MATCHINGS AND EAR-DECOMPOSITIONS

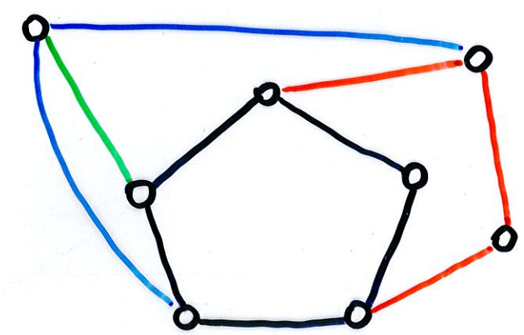
Zoltán SZIGETI

UNIVERSITY PARIS VI.
AND
UNIVERSITY OF BONN

DEFINITION: AN **EAR-DECOMPOSITION** OF G IS A SEQUENCE $G_0, G_1, \dots, G_t = G$ OF SUBGRAPHS OF G WHERE G_0 IS A VERTEX, AND EACH $G_i = G_{i-1} + P_i$, P_i IS A PATH AND ONLY ITS END-VERTICES BELONG TO G_{i-1}.

P_i IS CALLED **EAR**.

EXAMPLE:



P_1 —

P_2 —

P_3 —

P_4 —

$\#$ EARS $= |E| - |V| + 1$

ODD EARS: P_1, P_2, P_4

EVEN EAR: P_3

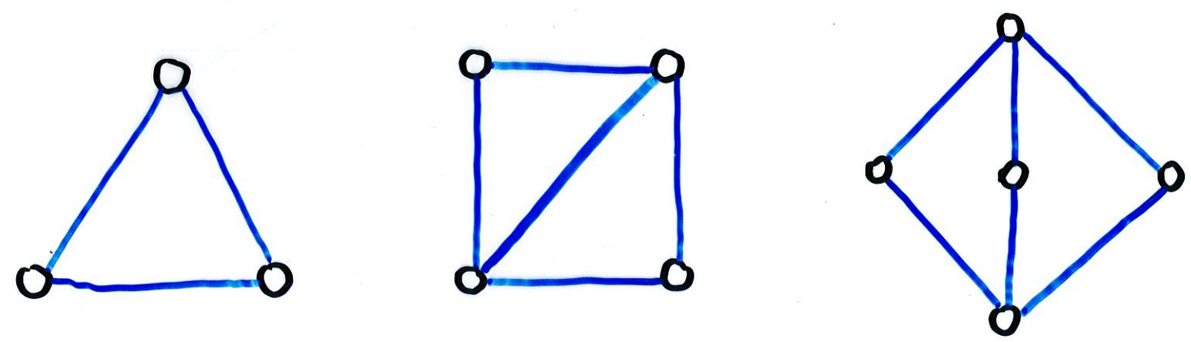
THEOREM: A GRAPH G POSSESSES AN EAR-DECOMPOSITION IF AND ONLY IF G IS 2-EDGE-CONNECTED. (WHITNEY)

DEFINITION: LET G BE A 2-EDGE-CONNECTED GRAPH.

$\varphi(G)$ = MINIMUM NUMBER OF EVEN EARS IN AN EAR-DECOMPOSITION OF G. (FRANK)

AN EAR-DECOMPOSITION IS **OPTIMAL** IF IT CONTAINS $\varphi(G)$ EVEN EARS.

EXAMPLES:



$\varphi = 0$ $\varphi = 1$ $\varphi = 2$

THEOREM (FRANK) AN OPTIMAL EAR-DECOMP. CAN BE CONSTRUCTED IN POLY. TIME.

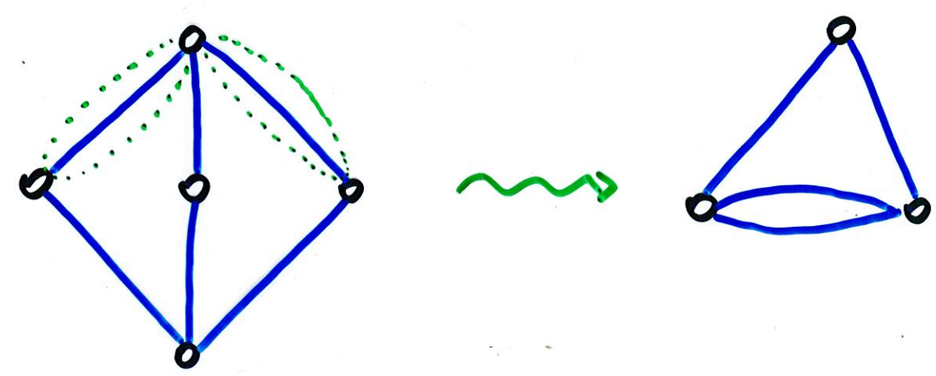
THEOREM: (LUCCHESI-YOUNGER)

FOR ANY DIRECTED GRAPH, THE MINIMUM NUMBER OF EDGES WHOSE CONTRACTION MAKES THE GRAPH STRONGLY CONNECTED IS EQUAL TO THE MAXIMUM NUMBER OF PAIRWISE EDGE-DISJOINT DIRECTED CUTS.

STATEMENT:

$\varphi(G)$ = MINIMUM NUMBER OF EDGES WHOSE CONTRACTION MAKES THE GRAPH G FACTOR-CRITICAL.

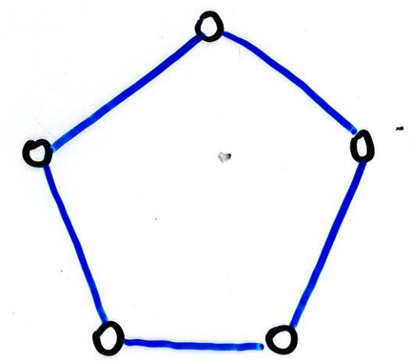
EXAMPLE:



$\varphi = 0$

DEFINITION: G IS **FACTOR-CRITICAL** IF FOR EVERY VERTEX $v \in V(G)$ $G - v$ HAS A PERFECT MATCHING.

EXAMPLE:



ODD CIRCUIT

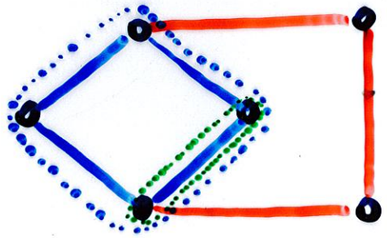
THEOREM: (LOVÁSZ)
A GRAPH G IS FACTOR-CRITICAL IF AND ONLY IF $\varphi(G) = 0$.

APPLICATION: IN EDMONDS MATCHING ALGORITHM.

DEFINITION: AN EDGE SET OF G IS **CRITICAL-MAKING** IF ITS CONTRACTION LEAVES A FACTOR-CRITICAL GRAPH.

THEOREM: (FRANK) THE MINIMUM CARDINALITY OF A CRITICAL-MAKING EDGE SET IS $\varphi(G)$.

THEOREM: (Z. S2.) ANY MINIMAL CRITICAL-MAKING EDGE SET IS OF SIZE $\varphi(G)$.



THEOREM: (Z. S2.) THE MINIMAL CRITICAL-MAKING EDGE SETS FORM THE BASES OF A MATROID $\mathcal{F}$.

DEFINITION: THE INDEPENDENT SETS F OF $\mathcal{F}$ ARE CALLED **EAR-EXTREME**, BECAUSE $\varphi(G/F) = \varphi(G) - |F|$.

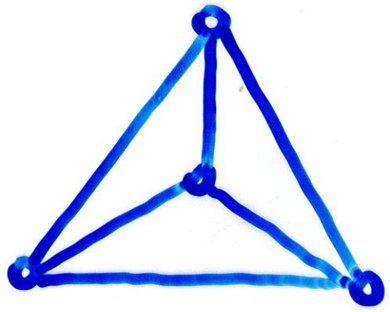
FACT: $\varphi(G/F) \geq \varphi(G) - |F| \quad \forall F \subseteq E(G)$.

DEFINITION: A CONNECTED GRAPH G IS **MATCHING-COVERED** IF EACH EDGE OF G BELONGS TO SOME PERFECT MATCHING.

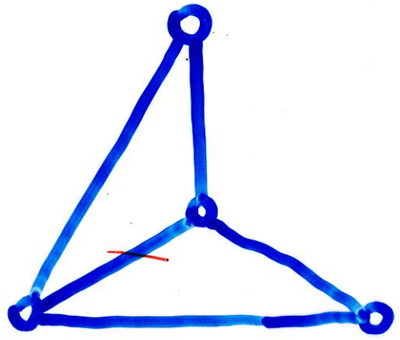
IN OTHER WORDS

FOR EACH PAIR OF ADJACENT VERTICES u AND v, $G-u-v$ HAS A PERFECT MATCHING.

EXAMPLE:



MATCHING-COVERED



NOT MATCHING-COVERED

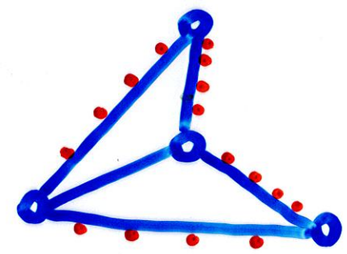
THEOREM (HETYEI)(LOVASZ - PLUMMER)

IF G IS MATCHING-COVERED THEN

$$\varphi(G) = 1.$$

DEFINITION: AN EDGE e OF G IS **φ-EXTREME** IF THERE EXISTS AN OPTIMAL EAR-DECOMPOSITION OF G SO THAT e BELONGS TO AN EVEN EAR.

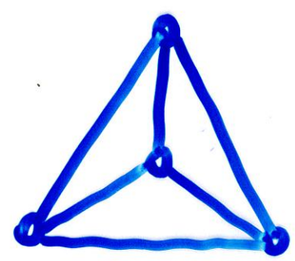
EXAMPLE:



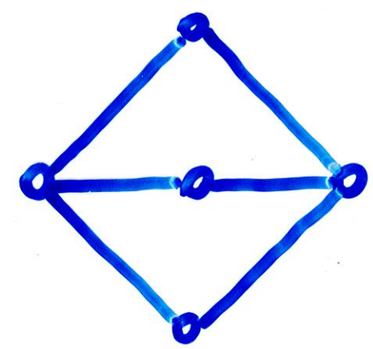
••• φ-EXTREME

DEFINITION: G IS **φ-COVERED** IF EVERY EDGE IS φ-EXTREME.

EXAMPLE:



$\varphi = 1$



$\varphi = 2$

THEOREM (LITTLE + LOVÁSZ-PLUMMER)

IF G IS MATCHING-COVERED AND $e, f \in E(G)$, THEN THERE EXISTS AN OPTIMAL EAR-DECOMPOSITION OF G SO THAT THE FIRST EAR IS EVEN (P_1) AND $e, f \in P_1$.

COROLLARY:

IF G IS MATCHING-COVERED THEN G IS φ-COVERED.

OBSERVATION:

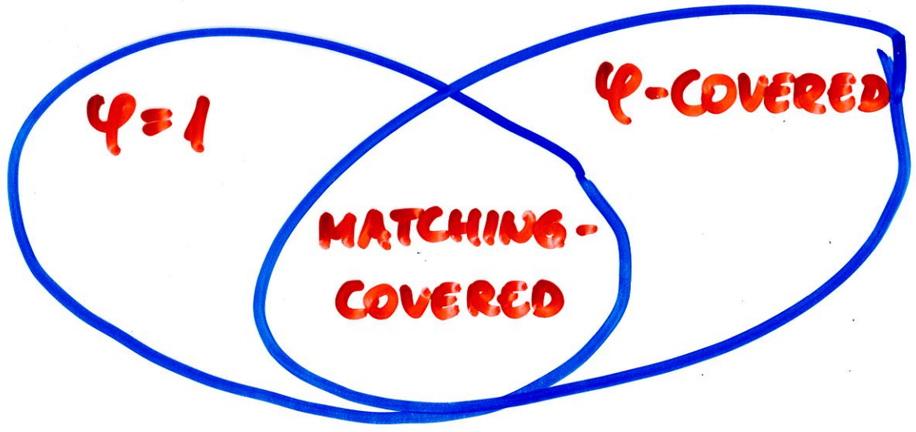
G IS MATCHING-COVERED

$\Updownarrow$

$\varphi(G) = 1$ AND G IS φ-COVERED

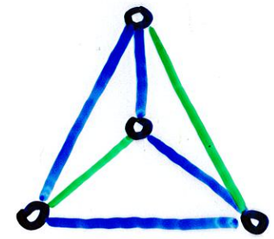
$\Updownarrow$

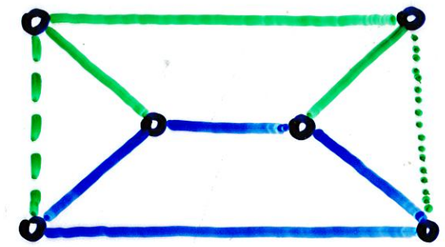
$\forall e \in E(G)$: G/e IS FACTOR-CRITICAL.



$\varphi = 1$

DEFINITION: G IS MATCHING-COVERED IF IT IS CONNECTED AND EACH EDGE BELONGS TO SOME PERFECT MATCHING OF G.

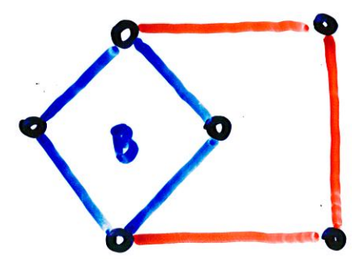




DEFINITION: $\varphi(G)=1$, AN EDGE e OF G IS CRITICAL-MAKING IF G/e IS FACTOR-CRITICAL.

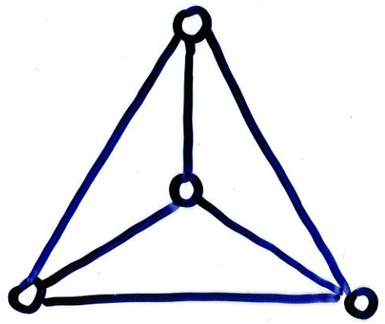
THEOREM: (Z.S2.) LET $\varphi(G)=1$ AND LET B BE THE SUBGRAPH OF G DEFINED BY THE CRITICAL-MAKING EDGES OF G. THEN

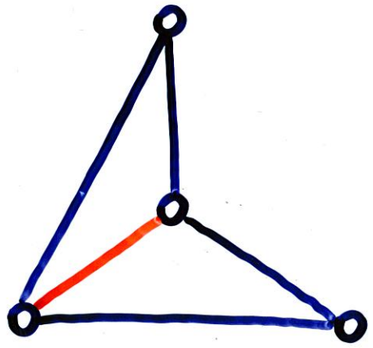
B IS MATCHING-COVERED,
G/B IS FACTOR-CRITICAL.



THEOREM: IF G IS MATCHING-COVERED THEN $\varphi(G)=1$. FURTHERMORE, THE FIRST EAR CAN BE CHOSEN TO BE THE EVEN EAR.

THEOREM: LET G BE BIPARTITE. THEN $\varphi(G)=1$ IFF G IS MATCHING-COVERED.





NOT MATCHING-COVERED

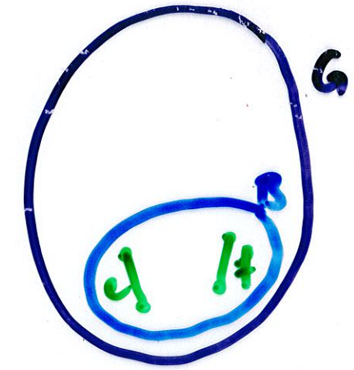
CLAIM: ANY EDGE CAN BE IN THE STARTING EAR OF AN EAR-DECOMPOSITION.

THEOREM: (FRANK) ANY EDGE CAN BE IN THE STARTING EAR OF AN OPTIMAL EAR-DECOMPOSITION.

THEOREM: (FRANK) AN EDGE CAN BE IN A STARTING EVEN EAR OF AN OPTIMAL EAR-DECOMPOSITION IF AND ONLY IF IT IS EAR-EXTREME.

THEOREM: (LITTLE) LET G BE A MATCHING-COVERED GRAPH. THEN $\varphi(G)=1$. MOREOVER, ANY TWO EDGES OF G CAN BE IN A STARTING EVEN EAR OF AN OPTIMAL EAR-DECOMPOSITION.

THEOREM: (Z. S2.) $\varphi(G)=1$, e AND f ARE TWO EDGES OF G. THEN e AND f TOGETHER CAN BE IN A STARTING EVEN EAR OF AN OPTIMAL EAR-DECOMPOSITION IF AND ONLY IF BOTH EDGES ARE EAR-EXTREME.



THEOREM: (2.52.) $\varphi(G) \geq 1$, e AND f ARE TWO EDGES OF G. THEN e AND f TOGETHER CAN BE IN A STARTING EVEN EAR OF AN OPTIMAL EAR-DECOMPOSITION IF AND ONLY IF THERE EXISTS A CIRCUIT IN THE MATROID $\mathcal{F}$ CONTAINING e AND f.

($\Leftrightarrow$ e AND f ARE IN THE SAME CONNECTED BLOCK OF $\mathcal{F}$.)

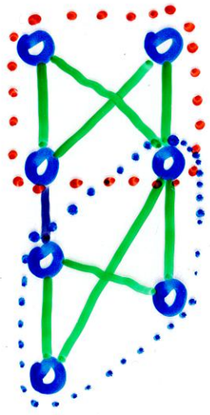
THEOREM: (2.52.) LET G BE FACTOR-CRITICAL AND LET e AND f BE TWO EDGES OF G. THEN G HAS AN ODD EAR-DECOMPOSITION SO THAT THE STARTING EAR CONTAINS e AND f IF AND ONLY IF $G/\{e,f\}$ IS FACTOR-CRITICAL.

DEF.: THE **BLOCKS** OF A MATROID $\mathcal{N}$ ARE THE EQUIVALENCE CLASSES OF $\sim$, WHERE $e \sim f$ IFF THERE EXISTS A CIRCUIT OF $\mathcal{N}$ CONTAINING e AND f.

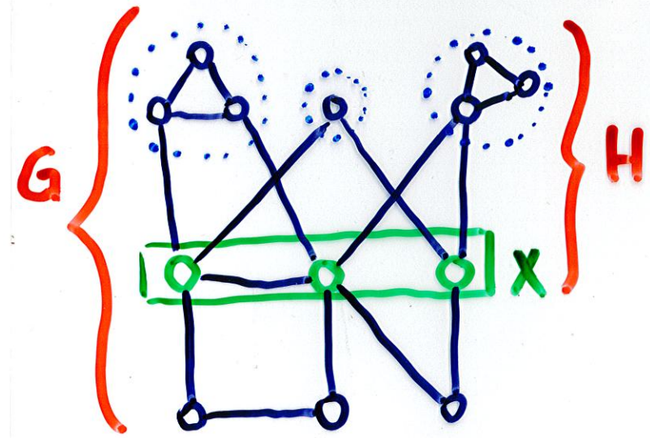
LEMMA (2.52.) THERE EXISTS AN OPTIMAL EAR-DECOMPOSITION OF G SO THAT THE FIRST EAR IS EVEN AND IT CONTAINS e AND f IFF e AND f BELONG TO THE SAME BLOCK OF $\mathcal{F}(G)$.

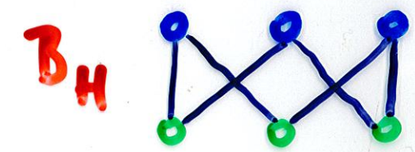
THEOREM (2.52.) IF G IS A 2-CONNECTED φ-COVERED GRAPH THEN $\mathcal{F}(G)$ HAS ONE BLOCK.

THEOREM (2.52.) BLOCKS OF $\mathcal{F}(G)$ = BLOCKS OF $D(G)$.



$D(G)$





H IS A STRONG SUBGRAPH OF G IF

(1) X SEPARATES $V(H)-X$ AND $V(G)-V(H)$ OR $V(G)=V(H)$,

(2) EACH COMPONENT OF $H-X$ IS FACTOR-CRITICAL,

(3) THE BIPARTITE GRAPH B_H IS MATCHING-COVERED.

THEOREM (FRANK)

G HAS A STRONG SUBGRAPH IFF IT IS NOT FACTOR-CRITICAL.

THEOREM (FRANK)

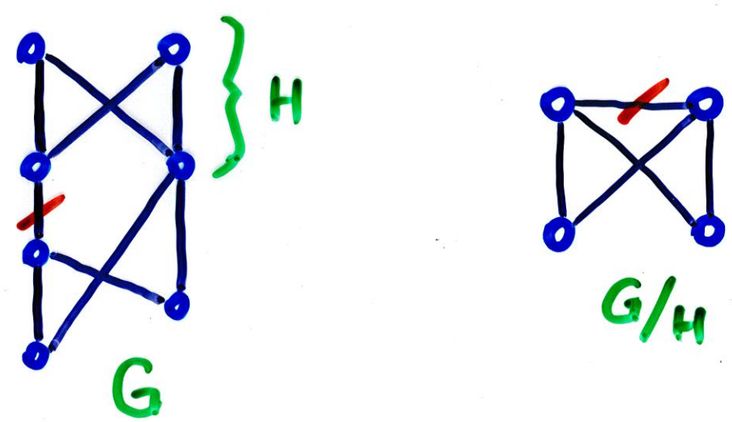
IF H IS A STRONG SUBGRAPH OF G THEN $\varphi(H)=1$ AND $\varphi(G/H)=\varphi(G)-1$.

DEF.: $e \in E(G)$ IS EAR-EXTREME IF $\varphi(G/e)=\varphi(G)-1$

DEF.: $D(G) := (V(G), E(D(G)))$, WHERE THE EDGES OF $D(G)$ ARE EXACTLY THE EAR-EXTREME EDGES OF G.

LEMMA (2.52.) LET H BE A STRONG SUBGRAPH OF G. THEN

$$E(D(G)) = E(D(H)) \cup E(D(G/H)).$$



THEOREM (FRANK)

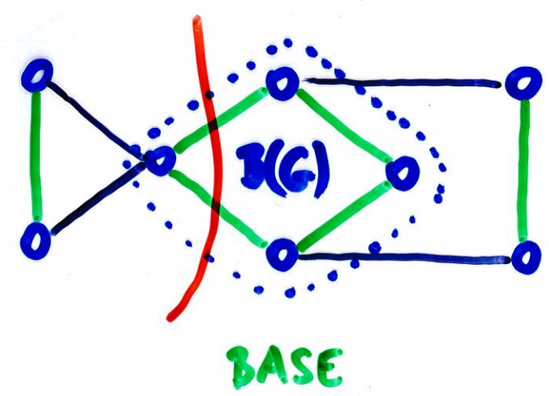
$\varphi(G)=1$ IFF $\exists$ A PERFECT MATCHING OF G AND $\nexists$ 2 VERTEX DISJOINT STRONG SUBGRAPHS IN G.

THEOREM (2.52.) LET $\varphi(G)=1$. THEN

(1) $E(D(G))$ COINCIDES WITH THE EDGE SET OF ONE OF THE CONNECTED COMPONENTS (SAY $B(G)$) OF THE SUBGRAPH OF G DEFINED BY THE ALLOWED EDGES OF G.

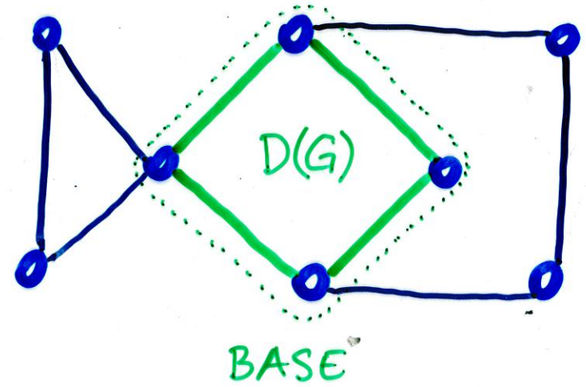
(2) $V(B(G)) = \bigcap \{V(H) : H \text{ IS A STRONG SUBGRAPH IN } G\}$

EX.:



THEOREM (Z. SZ.)

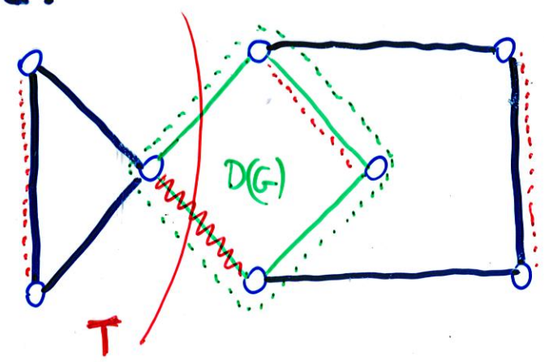
LET G BE A GRAPH WITH $\varphi(G)=1$.
LET $D(G)$ BE THE SUBGRAPH OF G DEFINED BY THE φ-EXTREME EDGES OF G.



G

THEN $D(G)$ IS CONNECTED AND EACH EDGE OF $D(G)$ BELONGS TO A PERFECT MATCHING OF G.

COROLLARY: IF $\varphi(G)=1$, THEN $\forall T \subset V(G)$ THAT SEPARATES $D(G)$ THERE EXISTS A PERFECT MATCHING M OF G SO THAT $|M \cap \delta(T)| \geq 1$.

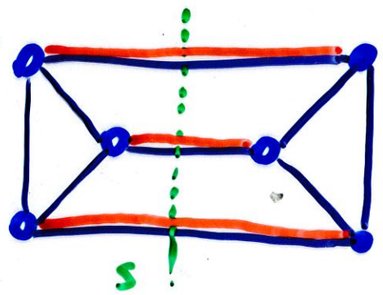


DEF.: G IS A **BRICK** IF IT IS 3-VERTEX-CONNECTED AND IT IS BICRITICAL, THAT IS $G-u-v$ HAS A PERFECT MATCHING $\forall u,v \in V(G)$.

TIGHT CUT LEMMA (EDMONDS, LOVÁSZ, PULLEYBLANK)

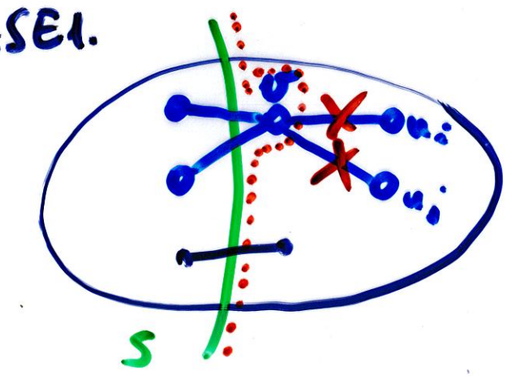
G IS A BRICK, $S \subset V(G)$: $3 \leq |S| \leq n-3$ ODD.

THEN $\exists$ A PERFECT MATCHING M : $|M \cap \delta(S)| \geq 3$.



SKETCH OF PROOF (2.52.)

CASE1.

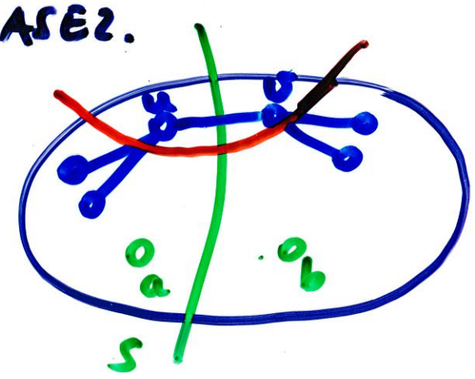


$\varphi(G') = 1$

$v \in B(G')$

$\exists u_i \in B(G')$

CASE2.



$\varphi(G'') = 1$

$\exists a,b \in B(G'')$

DEFINITION: A **GENERALIZED 2-GRADED EAR-DECOMPOSITION** OF G IS A SEQUENCE $G_0, G_1, \dots, G_t = G$

- G_1 IS AN EVEN CYCLE
- $G_{i+1} = G_i + P_{i1} + \dots + P_{i\ell}$, $1 \le \ell \le 2$ DISJOINT PATHS
- IF $\ell = 2$ THEN ODD PATHS
- G_i IS φ-COVERED $\forall i \geq 1$
- $\varphi(G_i) \le \varphi(G_{i+1})$ $\forall i$

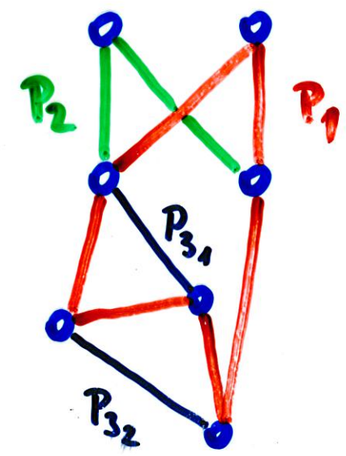
THEOREM (Z.SZ.)

A GRAPH G IS φ-COVERED

$\Updownarrow$

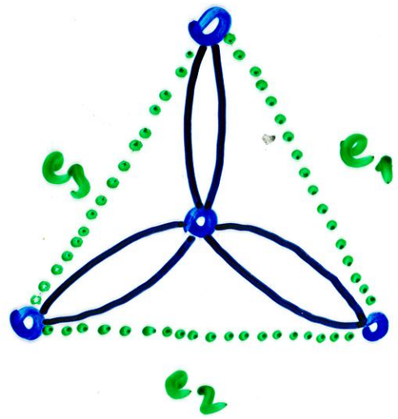
G HAS A GENERALIZED 2-GRADED EAR-DECOMPOSITION.

EXAMPLE:



THEOREM (Z. SZ.)

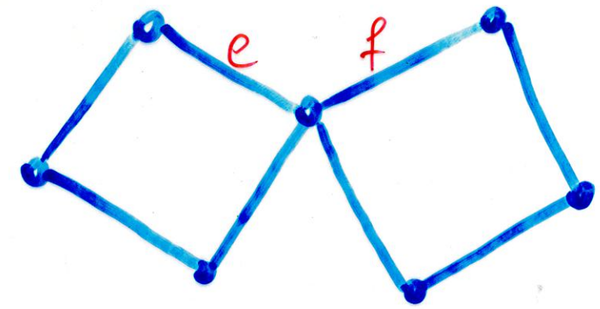
G IS φ-COVERED, $e_1, \dots, e_k \in E(\bar{G})$: $G+e_1+\dots+e_k$ IS φ-COVERED AND $\varphi(G) = \varphi(G+e_1+\dots+e_k)$. THEN $\exists\, i \le j$: $G+e_i+e_j$ IS φ-COVERED.



$\varphi(G) = 3$

$\varphi(G+e_1+e_2+e_3) = 1.$

EXAMPLE:

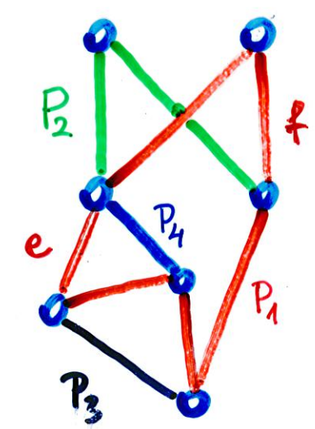


$\varphi = 2$, φ-COVERED

THEOREM (2.52.)

LET G BE A 2-VERTEX-CONNECTED φ-COVERED GRAPH. THEN FOR EVERY PAIR OF EDGES $\{e, f\}$ OF G THERE EXISTS AN OPTIMAL EAR-DECOMPOSITION OF G SO THAT THE FIRST EAR P_1 IS EVEN AND $e, f \in P_1$.

EXAMPLE:



$\varphi = 2$
φ-COVERED