Old and new results on packing arborescences

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Outline

- **Old results**
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    - Packing spanning arborescences
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  - **Dypergraphs**
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    - Matroid-based packing of rooted-arborescences
    - Maximal-rank packing of rooted-arborescences

- **New results**
  - **Matroid-based rooted-dypergraph**
    - Matroid-based packing of rooted-hyper-arborescences
    - Maximal-rank packing of rooted-hyper-arborescences
Reachability in digraph

**Definition**

Let $\vec{G} = (V, A)$ be a digraph and $X \subseteq V$.

1. $\rho_A(X)$ is the number of arcs entering $X$,
2. $P_A(X)$ is the set of vertices from which $X$ can be reached in $\vec{G}$,
3. $Q_A(X)$ is the set of vertices that can be reached from $X$ in $\vec{G}$.

$$\rho_A(X) = 2$$
Arborescences

Definition

Let $\vec{G} = (V, A)$ be a digraph and $r \in V$.

1. A subgraph $\vec{T} = (U, B)$ of $\vec{G}$ is an $r$-arborescence if
   1. $r \in U$ with $\rho_B(r) = 0$,
   2. $\rho_B(u) = 1$ for all $u \in U \setminus r$ and
   3. $\rho_B(X) \geq 1$ for all $X \subseteq V \setminus r$, $X \cap U \neq \emptyset$.

2. An $r$-arborescence $\vec{T}$ is
   1. spanning if $U = V$,
   2. maximal if $U = Q_A(r)$.

3. Packing of arborescences is a set of pairwise arc-disjoint arborescences.
Theorem (Edmonds 1973)

Let $\vec{G} = (V, A)$ be a digraph, $r \in V$ and $k$ a positive integer.

1. There exists a packing of $k$ spanning $r$-arborescences

2. $\rho_A(X) \geq k$ for all $\emptyset \neq X \subseteq V \setminus r$. 

\[ \vec{G} = (V, A) \text{ be a digraph, } r \in V \text{ and } k \text{ a positive integer.} \]

\[ \rho_A(X) \geq k \text{ for all } \emptyset \neq X \subseteq V \setminus r. \]
A packing of maximal arborescences is a set \( \{T_1, \ldots, T_t\} \) of pairwise arc-disjoint maximal \( r_i \)-arborescences \( T_i \) in \( \vec{G} \); that is for every \( v \in V \), \( \{r_i : v \in V(T_i)\} = \{r_i \in P_A(v)\} \).

For \( X \subseteq V \), \( p_A(X) = |\{r_i \in P_A(X) \setminus X\}| \).
Packing maximal arborescences

**Theorem (Kamiyama, Katoh, Takizawa 2009)**

Let $\vec{G} = (V, A)$ be a digraph and $(r_1, \ldots, r_t) \in V^t$.

1. There exists a packing of maximal arborescences $\iff$
2. $\rho_A(X) \geq p_A(X)$ for all $X \subseteq V$.

**Remark**

It implies Edmonds’ theorem.

1. Let $r_1 = \cdots = r_k = r$.
2. $\rho_A(X) \geq k$ for all $\emptyset \neq X \subseteq V \setminus r$ implies the above condition and that each vertex is reachable from $r$.
3. Hence there exists a packing of maximal $r$-arborescences that is a packing of spanning $r$-arborescences.
Directed hypergraph (shortly dypergraph) is $\vec{G} = (V, A)$, where
- $V$ denotes the set of vertices and
- $A$ denotes the set of hyperarcs of $\vec{G}$.

Hyperarc is a pair $(Z, z)$ such that $z \in Z \subseteq V$, where
- $z$ is the head of the hyperarc $(Z, z)$ and
- the elements of $Z \setminus z \neq \emptyset$ are the tails of the hyperarc $(Z, z)$.
Reachability in dypergraph

Definition

Let $\vec{G} = (V, A)$ be a dypergraph and $X \subseteq V$.

1. Hyperarc $(Z, z)$ enters $X$ if $z \in X$ and $(Z \setminus z) \cap (V \setminus X) \neq \emptyset$,
2. $\rho_A(X)$ is the number of hyperarcs entering $X$,
3. Path from $u$ to $x$ in $\vec{G}$ is $v_1(=u), (Z_1, v_2), v_2, \ldots, v_i, (Z_i, v_{i+1}), v_{i+1}, \ldots, v_j(=x)$ such that $v_i$ is a tail of $(Z_i, v_{i+1})$.
4. $P_A(X)$ is the set of vertices from which $X$ can be reached in $\vec{G}$,
5. $Q_A(X)$ is the set of vertices that can be reached from $X$ in $\vec{G}$.
**Definition**

Trimming the dypergraph $\vec{G}$ means replacing each hyperarc $(K, v)$ of $\vec{G}$ by an arc $uv$ where $u$ is one of the tails of the hyperarc $(K, v)$.

**Definition**

$h$ is supermodular: $h(X) + h(Y) \leq h(X \cap Y) + h(X \cup Y) \ \forall \ X, Y \subseteq V.$

**Theorem (Frank 2011)**

Let $\vec{G} = (V, A)$ be a dypergraph and $h$ an integer-valued, intersecting supermodular function on $V$ such that $h(\emptyset) = 0 = h(V)$.

If $\rho_{A}(X) \geq h(X)$ for all $X \subseteq V$, then $\vec{G}$ can be trimmed to a digraph $\vec{G}$ such that $\rho_{A}(X) \geq h(X)$ for all $X \subseteq V$. 
Hyper-arborescences

Definition

Let $\vec{G} = (V, \mathcal{A})$ be a dypergraph and $r \in V$.

1. A subgraph $\vec{T} = (U, B)$ of $\vec{G}$ is an $r$-hyper-arborescence if it can be trimmed to an $r$-arborescence on $U^* \cup r$, where $U^* = \{ u : \rho_B(u) \neq 0 \}$; that is
   1. $r \in U \setminus U^*$,  
   2. $\rho_B(u) = 1$ for all $u \in U^*$ and  
   3. $\rho_B(X) \geq 1$ for all $X \subseteq V \setminus r$, $X \cap U^* \neq \emptyset$.

2. The $r$-hyper-arborescence $\vec{T}$ is
   1. spanning if $U^* = V \setminus r$,  
   2. maximal if $U^* = Q_A(r) \setminus r$. 

\[ \text{Z. Szigeti (G-SCOP, Grenoble) On packing of arborescences 11 juin 2015 11 / 26} \]
Theorem (Frank, T. Király, Kriesell 2003)

Let $\vec{G} = (V, A)$ be a dypergraph, $r \in V$ and $k$ a positive integer.

1. There exists a packing of $k$ spanning $r$-hyper-arborescences.
2. $\rho_A(X) \geq k$ for all $\emptyset \neq X \subseteq V \setminus r$.

Remark

1. It is proved easily by trimming and Edmonds’ theorem.
2. It implies Edmonds’ theorem if $\vec{G}$ is a digraph.
Theorem (Bérczi, Frank 2008)

Let $\vec{G} = (V, A)$ be a dypergraph and $(r_1, \ldots, r_t) \in V^t$.

1. There exists a packing of maximal hyper-arborescences
2. $\rho_A(X) \geq p_A(X)$ for all $X \subseteq V$.

Remark

1. It is proved not easily by trimming and Kamiyama, Katoh, Takizawa’s theorem since $p_A(X)$ is not intersecting supermodular.
2. It implies
   1. Frank, T. Király, Kriesell’s theorem if $r_1 = \cdots = r_k = r$ and $\rho_A(X) \geq k$ for all $\emptyset \neq X \subseteq V \setminus r$,
   2. Kamiyama, Katoh, Takizawa’s theorem if $\vec{G}$ is a digraph.
Matroids

**Definition**

For $\mathcal{I} \subseteq 2^S$, $\mathcal{M} = (S, \mathcal{I})$ is a matroid if

1. $\mathcal{I} \neq \emptyset$,
2. If $X \subseteq Y \in \mathcal{I}$ then $X \in \mathcal{I}$,
3. If $X, Y \in \mathcal{I}$ with $|X| < |Y|$ then $\exists y \in Y \setminus X$ such that $X \cup y \in \mathcal{I}$.

**Examples**

1. Sets of linearly independent vectors in a vector space,
2. Edge-sets of forests of a graph,
3. $U_{n,k} = \{X \subseteq S : |X| \leq k\}$ where $|S| = n$, free matroid $= U_{n,n}$. 
Matroids

**Notion**

1. **independent sets** = $\mathcal{I}$,
   - any subset of an independent set is independent,

2. **base** = maximal independent set,
   - all basis are of the same size,

3. **rank function**: $r(X) = \max\{|Y| : Y \in \mathcal{I}, Y \subseteq X\}$.
   - non-decreasing,
   - submodular (that is $-r$ is supermodular),
   - $X \in \mathcal{I}$ if and only if $r(X) = |X|$.
A matroid-based rooted-digraph is a quadruple $(\vec{G}, \mathcal{M}, S, \pi)$:

1. $\vec{G} = (V, A)$ is a digraph,
2. $\mathcal{M}$ is a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. $\pi$ is a placement of the elements of $S$ at vertices of $V$ such that $S_v \in \mathcal{I}$ for every $v \in V$, where $S_X = \pi^{-1}(X)$, the elements of $S$ placed at $X$. 

$\vec{G}$
$S = \{s_1, s_2, s_3\}$
$\mathcal{M} = U_{3,2}$
Matroid-based packing of rooted-arborescences

**Definition**

A **rooted-arborescence** is a pair \((\vec{T}, s)\) where

1. \(\vec{T}\) is an \(r\)-arborescence for some vertex \(r\),
2. \(s \in S\), placed at \(r\).

**Definition**

A packing \(\{(\vec{T}_1, s_1), \ldots, (\vec{T}_{|S|}, s_{|S|})\}\) of rooted-arborescences is **matroid-based** if \(\{s_i \in S : v \in V(\vec{T}_i)\}\) forms a base of \(S\) for every \(v \in V\).

**Remark**

For the **free matroid** \(\mathcal{M}\) with all \(k\) roots at a vertex \(r\),

1. matroid-based packing of rooted-arborescences
2. packing of \(k\) spanning \(r\)-arborescences.

\[
\begin{align*}
\pi(s_1) & \quad \pi(s_2) \\
\pi(s_3) & \quad T_1 \quad T_2 \\
& \quad T_3 
\end{align*}
\]
Matroid-based packing of rooted-arborescences

**Theorem (Durand de Gevigney, Nguyen, Szigeti 2013)**

Let $(\tilde{G}, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph.

1. There is a matroid-based packing of rooted-arborescences

2. $\rho_A(X) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X)$ for all $\emptyset \neq X \subseteq V$.

**Remark**

It implies Edmonds’ theorem if $\mathcal{M}$ is the free matroid with all $k$ roots at the vertex $r$. 

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Z. Szigeti (G-SCOP, Grenoble)

On packing of arborescences

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Maximal-rank packing of rooted-arborescences

**Definition**

A packing \( \{(\vec{T}_1, s_1), \ldots, (\vec{T}_{|S|}, s_{|S|})\} \) of rooted-arborescences is of maximal rank if \( \{s_i \in S : v \in V(\vec{T}_i)\} \) forms a base of \( S_{PA(v)} \) for every \( v \in V \).

**Theorem (Cs. Király 2013)**

Let \( (\vec{G}, M, S, \pi) \) be a matroid-based rooted-digraph.

1. There exists a maximal-rank packing of rooted-arborescences \( \iff \)
2. \( \rho_A(X) \geq r_M(S_{PA(X)}) - r_M(S_X) \) for all \( X \subseteq V \).

**Remark**

1. It implies
   - DdG-N-Sz’ theorem if \( \rho_A(X) \geq r_M(S) - r_M(S_X) \) for all \( \emptyset \neq X \subseteq V \),
   - Kamiyama, Katoh, Takizawa’s theorem if \( M \) is the free matroid.
A matroid-based rooted-dypergraph is a quadruple $(\vec{G}, \mathcal{M}, S, \pi)$:

1. $\vec{G} = (V, A)$ is a dypergraph,
2. $\mathcal{M}$ is a matroid on a set $S = \{s_1, \ldots, s_t\}$.
3. $\pi$ is a placement of the elements of $S$ at vertices of $V$ such that $S_v \in \mathcal{I}$ for every $v \in V$. 
Matroid-based packing of rooted-hyper-arborescences

**Definition**

1. A rooted-hyper-arborescence is a triple \((\vec{T}, r, s)\) where \(\vec{T}\) is an \(r\)-hyper-arborescence and \(s\) is an element of \(S\) placed at \(r\).

2. A packing \(\{(\vec{T}_1, r_1, s_1), \ldots, (\vec{T}_{|S|}, r_{|S|}, s_{|S|})\}\) of rooted-hyper-arborescences is matroid-based if \(\{s_i \in S : v \in Q_{A(\vec{T}_i)}(r_i)\}\) forms a base of \(S\) for every \(v \in V\).

**Theorem (Léonard, Szigeti 2013)**

Let \((\vec{G}, \mathcal{M}, S, \pi)\) be a matroid-based rooted-dypergraph.

1. There is a matroid-based packing of rooted-hyper-arborescences \(\iff\)

2. \(\rho_A(X) \geq r_M(S) - r_M(S_X)\) for all \(\emptyset \neq X \subseteq V\).

**Remark**

1. It is proved easily by trimming and DdG-N-Sz’ theorem.
Maximal-rank packing of rooted-hyper-arborescences

Definition

Packing \( \{(\vec{T}_1, r_1, s_1), \ldots, (\vec{T}_{|S|}, r_{|S|}, s_{|S|})\} \) of rooted-hyper-arborescences is of maximal rank if \( \{s_i \in S : v \in Q_{\mathcal{A}(\vec{T}_i)}(r_i)\} \) forms a base of \( S_{\mathcal{P}\mathcal{A}(v)} \) for all \( v \in V \).

Theorem (Szigeti 2015-)

Let \((\vec{G}, \mathcal{M}, S, \pi)\) be a matroid-based rooted-dypergraph.

1. There is a maximal-rank packing of rooted-hyper-arborescences \( \iff \)
2. \( \rho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(S_{\mathcal{P}\mathcal{A}(X)}) - r_{\mathcal{M}}(S_X) \) for all \( X \subseteq V \).

Remark

1. It is proved not easily by trimming and Cs. Király’s theorem since \( r_{\mathcal{M}}(S_{\mathcal{P}\mathcal{A}(X)}) - r_{\mathcal{M}}(S_X) \) is not intersecting supermodular.
2. It implies all the previous results.
Proof of necessity

Proof

1. Let \( \{ (\vec{T}_1, r_1, s_1), \ldots, (\vec{T}_{|S|}, r_{|S|}, s_{|S|}) \} \) be a maximal-rank packing of rooted-hyper-arborescences in \((\vec{G}, \mathcal{M}, S, \pi)\).

2. Let \( B_v = \{ s_i \in S : \nu \in Q_{\mathcal{A}}(\vec{T}_i)(r_i) \} \) (base of \( S_{\mathcal{P}_\mathcal{A}(\nu)} \)) and \( X \subseteq V \).

3. For each root \( s_i \in \bigcup_{\nu \in X} B_v \setminus S_X \), there exists a vertex \( \nu \in X \) such that \( s_i \in B_v \) and then since \( \vec{T}_i \) is an \( r_i \)-hyper-arborescence, \( r_i \notin X \) and \( \nu \in Q_{\mathcal{A}}(\vec{T}_i)(r_i) \cap X \), there exists a hyperarc of \( \vec{T}_i \) that enters \( X \).

4. Since the hyper-arborescences are arc-disjoint,

\[
\rho_{\mathcal{A}}(X) \geq |\bigcup_{\nu \in X} B_v \setminus S_X| \\
\geq r_{\mathcal{M}}(\bigcup_{\nu \in X} B_v \setminus S_X) \\
\geq r_{\mathcal{M}}(\bigcup_{\nu \in X} B_v) - r_{\mathcal{M}}(S_X) \\
\geq r_{\mathcal{M}}(\bigcup_{\nu \in X} S_{\mathcal{P}_{\mathcal{A}}(\nu)}) - r_{\mathcal{M}}(S_X) \\
= r_{\mathcal{M}}(S_{\mathcal{P}_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(S_X).
\]
Thank you for your attention!
Motivation: Rigidity

Theorem (Tay 1984)

"Rigidity" of a Body-Bar Framework can be characterized by the existence of a spanning tree decomposition.

Theorem (Katoh, Tanigawa 2013)

"Rigidity" of a Body-Bar Framework with Bar-Boundary can be characterized by the existence of a matroid-based rooted-tree decomposition.