A CHARACTERIZATION OF SEYMOUR GRAPHS

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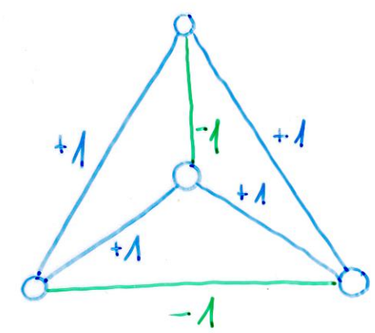
DEFINITION: A ± 1 VALUED WEIGHTING w OF THE EDGES OF G IS CONSERVATIVE IF THE WEIGHT OF ANY CIRCUIT IS NON-NEGATIVE.

NOTATION: LET $F \subseteq E(G)$.

$$w_F(e) = \begin{cases} -1 & \text{IF } e \in F \\ +1 & \text{IF } e \notin F \end{cases}$$

EXAMPLE:



THEOREM (MEI GU GUAN)

LET F BE A T-JOIN IN A GRAFT (G,T). THEN F IS A MINIMUM T-JOIN IF AND ONLY IF w_F IS CONSERVATIVE.

$$\nu(G,T,W) \leq \tau(G,T,W)$$

QUESTIONS:

1. GIVEN (G,T,W). DECIDE WHETHER EQUALITY HOLDS OR NOT.

2. CHARACTERIZE THE GRAPHS G FOR WHICH $\tau(G,T,W) = \nu(G,T,W)$ FOR EVERY T AND W.

3. CHARACTERIZE THE GRAFTS (G,T) FOR WHICH $\tau(G,T,W) = \nu(G,T,W)$ FOR EVERY W.

4. CHARACTERIZE THE GRAPHS G FOR WHICH $\tau(G,T) = \nu(G,T)$ FOR EVERY T.

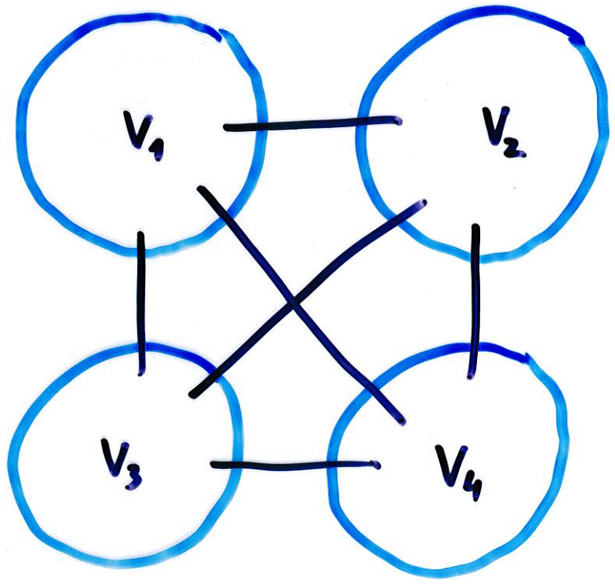
ANSWERS:

1. NP-COMPLETE EVEN FOR PLANAR GRAPHS (MIDDENDORF, PFEIFFER)

2. SERIES-PARALLEL GRAPHS (SEYMOUR)

3. GRAFTS (G,T) WHICH CAN NOT BE T-CONTRACTED TO $(K_4, V(K_4))$ (SEYMOUR)

   SIMPLE PROOF (FRANK, S2.)



$|V_i \cap T|$ ODD

$G(V_i)$ CONNECTED.

QUESTION:

WHICH GRAPHS HAVE THE PROPERTY THAT

$\tau(G,T) = \nu(G,T)$ FOR ALL T.

THEOREM (SEYMOUR)

IF G IS A BIPARTITE GRAPH THEN $\tau(G,T) = \nu(G,T)$ FOR EVERY T.

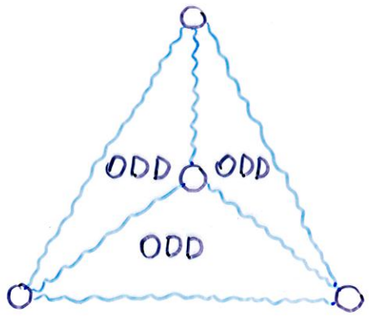
THEOREM (SEYMOUR)

IF G IS SERIES-PARALLEL THEN $\tau(G,T) = \nu(G,T)$ FOR EVERY T.

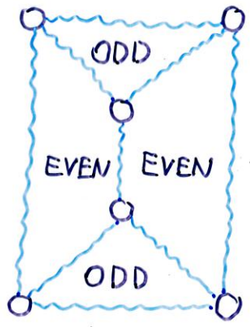
DEFINITION: A GRAPH G IS CALLED SEYMOUR-GRAPH IF $\tau(G,T) = \nu(G,T)$ FOR ALL T.

THEOREM (GERARDS)

IF A GRAPH G CONTAINS NEITHER AN ODD-K_4 NOR AN ODD PRISM AS A SUBGRAPH THEN G IS A SEYMOUR-GRAPH.



ODD-K_4



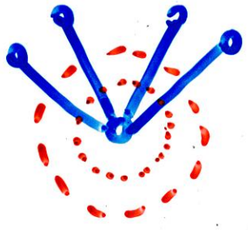
ODD-PRISM

THEOREM (AGEEV, KOSTOCHKA, SZ.)

A GRAPH G IS NOT A SEYMOUR GRAPH IF AND ONLY IF THERE EXIST A CONSERVATIVE WEIGHTING AND TWO CIRCUITS OF WEIGHT ZERO WHOSE UNION IS AN ODD K_4 OR AN ODD PRISM.

PROOF: (G,T) IS A MINIMUM COUNTER-EXAMPLE $(\tau(G,T) > \nu(G,T))$ AND F IS A MINIMUM T-JOIN.

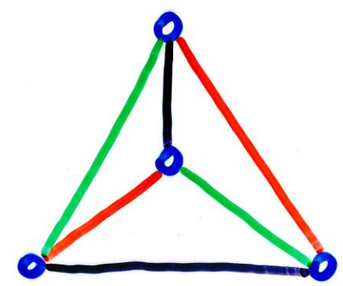
(1) IF $\{\delta(R) : R \in \mathcal{R}\}$ IS AN OPTIMAL 2-PACKING OF T-CUTS THEN IT DOES NOT CONTAIN A T-CUT $\delta(u)$ $(u \in V(G))$ TWICE.

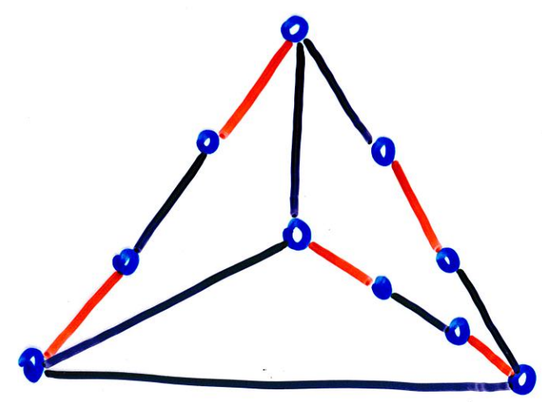


THEOREM (LOVÁSZ)

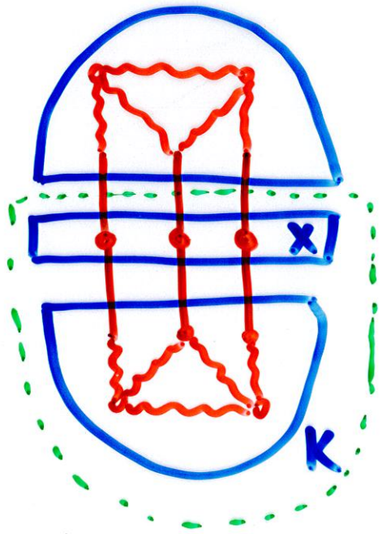
IF G IS A 1-EXTENDABLE, NON-BIPARTITE GRAPH, THEN IT CONTAINS AN EVEN SUBDIVISION OF EITHER THE K_4 OR THE TRIANGULAR PRISM.

DEFINITION: G IS 1-EXTENDABLE IF IT IS CONNECTED AND EACH EDGE OF G LIES IN SOME PERFECT MATCHING OF G.

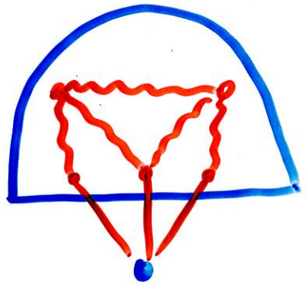


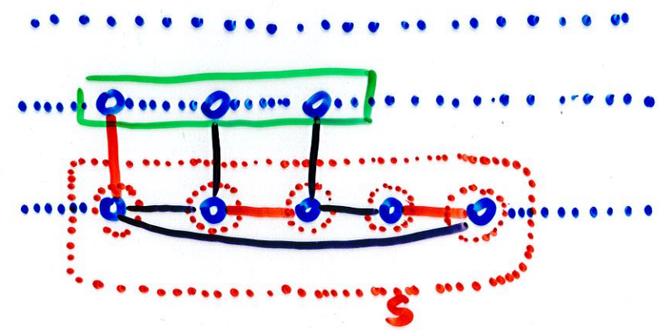


LEMMA: LET G BE A CONNECTED GRAPH AND LET $X \subset V(G)$ BE A CUT SET OF G. ASSUME THAT K IS A FACTOR CRITICAL COMPONENT OF $G-X$ SUCH THAT THE NEIGHBOURS OF K IS EXACTLY X. IF FOR $G/(X \cup V(K))$ THERE EXISTS A CONSERVATIVE WEIGHTING SO THAT THE UNION OF TWO CIRCUITS OF WEIGHT 0 IS AN ODD K_4 OR AN ODD PRISM THEN THE SAME IS TRUE FOR G.



G





(2) $T = S \cup \{x_0\}$ AND F IS A PERFECT MATCHING OF $G(T)$.

(3) $G(T)$ IS BICRITICAL.

(H IS BICRITICAL IF IT HAS AT LEAST ONE EDGE AND $H - u$ IS FACTOR-CRITICAL FOR ALL $u \in V(H)$)

(4) EVERY BICRITICAL GRAPH ON AT LEAST FOUR VERTICES IS 1-EXTENDABLE, AND NON-BIPARTITE. $\square$

DEFINITION: LET F BE A MINIMUM T-JOIN AND $x_0 \in V(G)$ IN (G,T).

$\lambda(x) := \min\{w_F(P) : P \text{ IS AN } (x,x_0) \text{ PATH}\}$

DISTANCE

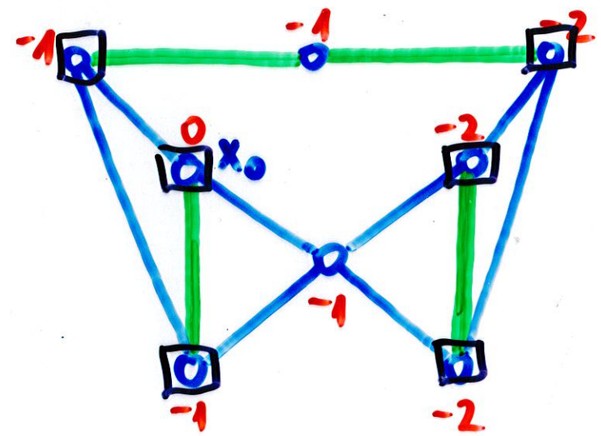
$\mathcal{D}_i$:= SET OF VERTEX SETS OF CONNECTED COMPONENTS OF $G(\{x \in V(G) : \lambda(x) \leq i\})$ TYPE 1.

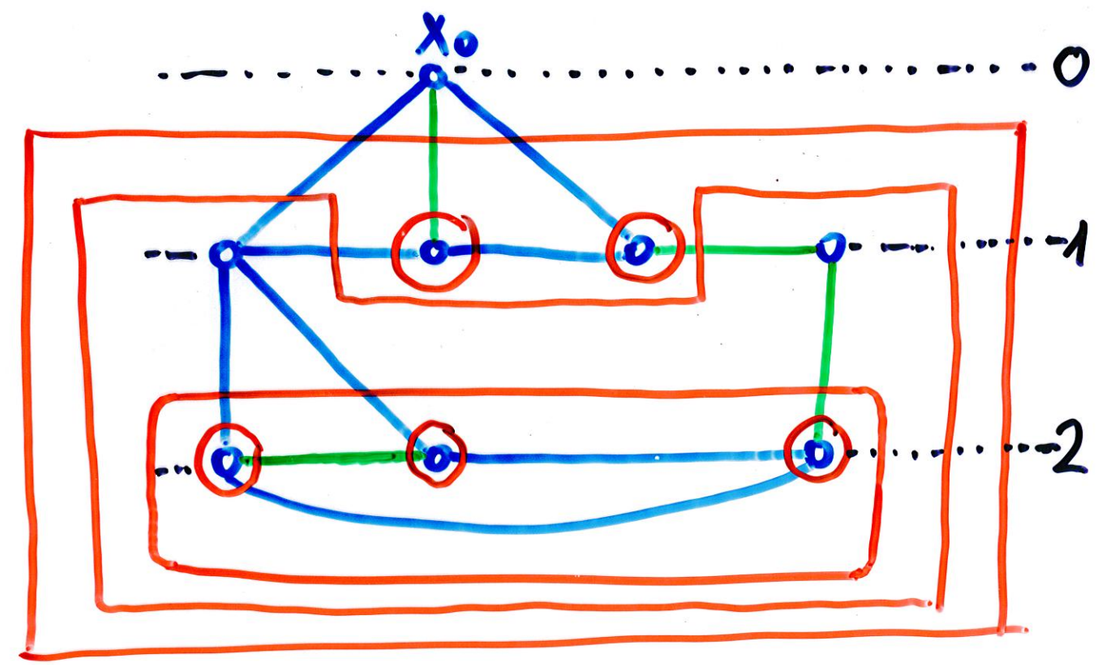
$\mathcal{Q}_i$:= SET OF VERTEX SETS OF CONNECTED COMPONENTS OF $G(\{x \in V(G) : \lambda(x) \leq i\}) - \{xy \in E(G) : \lambda(x) = \lambda(y) = i\}$. TYPE 2.

$$\mathcal{R} := \bigcup_{i=m}^{M} (\mathcal{D}_i \cup \mathcal{Q}_i)$$

WITH MULTIPLICITY.

EXAMPLE:





THEOREM (SEBŐ):

(1) IF $xy \in E(G)$: $|\lambda(x) - \lambda(y)| \leq 1$.

(2) IF $x_0 \notin D \in \mathcal{R}$: $|\delta(D) \cap F| = 1$,

IF $x_0 \in D \in \mathcal{R}$: $|\delta(D) \cap F| = 0$.

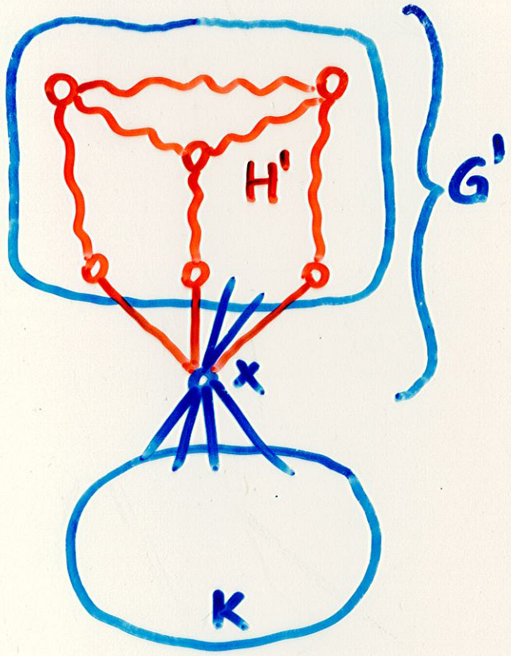
COROLLARY:

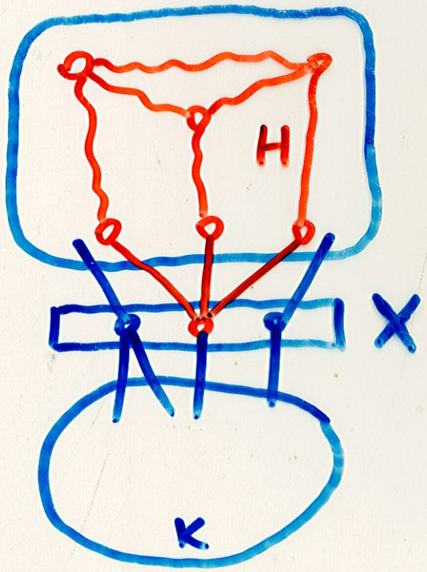
(1) $\tau(G,T) = \nu_2(G,T)/2$

$\{\delta(D) : x_0 \notin D \in \mathcal{R}\}$ IS A MAXIMUM 2-PACKING OF T-CUTS.

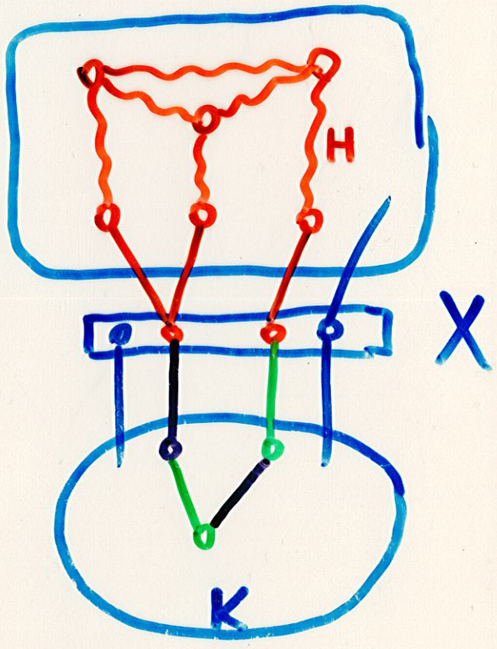
(2) IF $D \in \mathcal{D}_m$ THEN $D \subset T$ AND $G(D)$ IS FACTOR-CRITICAL.

(H IS FACTOR-CRITICAL IF $H - v$ HAS A PERFECT MATCHING FOR ALL $v \in H$)





I.



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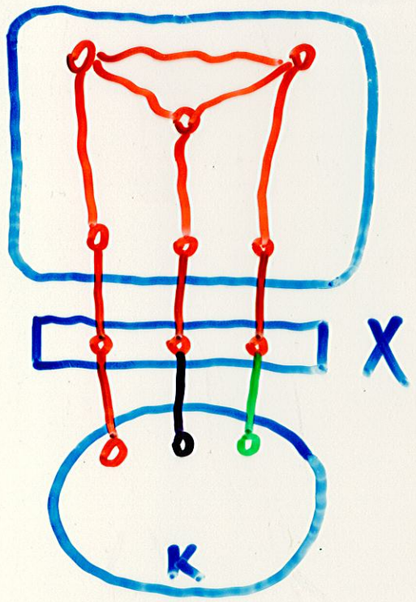
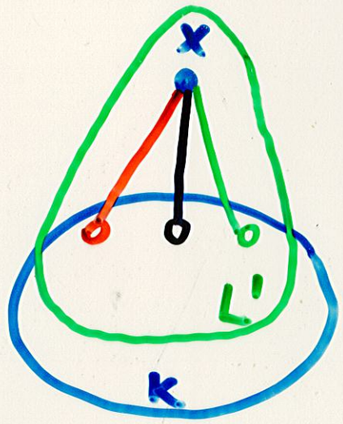
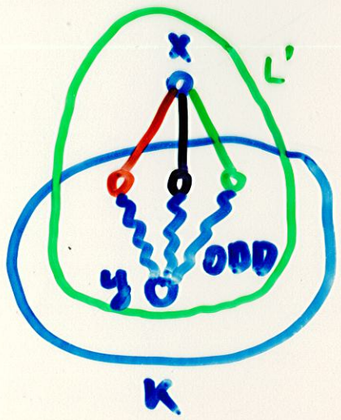


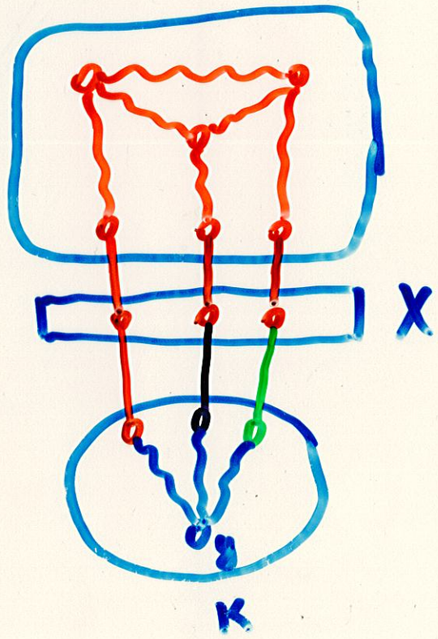


L' IS 1-EXTENDABLE

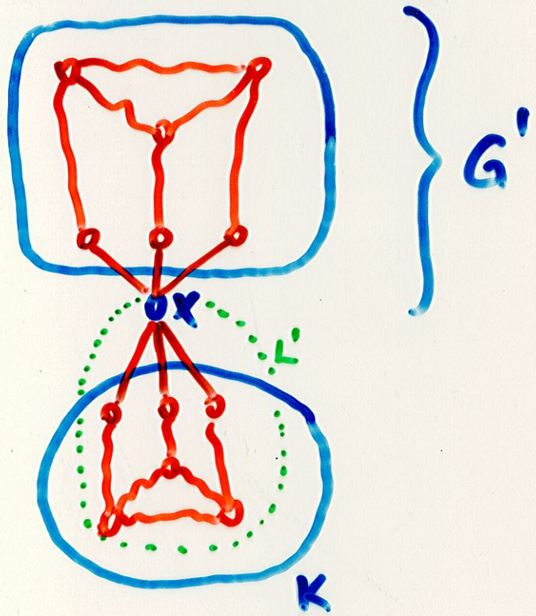
III.

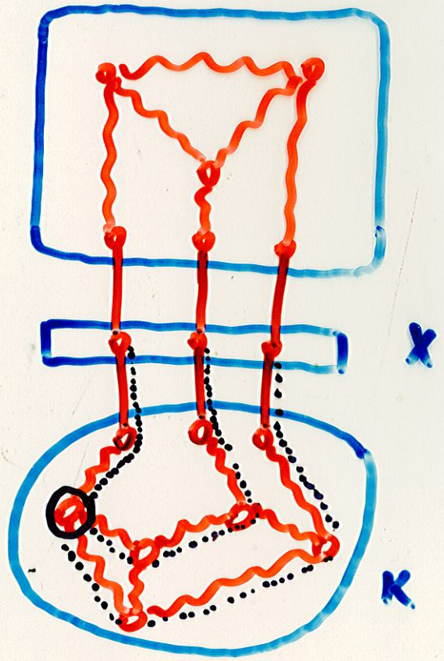
IF L' IS BIPARTITE



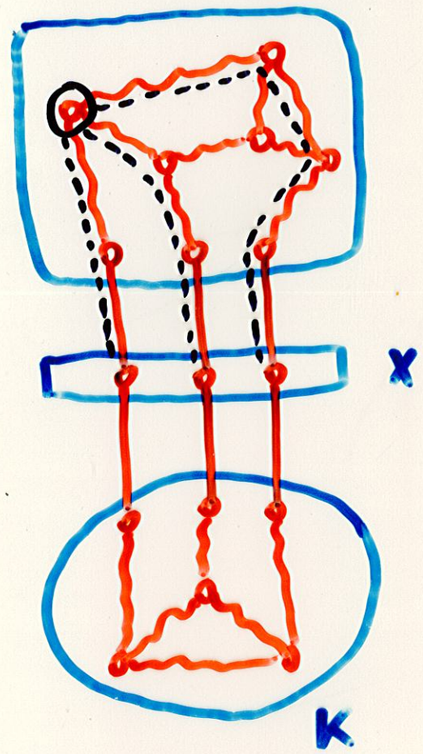


IF L' IS NON-BIPARTITE





1.



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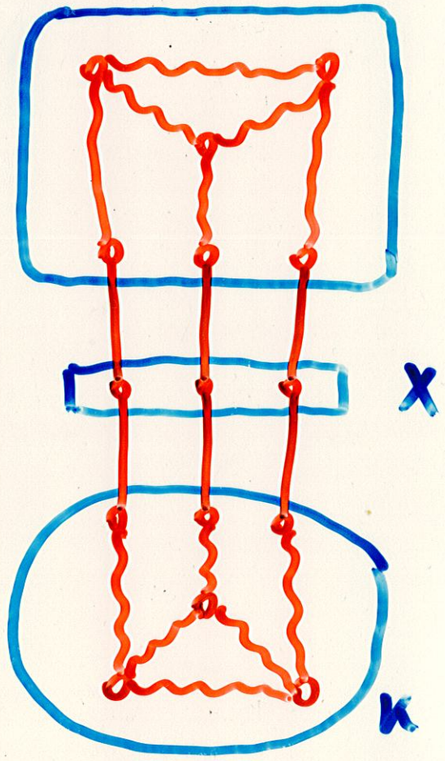

3.