A NOTE ON PACKING PATHS IN PLANAR GRAPHS

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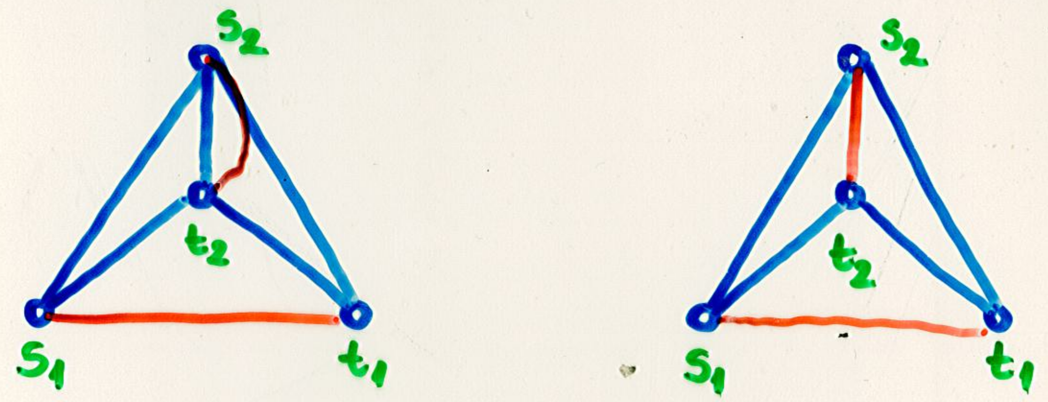
EDGE-DISJOINT PATHS PROBLEM

GIVEN A GRAPH G AND $s_i, t_i \in V(G)$ $i=1,\dots,\ell$.

FIND ℓ EDGE-DISJOINT PATHS CONNECTING THE CORRESPONDING PAIRS s_i, t_i.



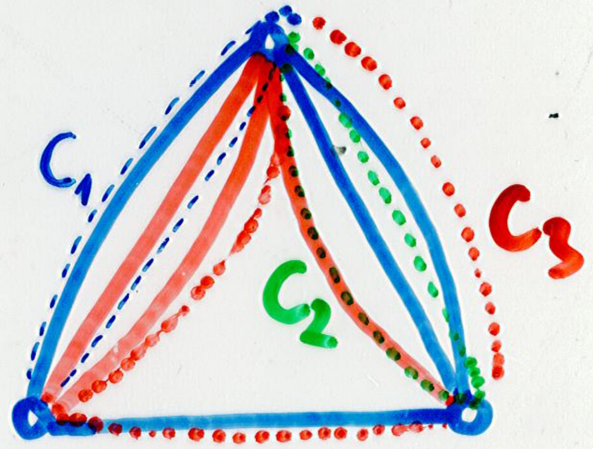
G SUPPLY GRAPH

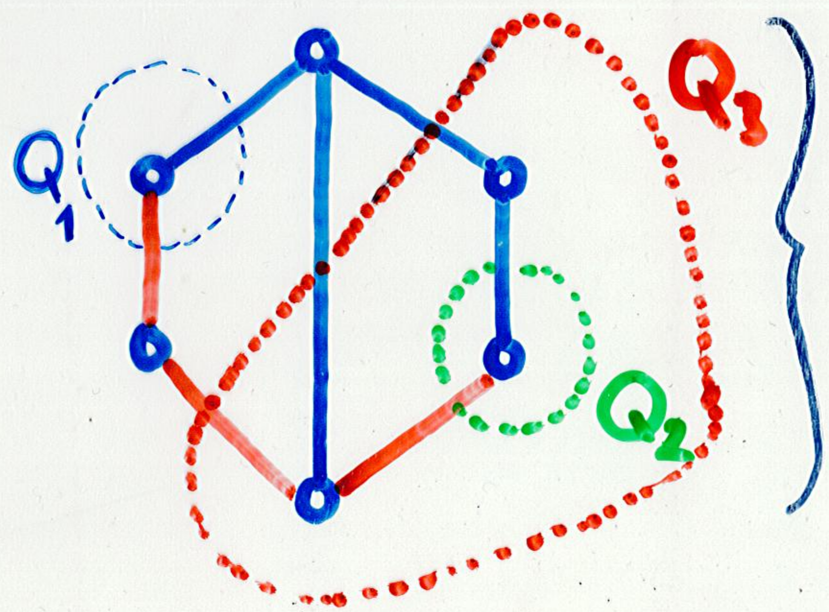
H DEMAND GRAPH

FIND ℓ EDGE-DISJOINT CIRCUITS IN $G+H$ SUCH THAT EACH OF THEM CONTAINS EXACTLY ONE DEMAND EDGE.

BY PLANAR DUALIZATION:

FIND ℓ EDGE-DISJOINT CUTS IN G' (DUAL OF $G+H$) SUCH THAT EACH OF THEM CONTAINS EXACTLY ONE EDGE OF F (CORRESPONDING TO H).





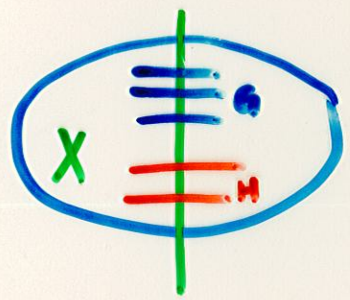
G+H IS PLANAR

EVEN IN THIS CASE THE PROBLEM IS NP-COMPLETE.

(MIDDENDORF, PFEIFFER '93)

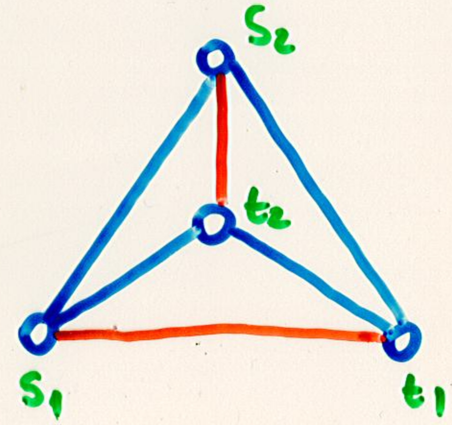
NECESSARY:

CUT CRITERION



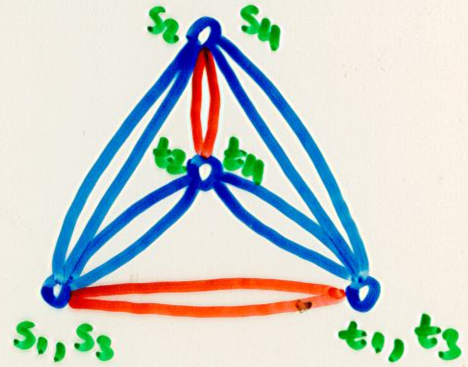
$$d_G(X) \geq d_H(X) \quad \forall X \subseteq V(G)$$

NOT SUFFICIENT:



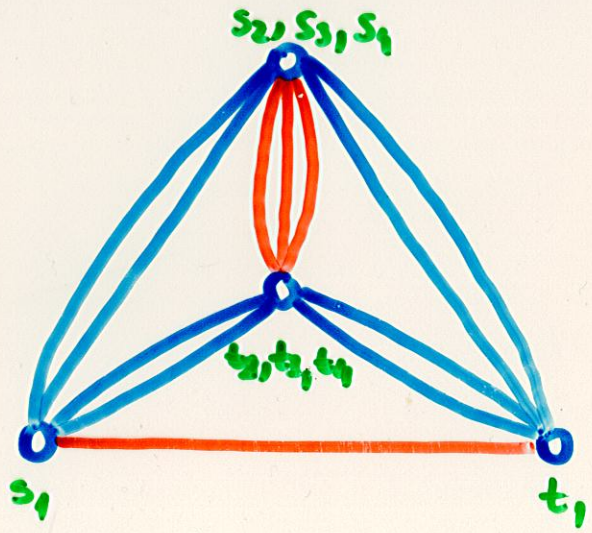
THEOREM (SEYMOUR '81)

WHEN G+H IS PLANAR AND EULERIAN, THE EDGE-DISJOINT PATHS PROBLEM HAS A SOLUTION IF AND ONLY IF THE CUT CRITERION HOLDS.



THEOREM (KORACH, PENN '92)

SUPPOSE G+H IS PLANAR AND THE CUT CRITERION HOLDS. THEN THERE IS AT MOST ONE DEMAND EDGE ON EACH BOUNDED FACE OF G SO THAT LEAVING OUT THESE EDGES FROM H THE PROBLEM HAS A SOLUTION.



REMARKS: 1. NO DEMAND EDGE IS LEFT OUT FROM THE INFINITE FACE OF G.

2. IF EACH BOUNDED FACE OF G CONTAINS AT MOST ONE DEMAND EDGE THEN THIS THEOREM SAYS NOTHING.

OUR **AIM** IS TO STRENGTHEN THE CUT CRITERION

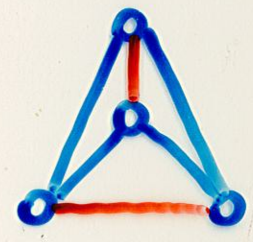
a.) TO BE A SUFFICIENT CONDITION FOR THE PROBLEM.

b.) TO HAVE 1. FOR MORE FACES OF G.

a.)

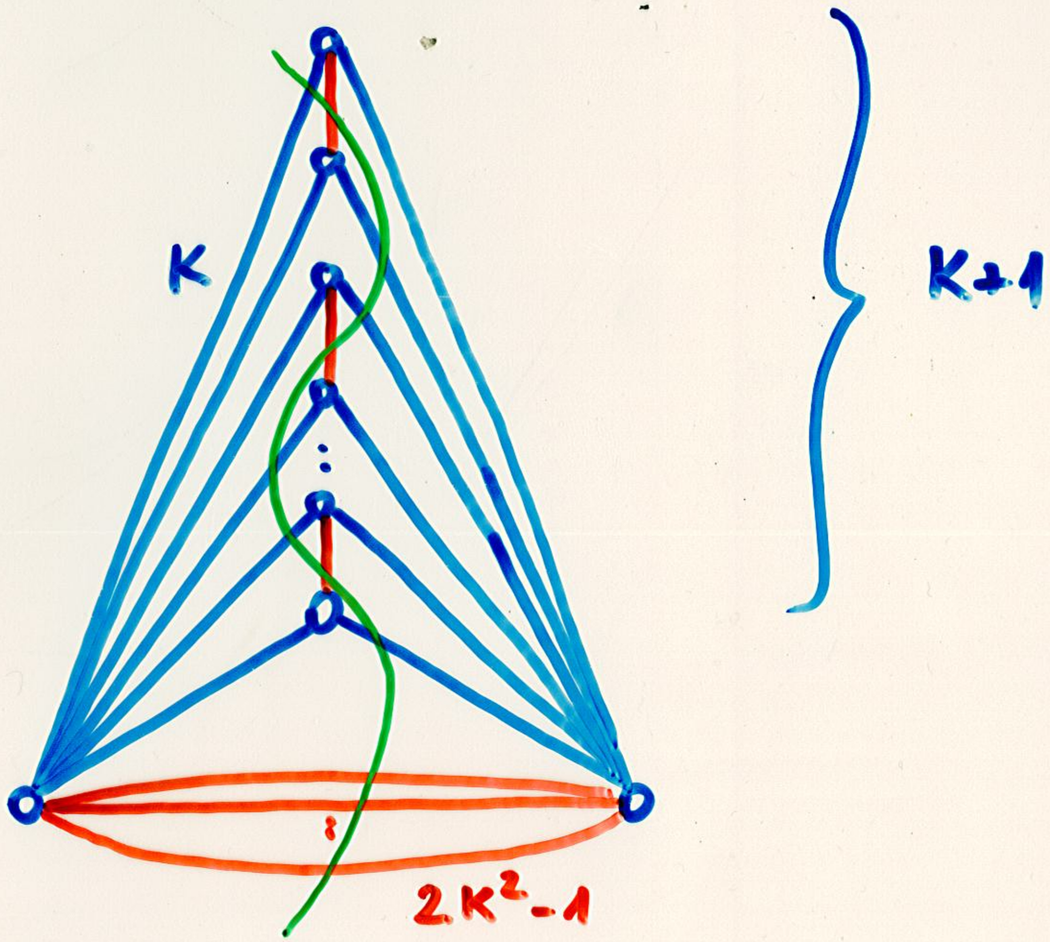
i.) $d_G(X) \geq d_H(X) \quad \forall X \subseteq V$

IS NOT SUFFICIENT



ii.) $d_G(X) \geq d_H(X) + K \quad \forall X \subsetneq V$

IS NOT SUFFICIENT



$$d_G(X) - d_H(X) = (k+1)2k - (2k^2-1+k+1) = k$$

THEOREM (FRANK, SZ. '93)

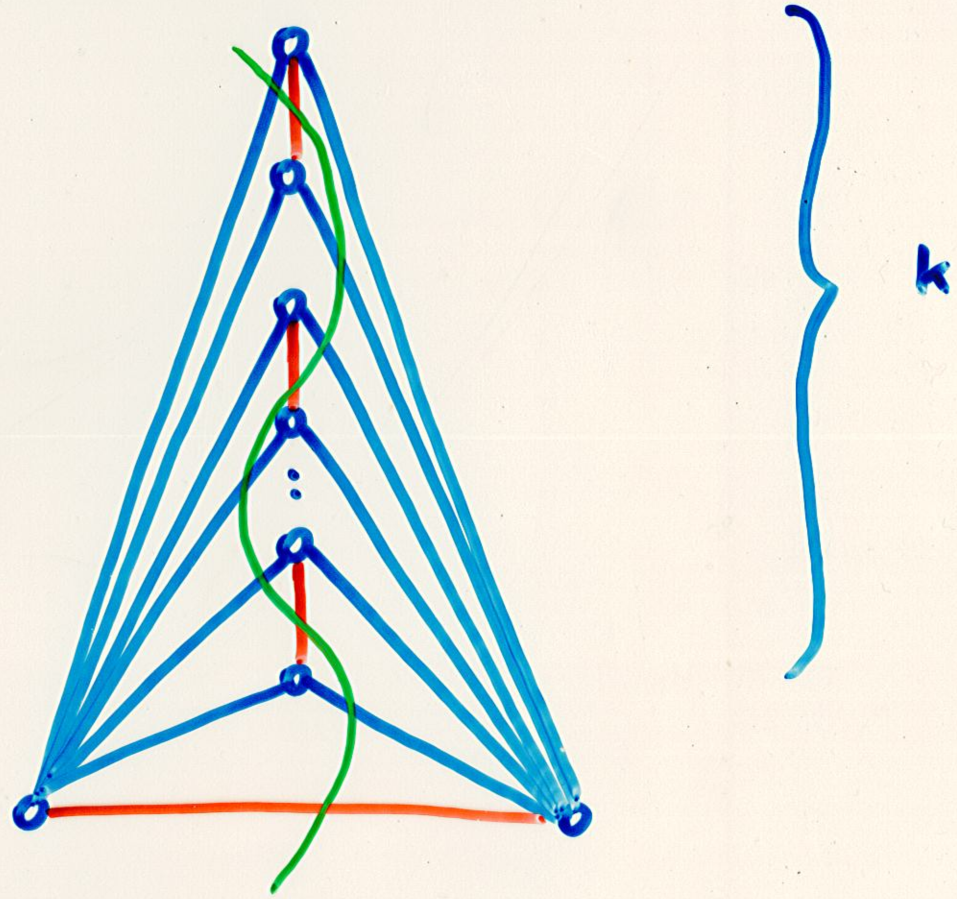
IF G+H IS PLANAR AND

$$d_G(X) \geq 2 \cdot d_H(X) \quad \forall X \subseteq V$$

THEN THE PROBLEM HAS A SOLUTION.

iv.) $d_G(X) \geq (2-\varepsilon) \cdot d_H(X) \quad \forall X \subseteq V, \ \varepsilon > 0$

IS NOT SUFFICIENT.



$d_G(X) = 2 \cdot k$

$d_H(X) = k+1$

$$\Rightarrow \quad d_G(X) = \left(2 - \frac{2}{k+1}\right) d_H(X)$$

b.) SUPPOSE, THAT THE FACES OF G CONTAINING DEMAND EDGES ARE PARTITIONED INTO TWO GROUPS:

$$C_0, C_1, \dots, C_k \qquad \begin{pmatrix} k \geq 0 \\ \ell \geq 0 \end{pmatrix}.$$
$$D_1, \dots, D_\ell$$

FOR A CUT $\delta_{G+H}(X)$ LET $\mu(X)$ BE THE NUMBER OF THOSE FACES C_i ($i \geq 1$!) FROM WHICH $\delta_{G+H}(X)$ CONTAINS AT LEAST ONE DEMAND EDGE.

THEOREM (FRANK, SZ. '93)

IF $G+H$ IS PLANAR AND

$$d_G(X) \geq d_H(X) + \mu(X) \qquad \forall X \subseteq V$$

THEN IT IS POSSIBLE TO LEAVE OUT AT MOST ONE DEMAND EDGE FROM EACH FACE $D_1, \dots, D_\ell$ SUCH THAT THE RESULTING PROBLEM HAS A SOLUTION.

THEOREM: IF $G+H$ IS PLANAR AND

$$d_G(X) \geq d_H(X) + \mu(X) \quad \forall X \subseteq V$$

THEN THE EDGE-DISJOINT PATHS PROBLEM HAS A SOLUTION.

REMARK: $d_G(X) \geq d_H(X) + \mu(X) - 2 \quad \forall X \subseteq V$

IS NOT SUFFICIENT.

OPEN PROBLEM:

$$d_G(X) \geq d_H(X) + \mu(X) - 1 \quad \forall X \subseteq V$$

IS SUFFICIENT OR NOT ?