ON MIN-MAX THEOREMS IN MATCHING THEORY

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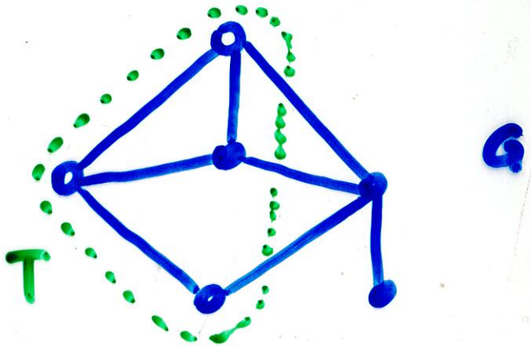
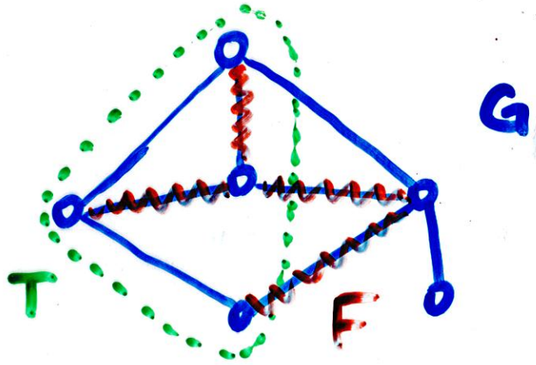
DEFINITION: FOR A CONNECTED GRAPH G AND $T \subseteq V(G)$, $|T|$ EVEN, (G,T) IS CALLED **GRAFT**.



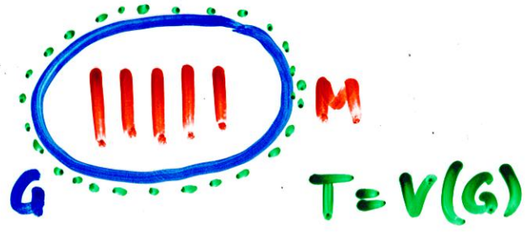
DEFINITION: FOR A GRAFT (G,T), $F \subseteq E(G)$ IS A **T-JOIN** IF

$$\deg_F(v) \text{ IS ODD} \iff v \in T.$$

EXAMPLE:



EXAMPLE: IF G HAS A PERFECT MATCHING M THEN M IS A **V-JOIN** IN (G,V).



PROBLEM: FIND A T-JOIN OF MIN. CARDINALITY, FOR A GIVEN GRAFT (G,T).

REMARKS:

I. G HAS A PERFECT MATCHING

$$\Updownarrow$$

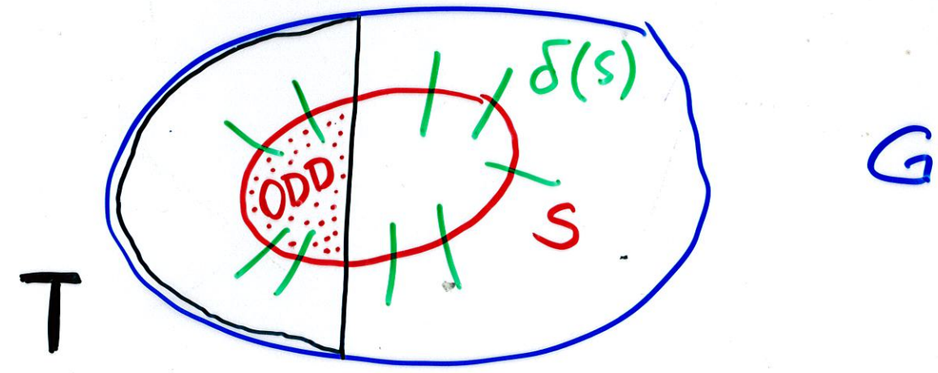
MIN. CARD. OF A V-JOIN $= \dfrac{|V(G)|}{2}$

$\Rightarrow$ MATCHING $\subseteq$ T-JOIN

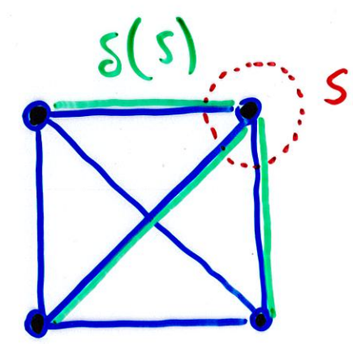
II. MIN. T-JOIN PROBLEM CAN BE REDUCED TO WEIGHTED MATCHING (EDMONDS, JOHNSON)

$\Rightarrow$ T-JOIN $\subseteq$ MATCHING

DEFINITION: FOR A GRAFT (G,T) A CUT $\delta(S)$ IS A **T-CUT** IF $|S \cap T|$ IS **ODD**.

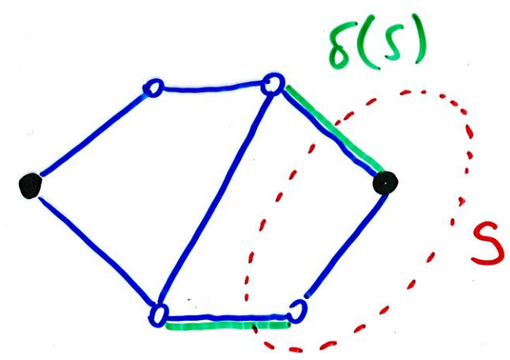


EXAMPLES:



$G = K_4$

$T = V(K_4)$



REMARK: IF F IS A T-JOIN AND $\delta(S)$ IS A T-CUT IN (G,T) THEN $|F \cap \delta(S)|$ IS ODD.

$$\sum_{v \in S} d_F(v) = 2 \cdot |E(G[S]) \cap F| + |F \cap \delta(S)|$$

$$\parallel$$

$$\underbrace{\sum_{v \in S \setminus T} \underbrace{d_F(v)}_{\text{EVEN}}}_{\text{EVEN}} + \underbrace{\sum_{v \in S \cap T} \underbrace{d_F(v)}_{\text{ODD}}}_{\text{ODD}} \quad \text{AND } |S \cap T| \text{ IS ODD}$$

REMARK: IF $F \subseteq E(G)$ THEN F IS A T-JOIN WITH $T = \{v \in V(G) : d_F(v) \text{ ODD}\}$.

REMARK: LET F BE A T-JOIN.

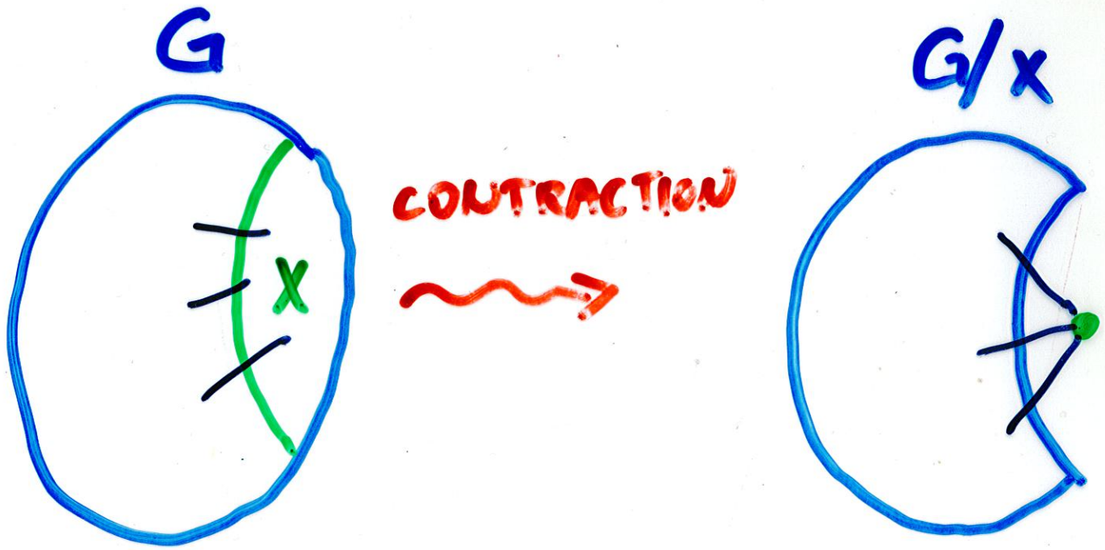
1. IF $|\delta(S) \cap F| = 1$ THEN $\delta(S)$ IS A T-CUT.

2. IF $\delta(S)$ IS A T-CUT THEN $|\delta(S) \cap F| \geq 1$.

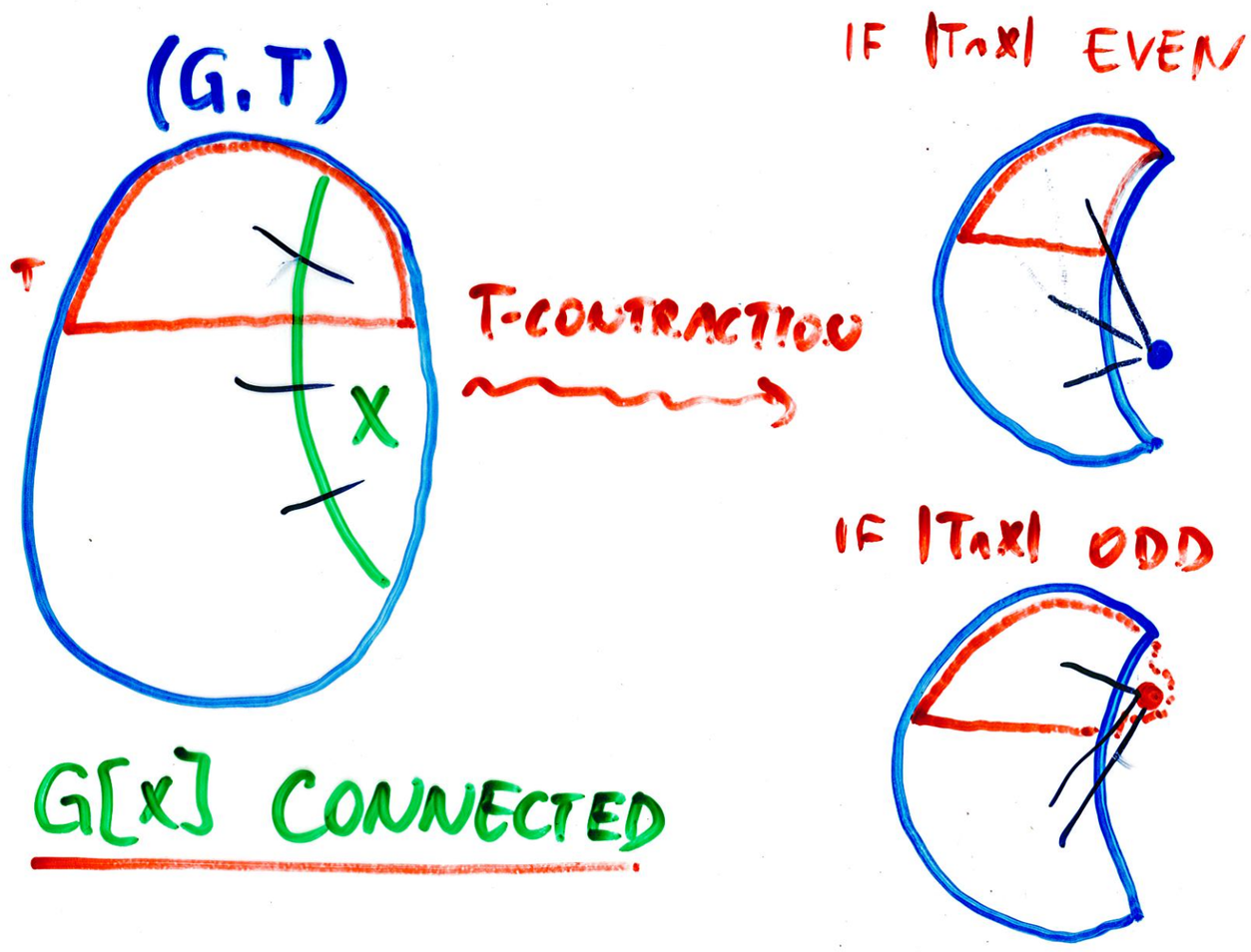
(NOT NECESSARILY 1)

BUT

3. IF $\{\delta(S_1), \dots, \delta(S_\ell)\}$ IS A SET OF EDGE DISJOINT T-CUTS WITH $\ell = |F|$ THEN $|\delta(S_i) \cap F| = 1$.



G[X] CONNECTED



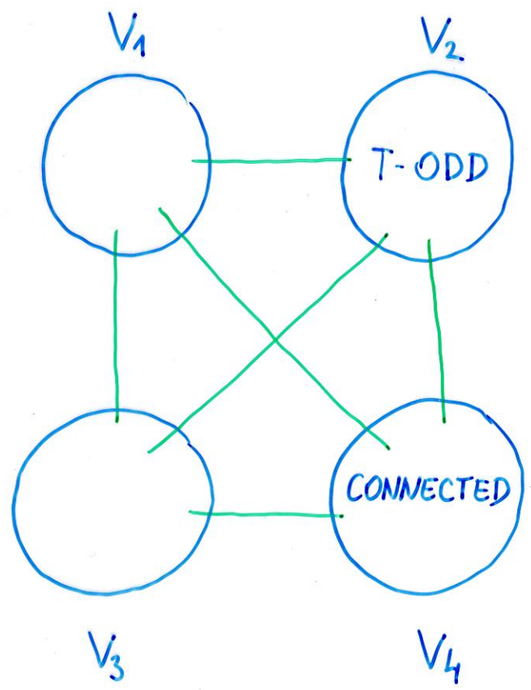
G[X] CONNECTED

DEFINITION: WE SAY THAT A GRAFT (G,T) CAN BE **T**-CONTRACTED TO $(K_4, V(K_4))$ IF THERE EXISTS A PARTITION OF $V(G)$ INTO FOUR SETS V_1, V_2, V_3 AND V_4 SUCH THAT

i) THE SUBGRAPHS INDUCED BY V_i IS CONNECTED

ii) $|V_i \cap T|$ IS ODD

iii) THERE IS AN EDGE CONNECTING V_i TO V_j $(1 \leq i < j \leq 4)$



DEFINITION: FOR A GRAFT (G,T)

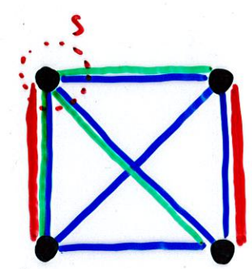
$\tau(G,T)$ = MINIMUM CARDINALITY OF A T-JOIN

$\nu(G,T)$ = MAXIMUM NUMBER OF EDGE-DISJOINT T-CUTS

(PACKING OF T-CUTS)

$$\boxed{\nu(G,T) \leq \tau(G,T).}$$

EXAMPLE:



$(K_4, V(K_4))$

$\nu = 1 < 2 = \tau$.

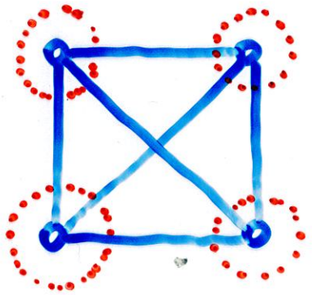
THEOREM: (SEYMOUR)

IF (G,T) CANNOT BE T-CONTRACTED TO $(K_4, V(K_4))$ THEN

$$\tau(G,T) = \nu(G,T).$$

DEFINITION: IN A GRAFT (G,T), A **2-PACKING OF T-CUTS** IS DEFINED TO BE A FAMILY OF T-CUTS SUCH THAT EACH EDGE IS CONTAINED IN AT MOST TWO OF THEM.

EXAMPLE:



$T = V(K_4)$

DEFINITION:

$\nu_2(G,T) :=$ MAXIMUM NUMBER OF T-CUTS IN A 2-PACKING.

$$\tau(G,T) \geq \nu_2(G,T)/2.$$

THEOREM (EDMONDS, JOHNSON) (LOVÁSZ)

$$\tau(G,T) = \nu_2(G,T)/2.$$

THEOREM: (SEYMOUR)

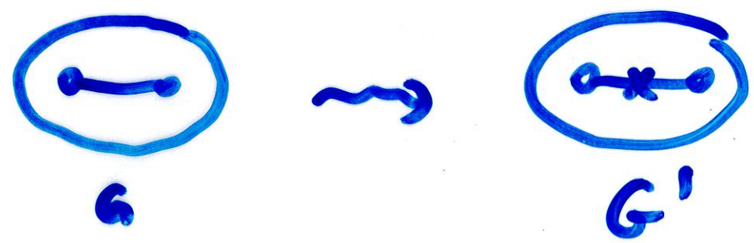
IF (G,T) IS A BIPARTITE GRAFT THEN

$$\tau(G,T) = \nu(G,T).$$

SEYMOUR'S THEOREM

$\Downarrow$

LOVÁSZ'S THEOREM



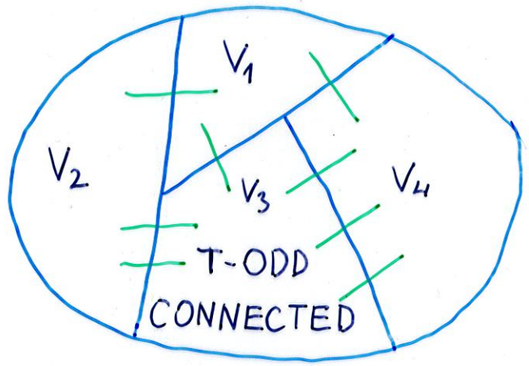
BIPARTITE

$$2\tau(G,T) \geq \nu_2(G,T) \geq \nu(G',T)$$

$$\| \text{ (SEYMOUR)}$$

$$\tau(G',T) = 2\tau(G,T).$$

DEFINITION: LET (G,T) BE A GRAFT.

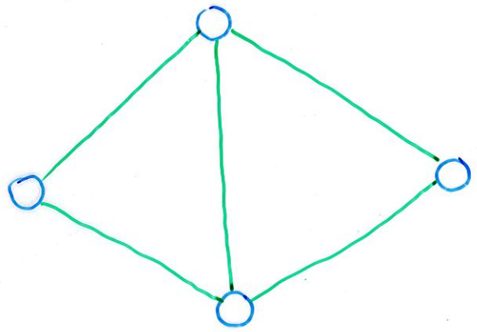


$\mathcal{P} = \{V_1, V_2, \dots, V_k\}$

B IS A T-BORDER

$val(B) = k/2$

IF $k=2$ THEN IT IS A T-CUT



BASIS GRAPH OF A T-BORDER

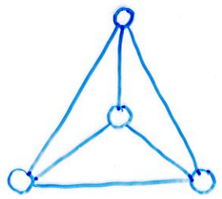
A PACKING OF T-BORDERS IS A LIST $\mathcal{B} = \{B_1, \dots, B_\ell\}$ OF EDGE-DISJOINT T-BORDERS.

$$val(\mathcal{B}) = \sum (val(B) : B \in \mathcal{B})$$

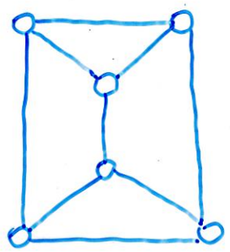
THEOREM (SEBŐ) IN A GRAFT (G, T) THE MINIMUM CARDINALITY OF A T-JOIN IS EQUAL TO THE MAXIMUM VALUE OF A T-BORDER PACKING $\{B_1, \dots, B_\ell\}$. FURTHERMORE, IF AN OPTIMAL PACKING IS CHOSEN IN SUCH A WAY THAT ℓ IS AS LARGE AS POSSIBLE, THEN EACH B_i IS BICRITICAL.

DEFINITION: A GRAPH G IS BICRITICAL IF IT HAS AN EDGE AND FOR ANY PAIR OF VERTICES x AND y $G-x-y$ HAS A PERFECT MATCHING.

EXAMPLE:

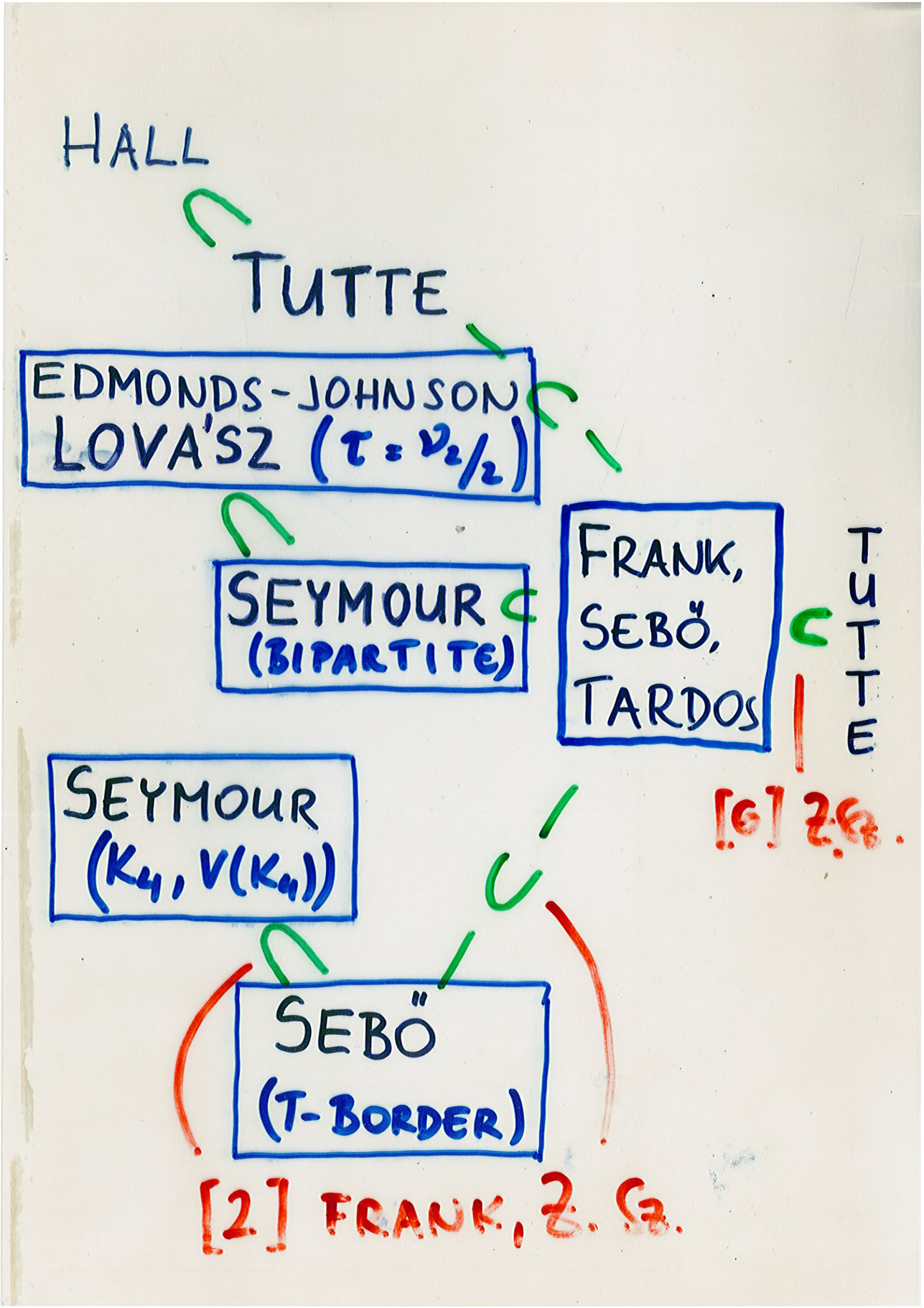


K_4



TRIANGULAR PRISM

STATEMENT (SEBŐ) IF G IS A BICRITICAL GRAPH $(V(G) \geq 4)$ THEN THE GRAFT $(G, V(G))$ CAN BE T-CONTRACTED TO $(K_4, V(K_4))$.

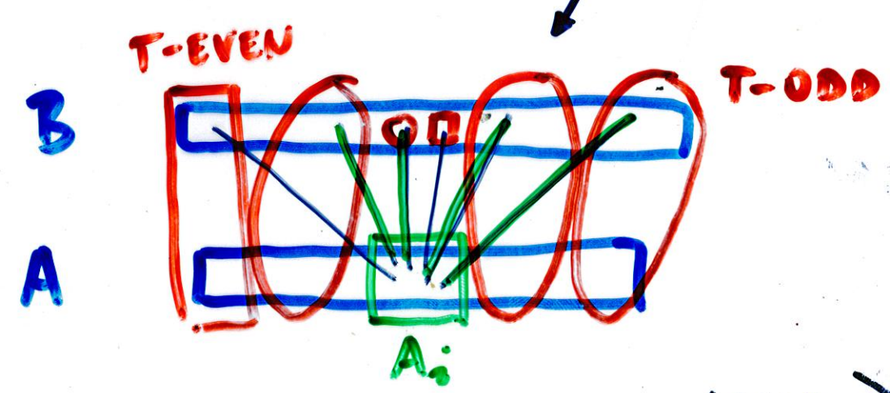

HALL
TUTTE
EDMONDS-JOHNSON
LOVÁSZ ($\tau = \nu_2/2$)
SEYMOUR
(BIPARTITE)
FRANK,
SEBŐ,
TARDOS
TUTTE
[6] Z.Gy.
SEYMOUR
(K_4, $V(K_4)$)
SEBŐ
(T-BORDER)
[2] FRANK, Z. Gy.

THEOREM: (FRANK, SEBŐ, TARDOS)

IF (G,T) IS A BIPARTITE GRAFT THEN

$$\tau(G,T) = \max\left\{ \sum_{A_i \in \mathcal{P}} q_T(G-A_i) : \mathcal{P} \text{ PARTITION OF } A \right\}$$

PACKING OF T-CUTS!



MIN $\geq$ MAX

1. F-S-T THEOREM

$\Downarrow$

SEYMOUR'S THEOREM

2. F MIN. T-JOIN, $\mathcal{P}$ MAX PARTITION $A_i \in \mathcal{P}$, K COMP. OF $G-A_i$

$$\Rightarrow |\delta(K) \cap F| = \begin{cases} 0 & K \text{ T-EVEN} \\ 1 & K \text{ T-ODD} \end{cases}$$

ORIGINAL PROOF:

MIN. T-JOIN F

$\updownarrow$

CONSTRUCTION OF A PARTITION $\mathcal{P}$ WITH VALUE $|F|$.

NEW APPROACH:

MAX. ADMISSIBLE PARTITION $\mathcal{P}$ OF A OF VALUE k

$\updownarrow$

CONSTRUCTION OF A T-JOIN F WITH $|F| = k$.

INDUCTION + HALL TH.

$\Downarrow$

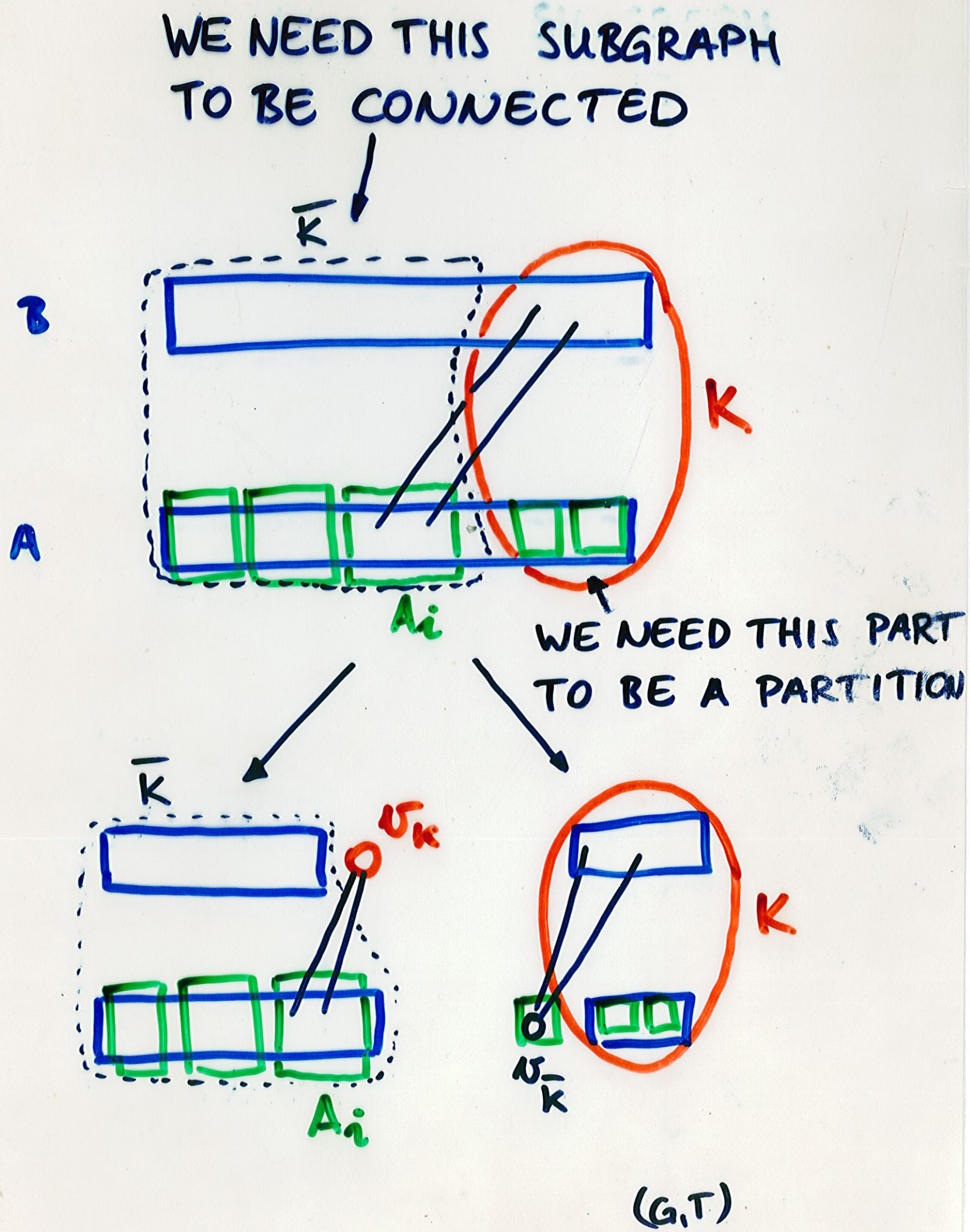
TUTTE TH.

SIMILARLY

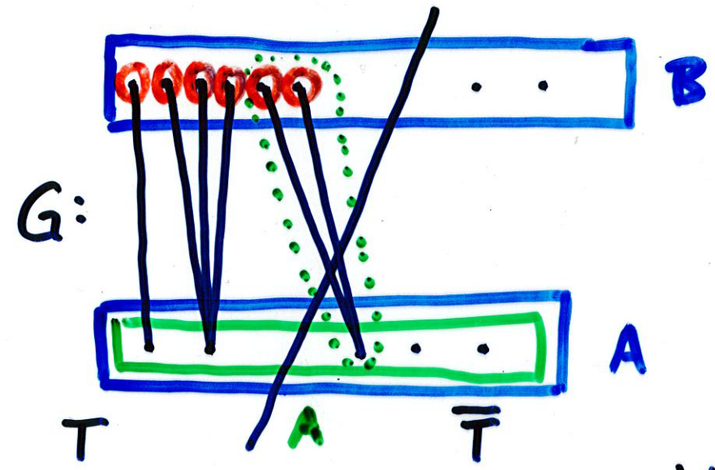
INDUCTION + TUTTE TH.

$\Downarrow$

"FST" TH.


WE NEED THIS SUBGRAPH
TO BE CONNECTED
$\bar{K}$
B
A
K
A_i
WE NEED THIS PART
TO BE A PARTITION
$\bar{K}$
v_K
A_i
K
$v_{\bar{K}}$
(G,T)

IF OPT. ADM. PARTITION $\mathcal{P} = \{A\}$



$\Downarrow$

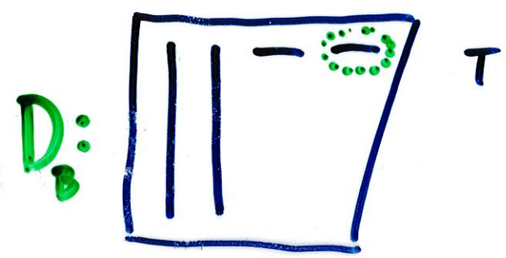
$val(\mathcal{P}) = |T \cap B|$

$\Downarrow$

WE NEED A T-JOIN F

$$\deg_F(v) = \begin{cases} 0 & v \in B \setminus T \\ 1 & v \in B \cap T \end{cases}$$

$\rightsquigarrow$



G HAS A T-JOIN OF CARD. $|T \cap B|$

$\Updownarrow$

D HAS A PERFECT MATCHING.

(IF OPT. ADM. PART. $\mathcal{P} = \{v : v \in A\}$
THEN SIMILAR)

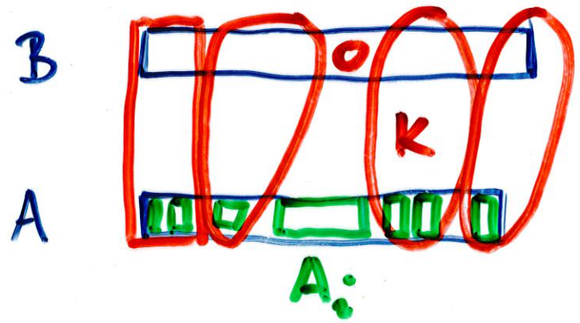
DEF.: A PARTITION $\mathcal{P} = \{A_1, \dots, A_k\}$ OF A IS **ADMISSIBLE** IF

$\forall\ 1 \leq i \neq j \leq k$ AND

FOR EVERY CONNECTED COMPONENT K OF $G - A_i$

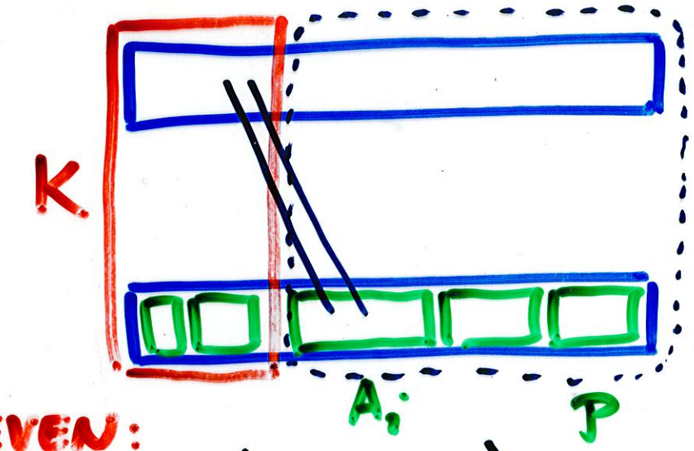
(1) $G - K$ IS CONNECTED,

(2) $A_j \subseteq K$ OR $A_j \cap K = \emptyset$.



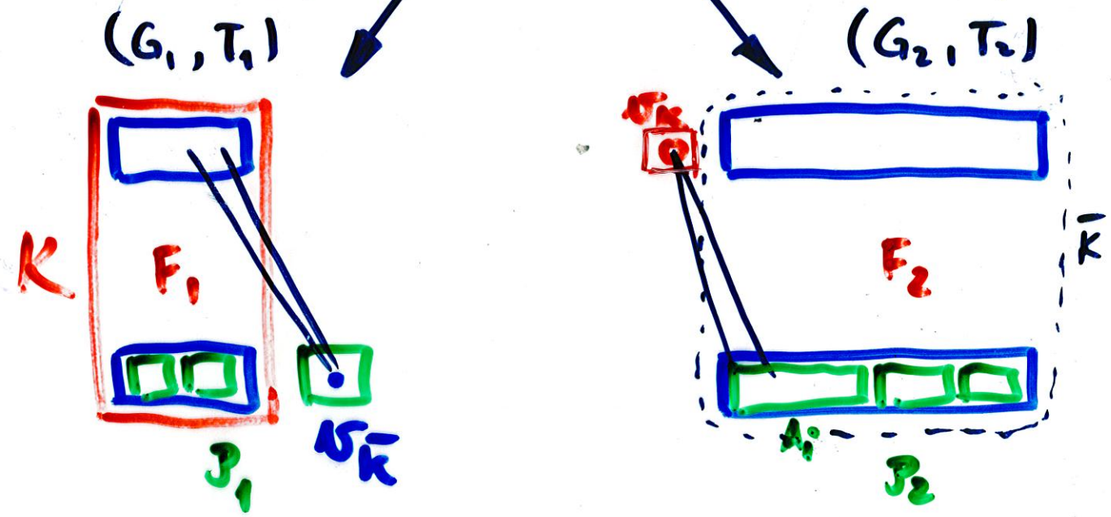
THEOREM: IF (G,T) IS A BIPARTITE GRAFT THEN

$$\tau(G,T) = \max\left\{ \sum_{A_i \in \mathcal{P}} q_T(G - A_i) : \mathcal{P} \text{ ADMISSIBLE PARTITION OF } A \right\}.$$



$|K|, |\bar{K}| > 1$

K T-EVEN:

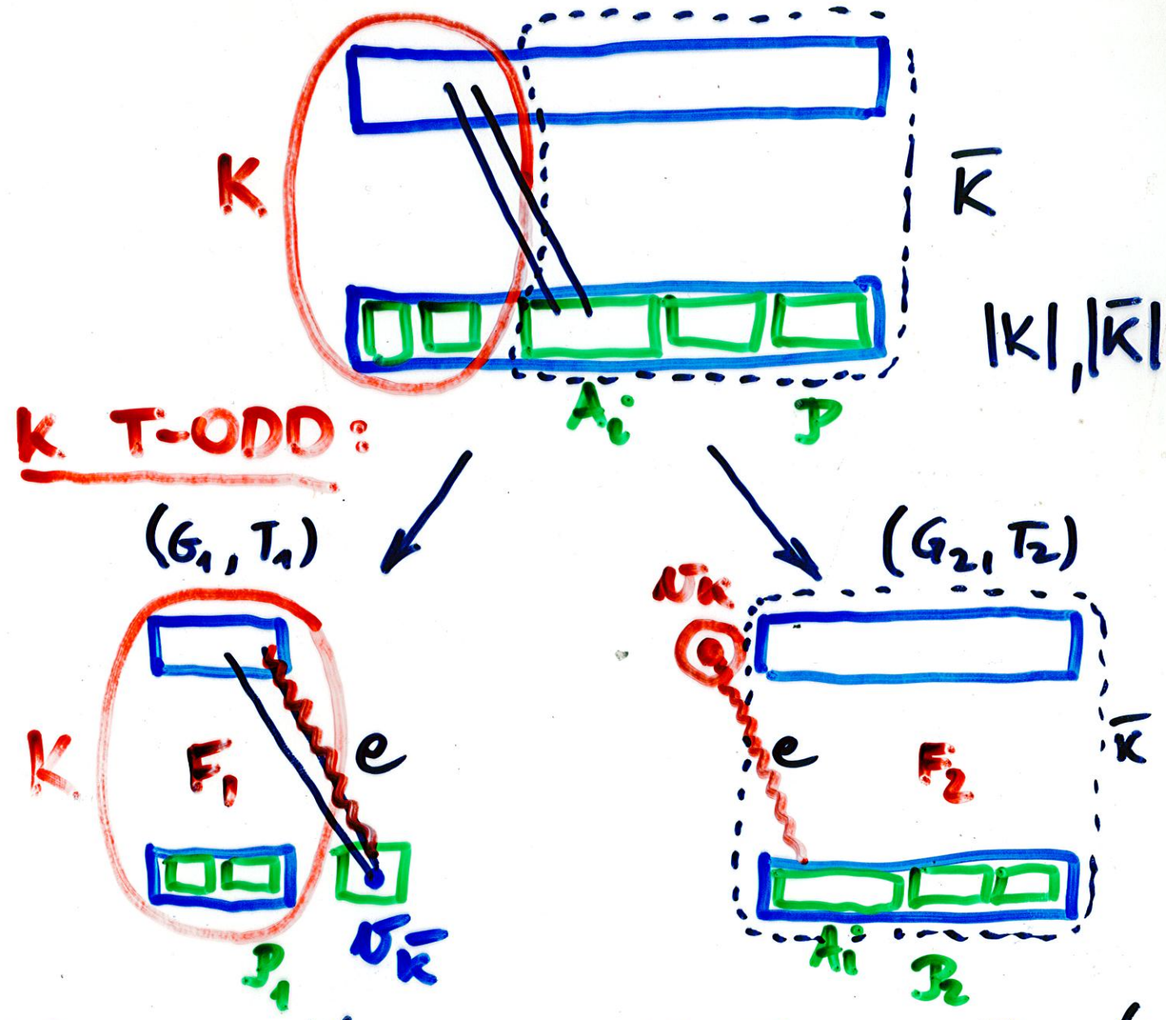


LEMMA: $\mathcal{P}_i$ IS AN OPT. ADM. PART. OF (G_i, T_i).

BY INDUCTION:

$\exists$ T_i-JOIN F_i IN (G_i, T_i) : $|F_i| = val(\mathcal{P}_i)$

$\Rightarrow$ $F = F_1 \cup F_2$ IS A T-JOIN IN (G, T) WITH $|F| = val(\mathcal{P})$.



$|K|, |\bar{K}| > 1$

K T-ODD:

LEMMA: $\forall e = v_{\bar{K}} u$ $\exists$ T_1-JOIN F_1 IN (G_1, T_1) s.t. $|F_1| = val(P_1)$ AND $e \in F_1$.

$\Rightarrow$ $F = F_1 \cup F_2$ IS A T-JOIN IN (G, T) WITH $|F| = val(P)$.