Partition Constrained Edge-Connectivity Augmentation of a Hypergraph

Zoltán Szigeti

Laboratoire G-SCOP
Grenoble INP, France

November 2009

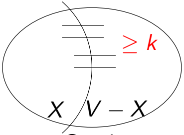
Outline

1. Edge-connectivity Augmentation Problems
2. Results
3. General Method
4. Ideas of the Proof

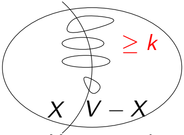
Edge-connectivity Augmentation

Definition

A (hyper)graph G is called k-edge-connected if each cut contains at least k (hyper)edges.



Graph

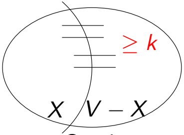


Hypergraph

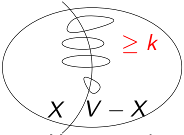
Edge-connectivity Augmentation

Definition

A (hyper)graph G is called k-edge-connected if each cut contains at least k (hyper)edges.



Graph



Hypergraph

Problems to be considered

1. Edge-connectivity augmentation of a graph
2. Edge-connectivity augmentation of a hypergraph
3. Partition constrained edge-connectivity augmentation of a graph
4. Partition constrained edge-connectivity augmentation of a hypergraph

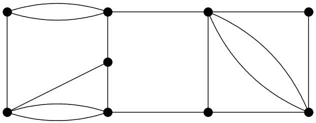
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $\gamma = \alpha_k(G) := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \text{ subpartition of } V(G)\}.$

2. Polynomially solvable (Cai, Sun (1989))



Graph $G, k = 4$

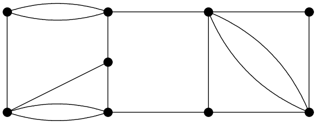
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $\gamma = \alpha_k(G) := \max\{\lceil \frac{1}{2}\sum_{X\in\mathcal{X}}(k - d(X))\rceil : \mathcal{X}$ subpartition of $V(G)\}$.

2. Polynomially solvable (Cai, Sun (1989))



Graph $G, k = 4$

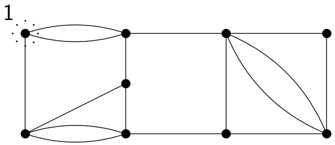
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $\gamma = \alpha_k(G) := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X}$ subpartition of $V(G)\}$.

2. Polynomially solvable (Cai, Sun (1989))



Graph $G, k = 4$

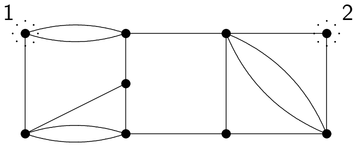
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

$$\gamma = \alpha_k(G) := \max\{\lceil \tfrac{1}{2} \textstyle\sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \text{ subpartition of } V(G)\}.$$

2. Polynomially solvable (Cai, Sun (1989))



Graph $G, k = 4$

Graphs

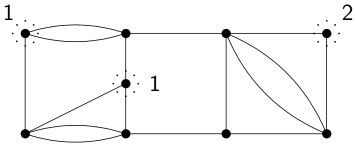
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $\gamma = \alpha_k(G) := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \text{ subpartition of } V(G)\}.$

2. Polynomially solvable (Cai, Sun (1989))



Graph $G, k = 4$

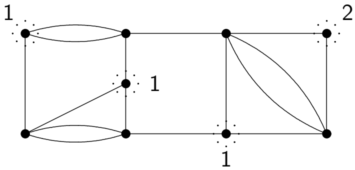
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $\gamma = \alpha_k(G) := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \text{ subpartition of } V(G)\}.$

2. Polynomially solvable (Cai, Sun (1989))



Graph $G, k = 4$

Graphs

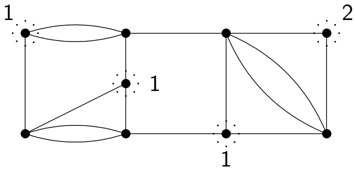
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $\gamma = \alpha_k(G) := \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X}$ subpartition of $V(G)\}$.

2. Polynomially solvable (Cai, Sun (1989))



$$\text{Opt} \geq \lceil \tfrac{5}{2} \rceil = 3$$

Graphs

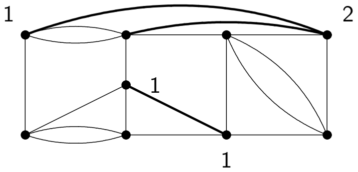
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $$\gamma = \alpha_k(G) := \max\{\lceil \tfrac{1}{2} \textstyle\sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \text{ subpartition of } V(G)\}.$$

2. Polynomially solvable (Cai, Sun (1989))



Graph $G + F$ is 4-edge-connected and $|F| = 3$

Graphs

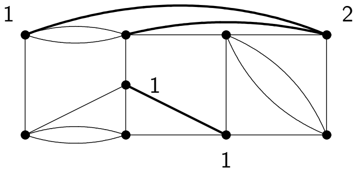
Edge-connectivity augmentation of a graph

Given a graph G and an integer $k \geq 2$, what is the minimum number γ of new edges whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Watanabe, Nakamura (1987))

   $$\gamma = \alpha_k(G) := \max\{\lceil \tfrac{1}{2} \sum_{X \in \mathcal{X}} (k - d(X)) \rceil : \mathcal{X} \text{ subpartition of } V(G)\}.$$

2. Polynomially solvable (Cai, Sun (1989))



Graph $G + F$ is 4-edge-connected and $|F| = 3$

Hypergraphs

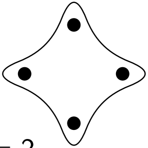
Edge-connectivity augmentation of a hypergraph

Given a hypergraph $\mathcal{G}$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bang-Jensen, Jackson (1999))

   $\gamma = \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G})\}$,

   where $c_k(\mathcal{G}) := \max\{c(\mathcal{G} - \mathcal{H}) - 1 : \mathcal{H} \subseteq E(\mathcal{G}), |\mathcal{H}| \leq k - 1\}$.
2. Polynomially solvable (Bang-Jensen, Jackson (1999))



$k = 2$

Edge-connectivity augmentation of a hypergraph

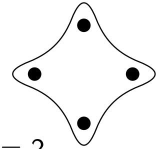
Given a hypergraph $\mathcal{G}$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bang-Jensen, Jackson (1999))

   $$\gamma = \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G})\},$$

   where $c_k(\mathcal{G}) := \max\{c(\mathcal{G} - \mathcal{H}) - 1 : \mathcal{H} \subseteq E(\mathcal{G}), |\mathcal{H}| \leq k - 1\}$.

2. Polynomially solvable (Bang-Jensen, Jackson (1999))



$k = 2$

Edge-connectivity augmentation of a hypergraph

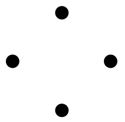
Given a hypergraph $\mathcal{G}$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bang-Jensen, Jackson (1999))

   $$\gamma = \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G})\},$$

   where $c_k(\mathcal{G}) := \max\{c(\mathcal{G} - \mathcal{H}) - 1 : \mathcal{H} \subseteq E(\mathcal{G}), |\mathcal{H}| \leq k-1\}$.

2. Polynomially solvable (Bang-Jensen, Jackson (1999))



$k = 1$

Hypergraphs

Edge-connectivity augmentation of a hypergraph

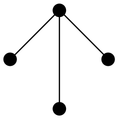
Given a hypergraph $\mathcal{G}$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bang-Jensen, Jackson (1999))

   $$\gamma = \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G})\},$$

   where $c_k(\mathcal{G}) := \max\{c(\mathcal{G} - \mathcal{H}) - 1 : \mathcal{H} \subseteq E(\mathcal{G}), |\mathcal{H}| \leq k-1\}$.

2. Polynomially solvable (Bang-Jensen, Jackson (1999))



$k = 1$

Hypergraphs

Edge-connectivity augmentation of a hypergraph

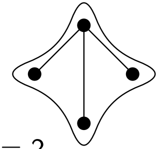
Given a hypergraph $\mathcal{G}$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bang-Jensen, Jackson (1999))

$$\gamma = \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G})\},$$

where $c_k(\mathcal{G}) := \max\{c(\mathcal{G} - \mathcal{H}) - 1 : \mathcal{H} \subseteq E(\mathcal{G}), |\mathcal{H}| \leq k - 1\}.$

2. Polynomially solvable (Bang-Jensen, Jackson (1999))



$k = 2$

Edge-connectivity augmentation of a hypergraph

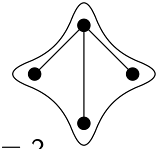
Given a hypergraph $\mathcal{G}$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bang-Jensen, Jackson (1999))

$$\gamma = \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G})\},$$

where $c_k(\mathcal{G}) := \max\{c(\mathcal{G} - \mathcal{H}) - 1 : \mathcal{H} \subseteq E(\mathcal{G}), |\mathcal{H}| \leq k-1\}$.

2. Polynomially solvable (Bang-Jensen, Jackson (1999))



$k = 2$

Graphs with partition constraints

Partition constrained edge-connectivity augmentation of a graph

Given a graph G, a partition $\mathcal{P}$ of $V(G)$ and an integer $k \geq 2$, what is the minimum number γ of new edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Bang-Jensen, Gabow, Jordán, Szigeti (1999))

$$\gamma = \begin{cases} \Phi & \text{if } G \text{ contains no } C_4\text{- and no } C_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

where $\Phi := \max\{\alpha_k(G), \beta_k(G,\mathcal{P})\}$ and
$\beta_k(G,\mathcal{P}) := \max\{\sum_{Y\in\mathcal{Y}}(k - d(Y)) : \mathcal{Y} \text{ subpartition of } P_j, P_j \in \mathcal{P}\}$.

2. Polynomially solvable (Bang-Jensen, Gabow, Jordán, Szigeti (1999))

Graph $G = (V, E)$
Partition $\mathcal{P}$ of V

G' k-E-C

Graphs with partition constraints

Partition constrained edge-connectivity augmentation of a graph

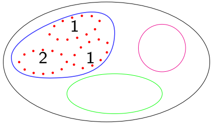
Given a graph G, a partition $\mathcal{P}$ of $V(G)$ and an integer $k \geq 2$, what is the minimum number γ of new edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Bang-Jensen, Gabow, Jordán, Szigeti (1999))

$$\gamma = \begin{cases} \Phi & \text{if } G \text{ contains no } C_4\text{- and no } C_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

where $\Phi := \max\{\alpha_k(G), \beta_k(G, \mathcal{P})\}$ and
$\beta_k(G, \mathcal{P}) := \max\{\sum_{Y \in \mathcal{Y}}(k - d(Y)) : \mathcal{Y}$ subpartition of $P_j, P_j \in \mathcal{P}\}$.

2. Polynomially solvable (Bang-Jensen, Gabow, Jordán, Szigeti (1999))



Graphs with partition constraints

Partition constrained edge-connectivity augmentation of a graph

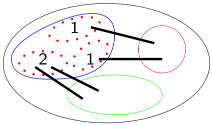
Given a graph G, a partition $\mathcal{P}$ of $V(G)$ and an integer $k \geq 2$, what is the minimum number γ of new edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Bang-Jensen, Gabow, Jordán, Szigeti (1999))

$$\gamma = \begin{cases} \Phi & \text{if } G \text{ contains no } C_4\text{- and no } C_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

where $\Phi := \max\{\alpha_k(G), \beta_k(G, \mathcal{P})\}$ and
$\beta_k(G, \mathcal{P}) := \max\{\sum_{Y \in \mathcal{Y}}(k - d(Y)) : \mathcal{Y}$ subpartition of $P_j, P_j \in \mathcal{P}\}$.

2. Polynomially solvable (Bang-Jensen, Gabow, Jordán, Szigeti (1999))



Partition constrained edge-connectivity augmentation of a graph

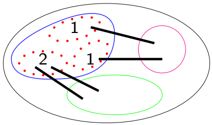
Given a graph G, a partition $\mathcal{P}$ of $V(G)$ and an integer $k \geq 2$, what is the minimum number γ of new edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected graph ?

1. Minimax theorem (Bang-Jensen, Gabow, Jordán, Szigeti (1999))

$$\gamma = \begin{cases} \Phi & \text{if } G \text{ contains no } C_4\text{- and no } C_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

   where $\Phi := \max\{\alpha_k(G), \beta_k(G, \mathcal{P})\}$ and
   $\beta_k(G, \mathcal{P}) := \max\{\sum_{Y \in \mathcal{Y}}(k - d(Y)) : \mathcal{Y}$ subpartition of $P_j, P_j \in \mathcal{P}\}$.

2. Polynomially solvable (Bang-Jensen, Gabow, Jordán, Szigeti (1999))



Hypergraphs with partition constraints

Partition constrained edge-conn. augmentation of a hypergraph

Given a hypergraph $\mathcal{G}$, a partition $\mathcal{P}$ of $V(\mathcal{G})$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bernáth, Grappe, Szigeti (2009))

$$\gamma = \begin{cases} \Phi & \text{if } \mathcal{G} \text{ contains no } \mathcal{C}_4\text{- and no } \mathcal{C}_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

where $\Phi := \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G}), \beta_k(\mathcal{G}, \mathcal{P})\}$.

2. Polynomially solvable (Bernáth, Grappe, Szigeti (2009))

Hypergraphs with partition constraints

Partition constrained edge-conn. augmentation of a hypergraph

Given a hypergraph $\mathcal{G}$, a partition $\mathcal{P}$ of $V(\mathcal{G})$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bernáth, Grappe, Szigeti (2009))

$$\gamma = \begin{cases} \Phi & \text{if } \mathcal{G} \text{ contains no } \mathcal{C}_4\text{- and no } \mathcal{C}_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

where $\Phi := \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G}), \beta_k(\mathcal{G}, \mathcal{P})\}$.

2. Polynomially solvable (Bernáth, Grappe, Szigeti (2009))

Hypergraphs with partition constraints

Partition constrained edge-conn. augmentation of a hypergraph

Given a hypergraph $\mathcal{G}$, a partition $\mathcal{P}$ of $V(\mathcal{G})$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bernáth, Grappe, Szigeti (2009))

$$\gamma = \begin{cases} \Phi & \text{if } \mathcal{G} \text{ contains no } \mathcal{C}_4\text{- and no } \mathcal{C}_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

   where $\Phi := \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G}), \beta_k(\mathcal{G}, \mathcal{P})\}$.

2. Polynomially solvable (Bernáth, Grappe, Szigeti (2009))

Hypergraphs with partition constraints

Partition constrained edge-conn. augmentation of a hypergraph

Given a hypergraph $\mathcal{G}$, a partition $\mathcal{P}$ of $V(\mathcal{G})$ and an integer $k \geq 1$, what is the minimum number γ of new graph edges, between different members of $\mathcal{P}$, whose addition results in a k-edge-connected hypergraph ?

1. Minimax theorem (Bernáth, Grappe, Szigeti (2009))

$$\gamma = \begin{cases} \Phi & \text{if } \mathcal{G} \text{ contains no } \mathcal{C}_4\text{- and no } \mathcal{C}_6\text{-configuration,} \\ \Phi + 1 & \text{otherwise,} \end{cases}$$

where $\Phi := \max\{\alpha_k(\mathcal{G}), c_k(\mathcal{G}), \beta_k(\mathcal{G}, \mathcal{P})\}$.

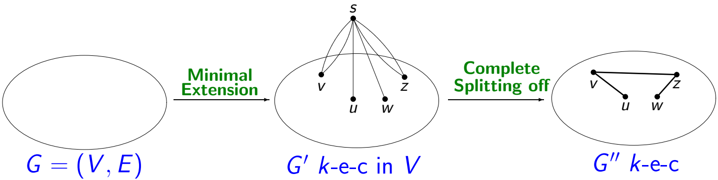
2. Polynomially solvable (Bernáth, Grappe, Szigeti (2009))

Ben Cosh (2000) solved the special case of bipartition.

General method

Frank's algorithm

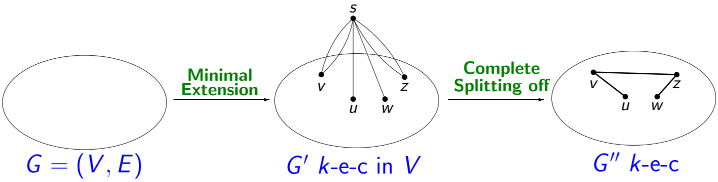
1. Minimal extension,
   (i) Add a new vertex s,
   (ii) Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   (iii) If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off.



General method

Frank's algorithm

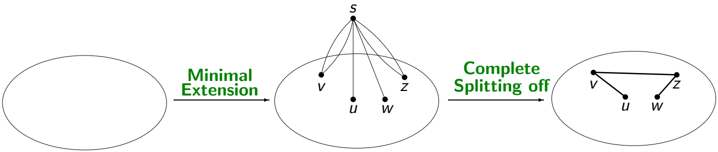
1. Minimal extension,
   - (i) Add a new vertex s,
   - (ii) Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   - (iii) If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off.



General method

Frank's algorithm

1. Minimal extension,
   (i) Add a new vertex s,
   (ii) Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   (iii) If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off.

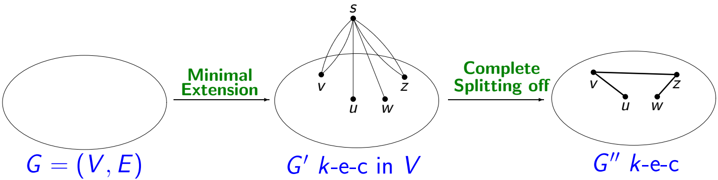


$G = (V, E)$ $\quad$ G' k-e-c in V $\quad$ G'' k-e-c

General method

Frank's algorithm

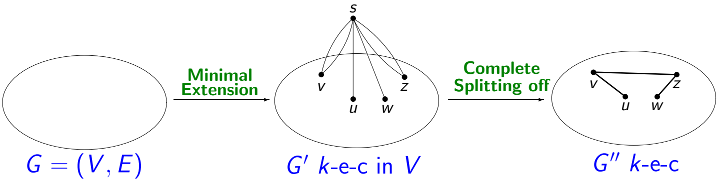
1. Minimal extension,
   - (i) Add a new vertex s,
   - (ii) Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   - (iii) If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off.



General method

Frank's algorithm

1. Minimal extension,
   (i) Add a new vertex s,
   (ii) Add a minimum number of new edges incident to s to satisfy the edge-connectivity requirements,
   (iii) If the degree of s is odd, then add an arbitrary edge incident to s.
2. Complete splitting off.

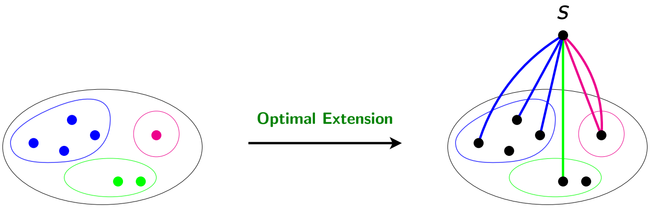


Optimal extension

Definion : Given $\mathcal{G} = (V, \mathcal{E})$, partition $\mathcal{P}$ of V, integer k,

optimal extension : $\hat{\mathcal{G}} = (V + s, \mathcal{E} + \delta(s))$, $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$

1. $\hat{\mathcal{G}}$ is k-edge-connected in V,
2. $\delta_{\hat{\mathcal{G}}}(s)$ consists of 2Φ graph edges,
3. $|\hat{P}| \leq \frac{1}{2} d_{\hat{\mathcal{G}}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$.

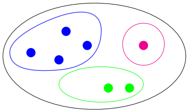


$\mathcal{G}, \mathcal{P}$ $\qquad$ $\hat{\mathcal{G}}, \hat{\mathcal{P}}$

How to find an optimal extension

Algorithm :

1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \not\subseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V-P$.
4. Let $\hat{\mathcal{G}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.

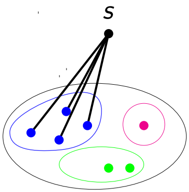


$\mathcal{G}, \mathcal{P}$

How to find an optimal extension

Algorithm :

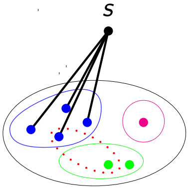
1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \not\subseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V-P$.
4. Let $\hat{\mathcal{G}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.



How to find an optimal extension

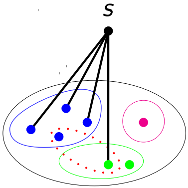
Algorithm :

1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \not\subseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V - P$.
4. Let $\hat{G}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.



Algorithm :

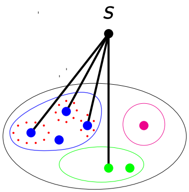
1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \nsubseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V - P$.
4. Let $\hat{\mathcal{G}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.



How to find an optimal extension

Algorithm :

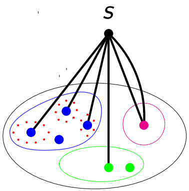
1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \not\subseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V - P$.
4. Let $\hat{\mathcal{G}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.



How to find an optimal extension

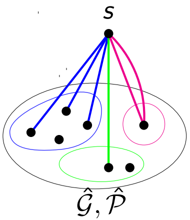
Algorithm :

1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \not\subseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V - P$.
4. Let $\hat{\mathcal{G}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.



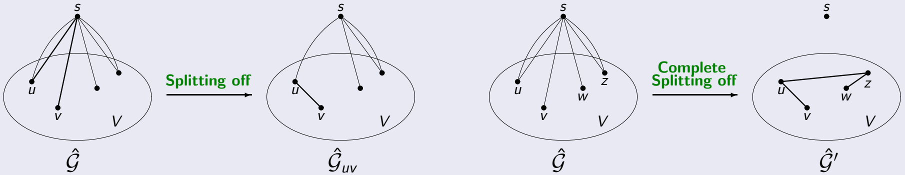
Algorithm :

1. Apply Frank's minimal extension to find $(V+s, \mathcal{E}+\delta(s))$ k-edge-connected in V and $d(s)$ is even. ($d(s) = 2\alpha_k(\mathcal{G})$.)
2. Add edges incident to s so that $d(s) = \max\{2\alpha_k(\mathcal{G}), 2c_k(\mathcal{G})\}$.
3. If some $P \in \mathcal{P}$ satisfies $d(s,P) > \frac{d(s)}{2}$, then
   If $\exists su \in \delta(s), u \in P$, $X_u \not\subseteq P$, replace su by su', $u' \in X_u - P$. Repeat 3.
   Otherwise, add $2d(s,P) - d(s)$ edges between s and $V-P$.
4. Let $\hat{\mathcal{G}}$ be the resulting hypergraph and $\hat{\mathcal{P}} = \{\delta_{\hat{\mathcal{G}}}(s) \cap \delta_{\hat{\mathcal{G}}}(P) : P \in \mathcal{P}\}$.



Splitting off

Definitions



Definitions

A splitting off is

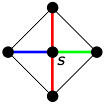
1. *k*-admissible if $\hat{\mathcal{G}}_{uv}$ is k edge-connected in V,
2. rainbow if the edges are of different colors and $|\hat{P}| \leq \frac{1}{2} d_{\hat{\mathcal{G}}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$ remains valid,
3. *k*-allowed if it is k-admissible and rainbow.

Splitting off theorem

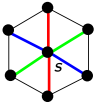
Theorem (Bernáth, Grappe, Szigeti (2009))

Let $\hat{\mathcal{G}} = (V+s, \mathcal{E}+\delta(s))$ be a hypergraph, where $\delta(s)$ consists of graph edges, and $\hat{\mathcal{P}}$ a partition of $\delta(s)$. There is a complete k-admissible rainbow (k-allowed) splitting off in $\hat{\mathcal{G}}$ if and only if

1. $\hat{\mathcal{G}}$ is k-edge-connected in V,
2. $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$ is even,
3. $|\hat{P}| \leq \frac{1}{2} d_{\hat{\mathcal{G}}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$ and
4. $\hat{\mathcal{G}}$ contains no obstacle.



C_4, $k = 3$



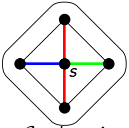
C_6, $k = 3$

Splitting off theorem

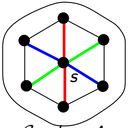
Theorem (Bernáth, Grappe, Szigeti (2009))

Let $\hat{\mathcal{G}} = (V+s, \mathcal{E}+\delta(s))$ be a hypergraph, where $\delta(s)$ consists of graph edges, and $\hat{\mathcal{P}}$ a partition of $\delta(s)$. There is a complete k-admissible rainbow (k-allowed) splitting off in $\hat{\mathcal{G}}$ if and only if

1. $\hat{\mathcal{G}}$ is k-edge-connected in V,
2. $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$ is even,
3. $|\hat{P}| \leq \frac{1}{2} d_{\hat{\mathcal{G}}}(s)$ for all $\hat{P} \in \hat{\mathcal{P}}$ and
4. $\hat{\mathcal{G}}$ contains no obstacle.



$\mathcal{C}_4$, $k = 4$



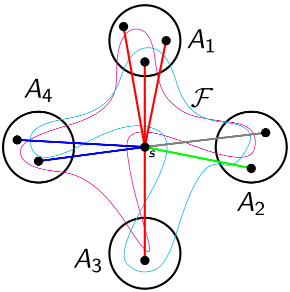
$\mathcal{C}_6$, $k = 4$

$\mathcal{C}_4$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \dots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \dots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2} d(s)$.

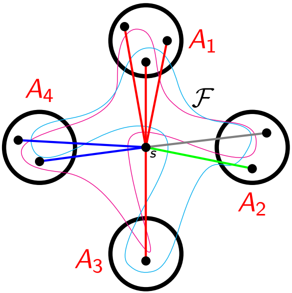


$\mathcal{C}_4$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \ldots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2}d(s)$.

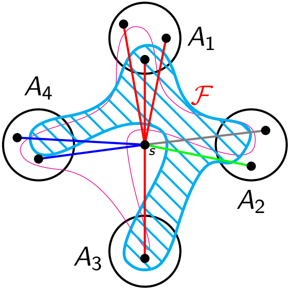


$\mathcal{C}_4$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \dots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \dots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2} d(s)$.

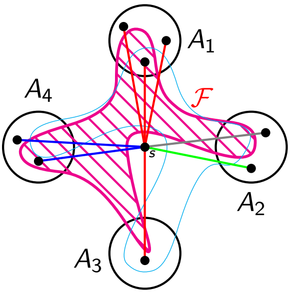


$\mathcal{C}_4$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \dots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \dots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2} d(s)$.

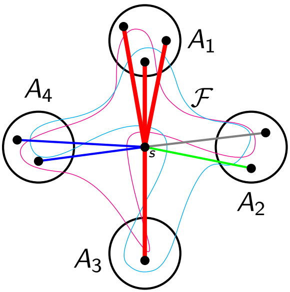


$\mathcal{C}_4$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \ldots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2} d(s)$.

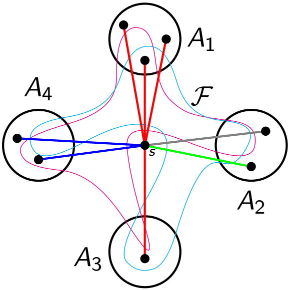


$\mathcal{C}_4$-obstacles

$\mathcal{C}_4$-obstacle rests $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \ldots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2} d(s)$.

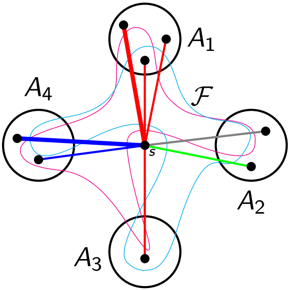


$\mathcal{C}_4$-obstacles

$\mathcal{C}_4$-obstacle rests $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \ldots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2} d(s)$.

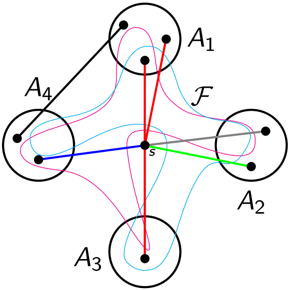


$\mathcal{C}_4$-obstacles

$\mathcal{C}_4$-obstacle rests $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$ for $i = 1, \ldots, 4$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_i) \cap \delta(A_{i+2})$ for $i = 1, 2$,
3. $\exists l \in \{1, 2\}$, $\hat{P} \in \hat{\mathcal{P}}$ s.t. $\delta(A_l \cup A_{l+2}) \cap \delta(s) = \hat{P}$ and $|\hat{P}| = \frac{1}{2}d(s)$.

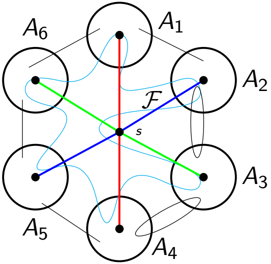


$\mathcal{C}_6$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

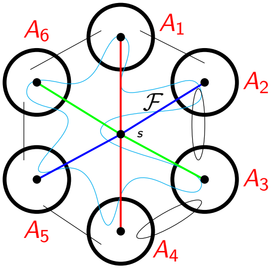


$\mathcal{C}_6$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k+1$ for $i = 1, \ldots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

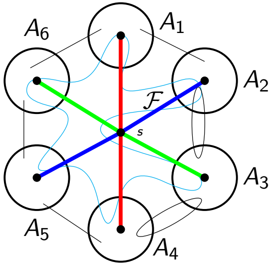


$\mathcal{C}_6$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \dots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \dots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

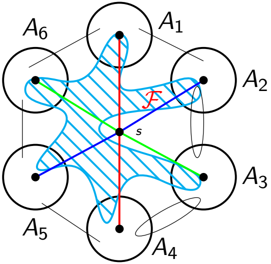


$\mathcal{C}_6$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \dots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \dots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

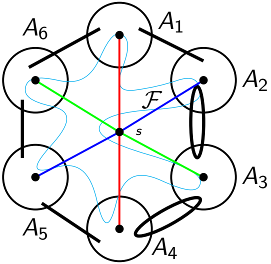


$\mathcal{C}_6$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

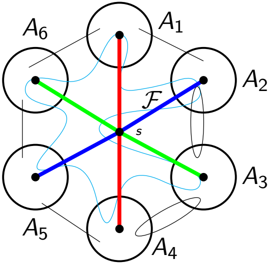


$\mathcal{C}_6$-obstacles

Definition

A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

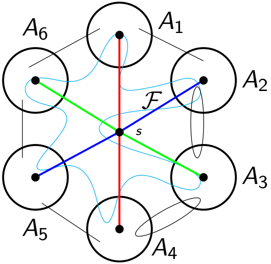


$\mathcal{C}_6$-obstacles

$\mathcal{C}_6$-obstacle becomes $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

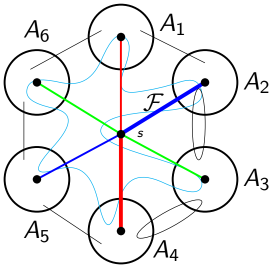


$\mathcal{C}_6$-obstacles

$\mathcal{C}_6$-obstacle becomes $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \dots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \dots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

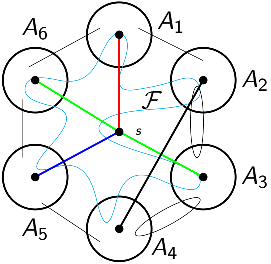


$\mathcal{C}_6$-obstacles

$\mathcal{C}_6$-obstacle becomes $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \dots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \dots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.

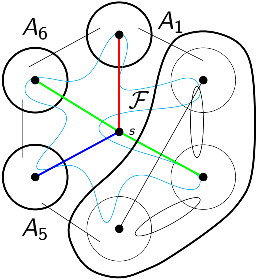


$\mathcal{C}_6$-obstacles

$\mathcal{C}_6$-obstacle becomes $\mathcal{C}_4$-obstacle after a k-allowed splitting off

A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\hat{\mathcal{G}}$ if

1. $d(A_i) = k$, $d(s, A_i) = 1$, $d(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,
2. $\exists \mathcal{F} \subseteq \mathcal{E}$ s.t. $k - |\mathcal{F}| \neq 1$ is odd and $\mathcal{F} = \delta(A_j) \cap \delta(A_l)$ for all distinct non consecutive A_j and A_l,
3. $\exists$ distinct $\hat{P}_j \in \hat{\mathcal{P}}$ s.t. $\delta(A_j \cup A_{j+3}) \cap \delta(s) = \hat{P}_j$ for $j = 1, 2, 3$.



Proof of the splitting theorem

1. Let G be the hypergraph obtained from $\hat{\mathcal{G}}$ by performing any longest sequence of allowed splittings.
2. G contains an admissible pair (otherwise, $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$).
3. G contains a $\mathcal{C}_4$-obstacle and $d_G(s) = 4$ (otherwise, $\exists st$ that belongs to $\geq \frac{d(s)}{2}$ distinct admissible pairs, so to an allowed pair).
4. For every split edge e, G^e contains an obstacle.
5. There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Proof of the splitting theorem

1. Let G be the hypergraph obtained from $\hat{\mathcal{G}}$ by performing any longest sequence of allowed splittings.
2. G contains an admissible pair (otherwise, $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$).
3. G contains a $\mathcal{C}_4$-obstacle and $d_G(s) = 4$ (otherwise, $\exists st$ that belongs to $\geq \frac{d(s)}{2}$ distinct admissible pairs, so to an allowed pair).
4. For every split edge e, G^e contains an obstacle.
5. There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Proof of the splitting theorem

1. Let G be the hypergraph obtained from $\hat{\mathcal{G}}$ by performing any longest sequence of allowed splittings.
2. G contains an admissible pair (otherwise, $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$).
3. G contains a $\mathcal{C}_4$-obstacle and $d_G(s) = 4$ (otherwise, $\exists st$ that belongs to $\geq \frac{d(s)}{2}$ distinct admissible pairs, so to an allowed pair).
4. For every split edge e, G^e contains an obstacle.
5. There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Proof of the splitting theorem

1. Let G be the hypergraph obtained from $\hat{\mathcal{G}}$ by performing any longest sequence of allowed splittings.
2. G contains an admissible pair (otherwise, $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$).
3. G contains a $\mathcal{C}_4$-obstacle and $d_G(s) = 4$ (otherwise, $\exists st$ that belongs to $\geq \frac{d(s)}{2}$ distinct admissible pairs, so to an allowed pair).
4. For every split edge e, G^e contains an obstacle.
5. There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

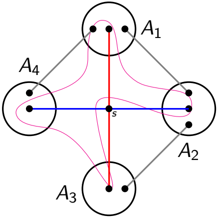
Proof of the splitting theorem

1. Let G be the hypergraph obtained from $\hat{\mathcal{G}}$ by performing any longest sequence of allowed splittings.
2. G contains an admissible pair (otherwise, $d_{\hat{\mathcal{G}}}(s) \geq 2c_k(\hat{\mathcal{G}} - s)$).
3. G contains a $\mathcal{C}_4$-obstacle and $d_G(s) = 4$ (otherwise, $\exists st$ that belongs to $\geq \frac{d(s)}{2}$ distinct admissible pairs, so to an allowed pair).
4. For every split edge e, G^e contains an obstacle.
5. There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

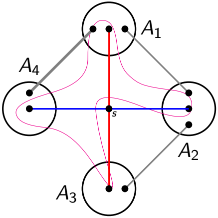
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

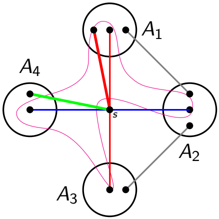
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

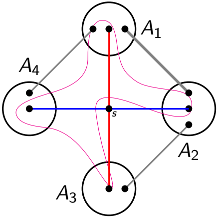
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

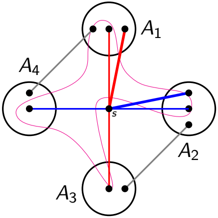
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

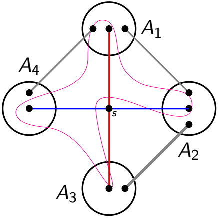
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

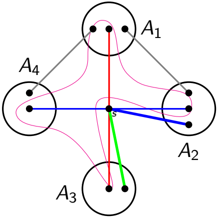
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

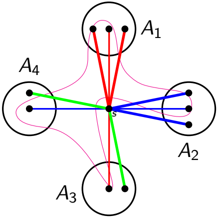
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

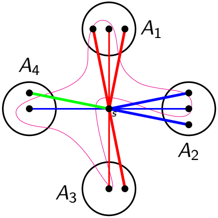
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

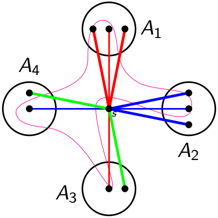
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

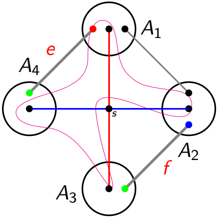
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

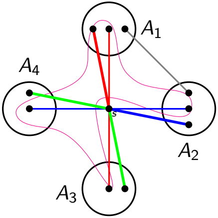
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

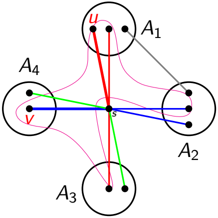
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

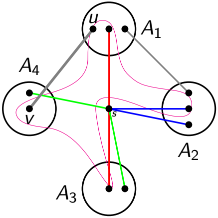
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

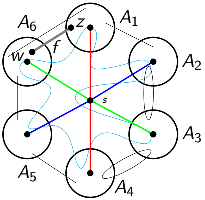
Case 1 : For every split edge e, G^e contains a $\mathcal{C}_4$-obstacle.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

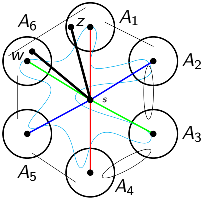
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

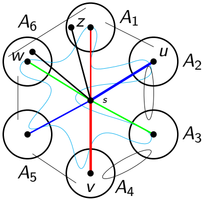
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

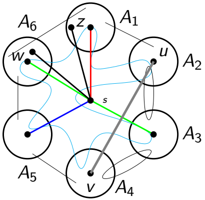
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

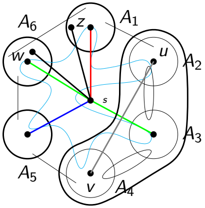
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



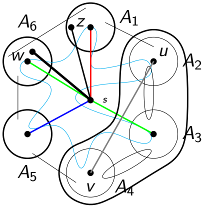
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

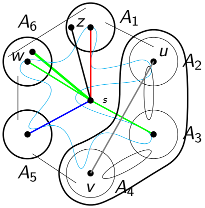
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

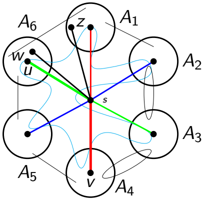
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

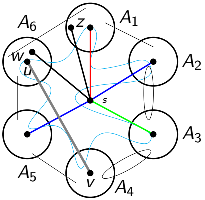
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

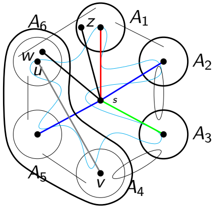
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

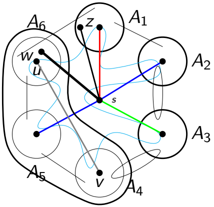
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

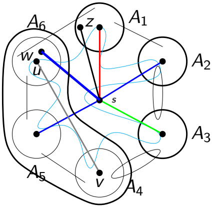
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

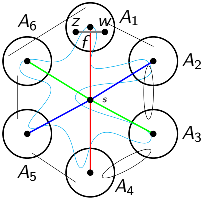
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case a : w and z belongs to different A_i's.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

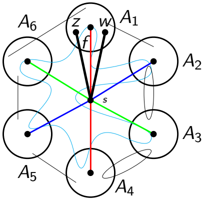
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

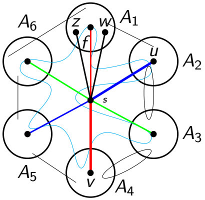
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

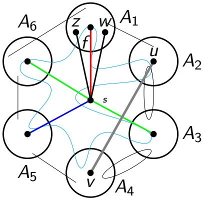
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

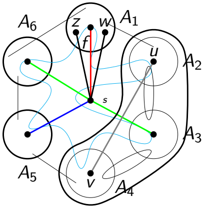
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

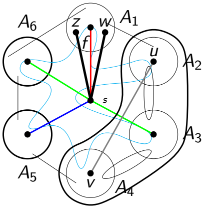
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

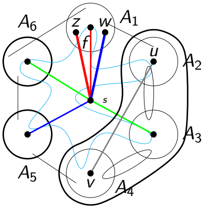
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

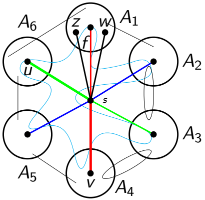
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

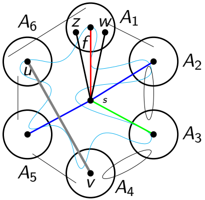
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



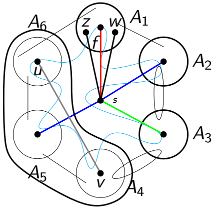
5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

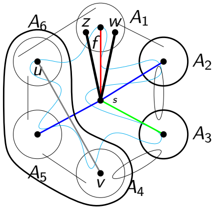
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

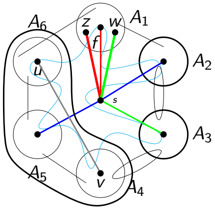
Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the splitting theorem

5 There exist two split edges e and f in G and an allowed pair su, sv in $G^{e,f}$ such that $G^{e,f}_{uv}$ contains no obstacle.

Case 2 : For a split edge e, G^e contains a $\mathcal{C}_6$-obstacle.
Then G^e contains a split edge $f = wz$.
Case b : w and z belongs to the same A_i.



Proof of the augmentation theorem

Lemma

1. *Every optimal extension of* $(\mathcal{G}, \mathcal{P})$ *contains an obstacle if and only if* $\mathcal{G}$ *contains a configuration.*
2. $\Phi \leq OPT$ *and if equality holds, then no configuration exists.*
3. $OPT \leq \Phi + 1$ *and if no configuration exists, then strict inequality holds.*

Thank you for your attention !