) - \sum_1^p \lfloor \frac{d_i}{2} \rfloor. \qquad (3)$$

Attention!: It does NOT generalize Fleiner's theorem!

$\lambda_{G-s}(u, v) \geq k - \Sigma_1^p \lfloor \frac{d_i}{2} \rfloor$ does not imply that
$\lambda_{G-s}(u, v) \geq \lambda_G(u, v) - \Sigma_1^p \lfloor \frac{d_i}{2} \rfloor$.

Local edge-connectivity II:

For $\underline{r} : V \times V \to \mathbb{Z}_+$, $G = (V + s, E)$ is **$\underline{r}$-edge-connected in** V, if

$$\lambda_G(u, v) \geq r(u, v) \quad \forall u, v \in V.$$

Theorem 5 (Jordán, Szigeti) *Given*

- $G = (V + s, E)$ *graph,*
- $\boldsymbol{r(u, v) \geq 2}$ $\forall u, v \in V$ *requirem. function,*
- $f(s) = \{d_1, ..., d_p\}$ *degree spec.* $\boldsymbol{d_i \geq 2}$ $\forall i$.

Then there exists an **$\underline{r}$-edge-connected in V** *$f(s)$-detachment of G iff* $\forall u, v \in V$,

$$\lambda_G(u, v) \geq r(u, v), \tag{4}$$

$$\lambda_{G-s}(u, v) \geq r(u, v) - \sum_1^p \lfloor \frac{d_i}{2} \rfloor. \tag{5}$$

Necessity is the same as for the global case.
Sufficiency: short proof (Szigeti (2004)).

Special case: at most one d_i is odd in $f(s)$.

(1) Then (4) implies (5) by Remark 2.

$$\lambda_{G-s}(u,v) \geq \lambda_G(u,v) - \lfloor \frac{d_G(s)}{2} \rfloor$$
$$\geq r(u,v) - \Sigma_1^p \lfloor \frac{d_i}{2} \rfloor.$$

(2) Thus this special case of Theorem 5 implies Mader's theorem.

(3) Mader's theorem implies this special case of Theorem 5.

Theorem 5 implies

- Fleiner's theorem ($r(u,v) = k \ \forall u, v \in V$),
- Mader's theorem ($f(s) = (2, d_G(s) - 2)$, $r(u,v) = \lambda_G(u,v) \ u, v \in V$),
- Theorem 4 ($r(u,v) = \lambda_G(u,v) \ u, v \in V$).

The method to solve edge-connectivity augmentation problems developed by András Frank consists of two phases:

(1) extension minimally,

(2) complete splitting off.

Theorem 6 (Frank) *Given* $p : V \to \mathbb{Z}$,

- **symmetric**, $p(X) = p(V - X)\ \forall X \subseteq V$.
- **skew-supermodular**, $\forall X, Y \subseteq V$,

$$\begin{aligned} p(X) + p(Y) &\leq p(X \cap Y) + p(X \cup Y) \ \textit{or} \\ p(X) + p(Y) &\leq p(X - Y) + p(Y - X). \end{aligned}$$

Then the empty graph $(V, \emptyset)$ *can be extended to a graph* H *by adding a new vertex* s *and* γ *edges incident to* s *so that* H **covers** p $(d_H(X) \geq p(X)\ \forall\ \emptyset \neq X \subset V)$ *if and only if*

$$\sum_{X_i \in \mathcal{X}} p(X_i) \leq \gamma \quad \forall \mathcal{X} \in \mathcal{S}(V).$$

Global edge-connectivity:

$$d_{G+H}(X) \geq k \quad \forall\, \emptyset \neq X \subset V \tag{6}$$

if and only if

$$d_H(X) \geq \underbrace{k - d_G(X)}_{p(X)} \quad \forall\, \emptyset \neq X \subset V. \tag{7}$$

As $p(X)$ is symmetric and skew-supermodular, Theorems 6 and 1 imply the following theorem.

Theorem 7 (Watanabe, Nakamura; Cai, Sun) *Given*

- $G = (V, E)$ *graph,*
- $\boldsymbol{k \geq 2}$ *integer.*

*Then G can be made k-**edge-connected** by adding at most γ new edges if and only if*

$$\sum_{X_i \in \mathcal{X}} (k - d(X_i)) \leq 2\gamma \quad \forall \mathcal{X} \in \mathcal{S}(V).$$

Local edge-connectivity:

$$R(X) := \max\{r(x,y) : x \in X, y \in V - X\}.$$

is symmetric and skew-supermodular.

$$\begin{array}{rcll} \lambda_{G+H}(u,v) & \geq & r(u,v) & \forall\, u,v \in V, \\ & \Longleftrightarrow & & \\ d_{G+H}(X) & \geq & R(X) & \forall\, \emptyset \neq X \subset V, \\ & \Longleftrightarrow & & \\ d_H(X) & \geq & \underbrace{R(X) - d_G(X)}_{p(X)} & \forall\, \emptyset \neq X \subset V. \end{array}$$

As $p(X)$ is symmetric and skew-supermodular, Theorems 6 and 2 imply the following theorem.

Theorem 8 (Frank) *Given*

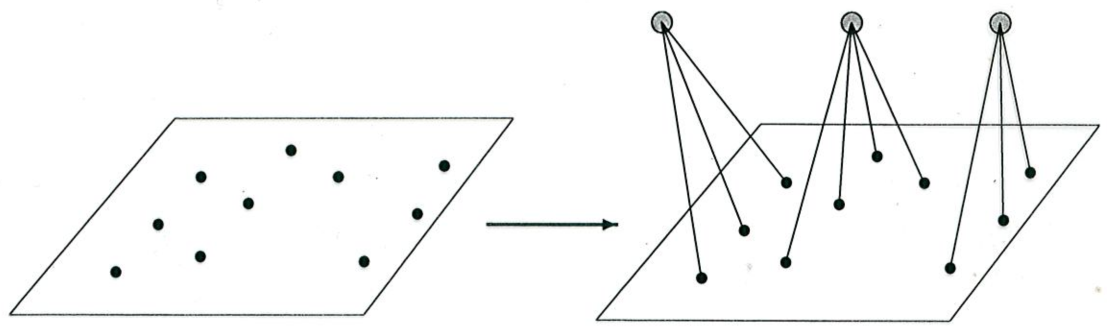
- $G = (V, E)$ *graph,*
- $\boldsymbol{r(u,v) \geq 2}$ $\forall u, v \in V$ *local edge-connectivity requirements.*

Then G can be made **$\underline{\mathbf{r}}$-edge-connected** *by adding at most γ new edges if and only if*

$$\sum_{X_i \in \mathcal{X}} (R(X_i) - d(X_i)) \leq 2\gamma \quad \forall \mathcal{X} \in \mathcal{S}(V).$$

Theorem 8 implies Theorem 7 of Watanabe and Nakamura.

Global edge-connectivity augmention by stars:



Theorems 6 and 3 imply the following theorem.

Theorem 9 (Fleiner B.) *Given*

- $G = (V, E)$ *graph,*
- $\boldsymbol{k} \geq \mathbf{2}$ *integer,*
- $d_1, \ldots, d_p$ $\;(\boldsymbol{d_i} \geq \mathbf{2}\;\; \forall i)$

Then G *can be made* k**-edge-connected in** V *by adding* p *stars of degrees* $d_1, \ldots, d_p$ *iff*

$$k - \sum_{1}^{p} \lfloor \frac{d_i}{2} \rfloor \leq \lambda_G(u, v) \quad \forall u, v \in V,$$

$$\sum_{X_j \in \mathcal{X}} (k - d(X_j)) \leq \sum_{1}^{p} d_i \qquad \forall \mathcal{X} \in \mathcal{S}(V).$$

Theorem 3 implies Theorem 7 of Watanabe and Nakamura.

Local edge-connectivity augmention by stars:

Theorems 6 and 5 imply the following theorem.

Theorem 10 (Jordán, Szigeti) *Given*

- $G = (V, E)$ *graph,*
- $\boldsymbol{r(u, v) \geq 2}$ $\forall u, v \in V$,
- $d_1, ..., d_p$ $(\boldsymbol{d_i \geq 2}$ $\forall i)$.

Then G can be made $\underline{\mathbf{r}}$**-edge-connected in** V *by attaching p stars of degrees $d_1, ..., d_p$ iff*

$$r(u, v) - \sum_1^p \lfloor \frac{d_i}{2} \rfloor \leq \lambda_G(u, v) \quad \forall u, v \in V,$$

$$\sum_{X_j \in \mathcal{X}} (R(X_j) - d(X_j)) \leq \sum_1^p d_i \quad \forall \mathcal{X} \in \mathcal{S}(V).$$

Theorem 10 implies

- Theorem 3 of Fleiner and
- Theorem 6 of Frank.

Theorem 11 (Bang-Jensen, Gabow, Jordán, Szigeti)
Given

- $G = (V, E)$ *graph,*
- $\mathcal{P} = \{P_1, ..., P_r\}$ *partition of* V,
- $\boldsymbol{k \geq 2}$.

Then G *can be made* k**-edge-connected** *by adding* γ *edges between different elements of* $\mathcal{P}$ *if and only if*

$$\sum_{X_i \in \mathcal{X}} (k - d(X_i)) \leq 2\gamma \quad \forall \mathcal{X} \in \mathcal{S}(V),$$

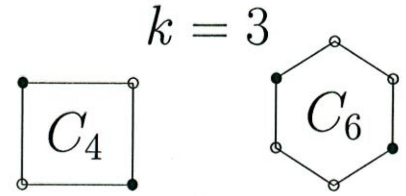
$$\sum_{Y_i \in \mathcal{Y}} (k - d(Y_i)) \leq \gamma \quad \forall \mathcal{Y} \in \mathcal{S}(P_j) \forall 1 \leq j \leq r,$$

$$G \text{ contains no } C_4 \text{ or } C_6\text{-configuration.} \quad (8)$$

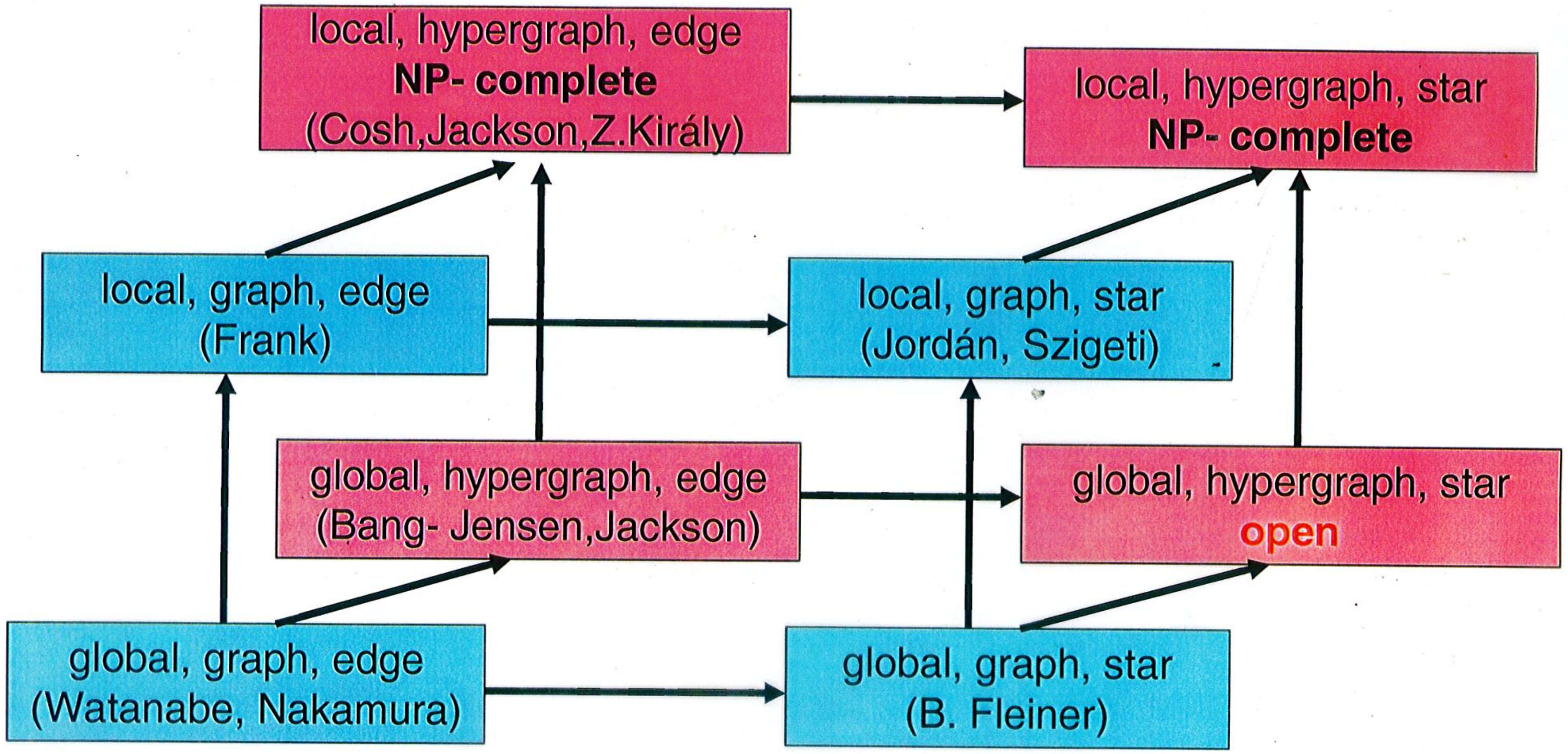
If G *contains a* C_4 *or* C_6*-configuration then we need one more edge.*

If k *is even then (8) is always satisfied.*

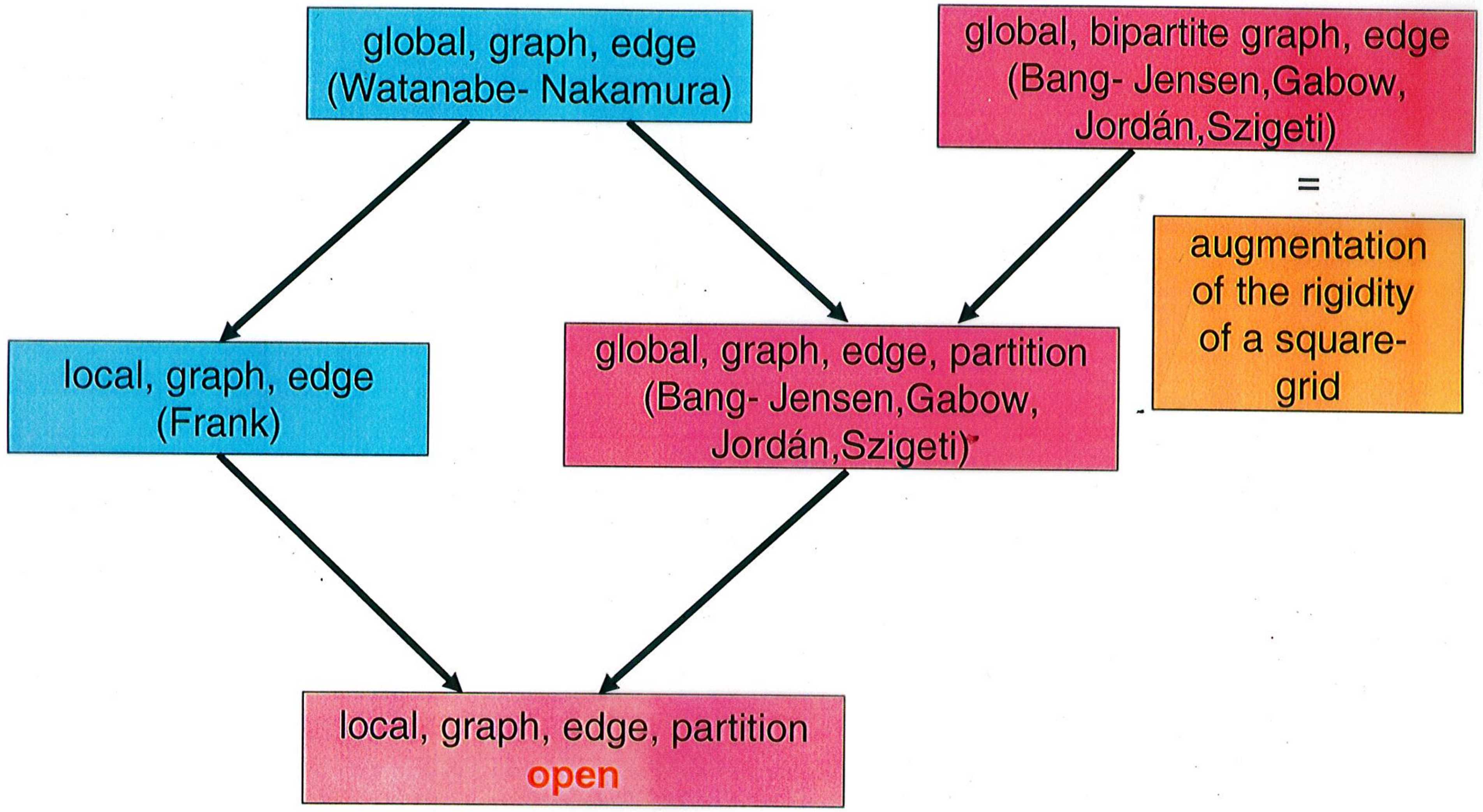
$k = 3$



Problems I.


local, hypergraph, edge
NP- complete
(Cosh,Jackson,Z.Király)
local, hypergraph, star
NP- complete
local, graph, edge
(Frank)
local, graph, star
(Jordán, Szigeti)
global, hypergraph, edge
(Bang- Jensen,Jackson)
global, hypergraph, star
open
global, graph, edge
(Watanabe, Nakamura)
global, graph, star
(B. Fleiner)

Problems II.


global, graph, edge
(Watanabe- Nakamura)
global, bipartite graph, edge
(Bang- Jensen,Gabow,
Jordán,Szigeti)
=
augmentation
of the rigidity
of a square-
grid
local, graph, edge
(Frank)
global, graph, edge, partition
(Bang- Jensen,Gabow,
Jordán,Szigeti)
local, graph, edge, partition
open

Problems III.