Recent results on packing arborescences

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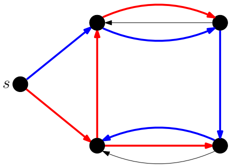
Packing spanning s-arborescences

Definitions

1. s-arborescence: tree + in-degree of every vertex except s is 1.
2. spanning subgraph of D: subgraph containing all the vertices of D.
3. packing subgraphs: pairwise arc-disjoint subgraphs.



Tool 1: Matroid intersection

$\vec{B} \subseteq \vec{E}$ is the arc set of a packing of k spanning s-arborescences in $\vec{G} \Longleftrightarrow$

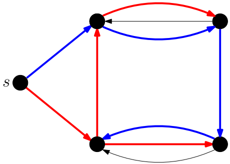
(a) B is the edge set of a packing of k spanning trees in G,

(b) $d^-_{\vec{B}}(v) = k$ for all $v \in V - s$.

Packing spanning arborescences : fixed root

Theorem 1 (Edmonds 1973)

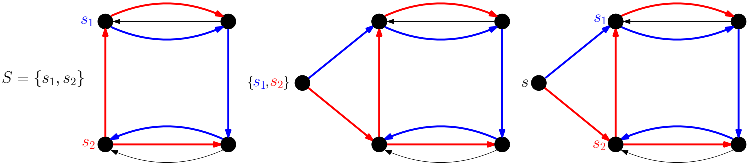
Let $D = (V, A)$ be a digraph, $s \in V$ and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning s-arborescences $\iff$
$d_A^-(X) \geq k$ *for all* $\emptyset \neq X \subseteq V \setminus s$.



Packing spanning arborescences : fixed roots

Theorem 2 (Edmonds 1973)

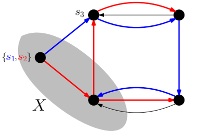
Let $D = (V, A)$ be a digraph and S a multiset of vertices in V.
There exists a packing of spanning s-arborescences ($s \in S$) $\Longleftrightarrow$
$|S_X| + d_A^-(X) \geq |S|$ *for all* $\emptyset \neq X \subseteq V$.



Theorems 1 and 2 are equivalent.

Definitions

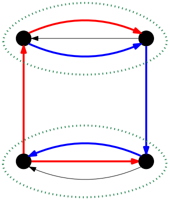
1. multiset S of V: vertex set with multiplicities.
2. S_X for $X \subseteq V$: restriction of S in X.



Packing spanning arborescences : flexible roots

Theorem 3 (Frank 1978)

Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning arborescences $\Longleftrightarrow$
$k \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X))$ *for every subpartition $\mathcal{P}$ of V.*



Theorem 3 implies Theorem 1.

Tool 2: generalized polymatroids (Frank)

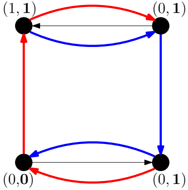
1. The root vectors of packings of k spanning arborescences are exactly the integer points of a generalized polymatroid.
2. Non-empty intersection of a generalized polymatroid and a box is a generalized polymatroid.

root vector $x : V \to \mathbb{Z}_+$ of a packing:
$x(s) =$ number of s-arborescences in the packing for all $s \in V$.

(f,g)-bounded packing of spanning arborescences

Definition

(f,g)-bounded packing of arborescences: number of v-arborescences in the packing is at least $f(v)$ and at most $g(v)$ $\forall v \in V$.



Theorem 4 (Frank 1978, Cai 1983)

Let $D=(V,A)$ be a digraph, $f,g: V \to \mathbb{Z}_+$ functions and $k \in \mathbb{Z}_+$. There exists an (f,g)-bounded packing of k spanning arborescences $\Longleftrightarrow$

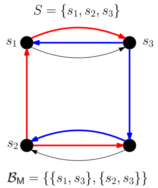
1. $g(v) \geq f(v)$ *for every* $v \in V$,
2. $k \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X)) + f(\overline{\cup \mathcal{P}})$ *for every subpartition* $\mathcal{P}$ *of* V,
3. $g(X) + d_A^-(X) \geq k$ *for every* $\emptyset \neq X \subseteq V$.

Theorem 4 implies Theorem 3.

Matroid-rooted packing of spanning arborescences

Definition

M-rooted packing of arborescences:
roots of the arborescences form a basis of a given matroid M.

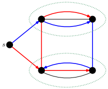


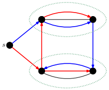
Theorem 5

Let $D = (V, A)$ be a digraph, S a multiset in V, $\mathrm{M} = (S, r_{\mathrm{M}})$ a matroid. There exists an M-rooted packing of spanning arborescences $\Longleftrightarrow$ $r_{\mathrm{M}}(S_{\cup\mathcal{P}}) \geq \sum_{X \in \mathcal{P}}(r_{\mathrm{M}}(S) - d_A^-(X))$ *for every subpartition $\mathcal{P}$ of V.*

Theorem 5 implies Theorem 3.

Packing spanning mixed arborescences : fixed root





Definition

1. mixed s-arborescence:
   it can be oriented to obtain an s-arborescence.
2. $e_E(\mathcal{P})$: number of edges entering at least one member of a subpartition $\mathcal{P}$ of V.

Theorem 6 (Frank 1978)

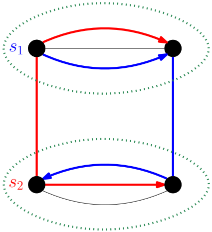
Let $F = (V, E \cup A)$ be a mixed graph, $s \in V$ and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning mixed s-arborescences $\iff$
$e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X))$ *for every subpartition $\mathcal{P}$ of $V - s$.*

Theorem 6 implies Theorem 1.

Packing spanning mixed arborescences : fixed roots

Theorem 7

Let $F = (V, E \cup A)$ be a mixed graph and S a multiset of vertices in V. There exists a packing of spanning mixed s-arborescences $(s \in S)$ $\Longleftrightarrow$ $e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}}(|S_{\overline{X}}| - d_A^-(X))$ for every subpartition $\mathcal{P}$ of V.

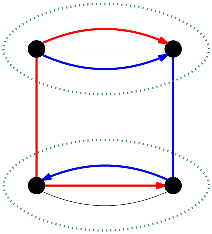


Theorems 6 and 7 are equivalent.

Packing spanning mixed arborescences : flexible roots

Theorem 8 (Gao, Yang 2021)

Let $F = (V, E \cup A)$ be a mixed graph and $k \in \mathbb{Z}_+$.
There exists a packing of k spanning mixed arborescences $\iff$
$e_E(\mathcal{P}) + k \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X))$ *for every subpartition $\mathcal{P}$ of V.*



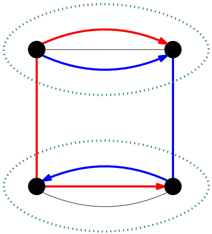
Theorem 8 implies Theorem 6.

(f,g)-bounded packing spanning mixed arborescences

Theorem 9 (Gao, Yang 2021)

Let $F = (V, E \cup A)$ be a mixed graph, $f, g : V \to \mathbb{Z}$ functions and $k \in \mathbb{Z}_+$. An (f,g)-bounded packing of k spanning mixed arborescences exists $\iff$

1. $g(v) \geq f(v)$ *for every* $v \in V$,
2. $e_E(\mathcal{P}) + \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \geq \sum_{X \in \mathcal{P}}(k - d_A^-(X))$ *for every subpartition* $\mathcal{P}$ *of* V.

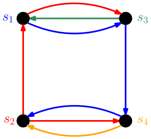


Theorem 9 implies Theorems 4 and 8.

Regular packing of arborescences

Definition

k-regular packing of arborescences:
each vertex belongs to k arborescences in the packing.



Theorem 10

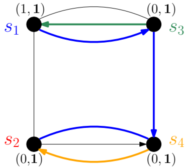
Let $D = (V, A)$ be a digraph, S a multiset of vertices in V and $k \in \mathbb{Z}_+$. There exists a k-regular packing of s-arborescences $(s \in S' \subseteq S)$ $\iff$
$|S_X| + |d_A^-(X)| \geq k$ *for all* $\emptyset \neq X \subseteq V$.

Theorem 10 implies Theorem 2.

Theorem 11 (Szigeti)

Let $F = (V, E \cup A)$ be a mixed graph, $k \in \mathbb{Z}_+$, $f, g : V \to \mathbb{Z}$ functions, S a multiset of vertices in V and $\mathrm{M} = (S, r_{\mathrm{M}})$ a matroid. There exists an M-rooted (f,g)-bounded k-regular packing of mixed arborescences $\Longleftrightarrow$

1. $g_k(v) = \min\{k, g(v)\} \geq f(v)$ *for all* $v \in V$,
2. $e_E(\mathcal{P}) + r_{\mathrm{M}}(S_X) + g_k(Y - X) \geq \sum_{X' \in \mathcal{P}}(k - d_A^-(X')) + f(X - Y)$ *for all* $X, Y \subseteq V$ *and subpartition* $\mathcal{P}$ *of* Y,
3. $r_{\mathrm{M}}(S_X) + g_k(\overline{X}) \geq r_{\mathrm{M}}(S)$ *for all* $X \subseteq V$.



$\mathcal{B}_{\mathrm{M}} = \{\{s_1, s_2, s_3\}, \{s_1, s_3, s_4\}\}$

Theorem 11 implies Theorems 1-10.

Sketch of the proof

1. The root vectors of M-rooted (f, g)-bounded k-regular packings of arborescences are exactly the integer points of the intersection of two generalized polymatroids $C(k - d_A^-(\cdot)) \cap T(-\infty, k) \cap T(f, g)$ and $B(r_{\mathrm{M}}(S_{(\cdot)}))$.
2. Frank's theorem on intersection of two generalized polymatroids provides the result for digraphs.
3. The mixed graph version is obtained by a new orientation theorem.

Further results

$M_\oplus$-restricted
(f,g)-bounded
regular
matroid-rooted
hypergraph
Szigeti

$M_\oplus$-restricted
(f,g)-bounded
regular
bounded number
hypergraph
Szigeti

mixed
(f,g)-bounded
regular
bounded number
hypergraph
Szigeti

mixed
(f,g)-bounded
regular
matroid-rooted
Szigeti

$M_\oplus$-restricted
matroid-based
(f,g)-bounded
hypergraph
Szigeti

mixed
matroid-based
reachability
hypergraph
Hörsch-Szigeti

$M_\oplus$-restricted
matroid-based
reachability
hypergraph
Szigeti

(f,g)-bounded
regular
bounded number
Bérczi-Frank

(f,g)-bounded
spanning
mixed
hypergraph
Hörsch-Szigeti

matroid-based
(f,g)-bounded
mixed
Szigeti

mixed
matroid-based
reachability
Gao-Yang

matroid-based
reachability
hypergraph
Fortier-Cs.Király-Léonard-Szigeti-Talon

$M_\oplus$-restricted
matroid-based
reachability
Cs.Király-Szigeti-Tanigawa

(f,g)-bounded
spanning
mixed
Gao-Yang

mixed
reachability
Matsuoka-Tanigawa

reachability
hypergraph
Bérczi-Frank

matroid-based
reachability
Cs.Király

matroid-rooted
spanning
Szigeti

bounded number

regular

(f,g)-bounded
spanning
Frank, Cai

mixed
spanning
Frank
Gao-Yang

$M_\oplus$-restricted
spanning
Frank
Bernáth-T.Király

matroid-based
Durand de Gevegny
Nguyen-Szigeti

hypergraph
spanning
Frank-T.Király-Z.Király

reachability
Kamiyama-Katoh-Takizawa
(Hörsch-Szigeti)

spanning
Edmonds
Frank

Results seen so far

$M_\oplus$-restricted, (f,g)-bounded, regular, matroid-rooted, hypergraph — Szigeti

$M_\oplus$-restricted, (f,g)-bounded, regular, bounded number, hypergraph — Szigeti

mixed, (f,g)-bounded, regular, bounded number, hypergraph — Szigeti

mixed, (f,g)-bounded, regular, matroid-rooted — Szigeti

$M_\oplus$-restricted, matroid-based, (f,g)-bounded, hypergraph — Szigeti

mixed, matroid-based, reachability, hypergraph — Hörsch-Szigeti

$M_\oplus$-restricted, matroid-based, reachability, hypergraph — Szigeti

(f,g)-bounded, regular, bounded number — Bérczi-Frank

(f,g)-bounded, spanning, mixed, hypergraph — Hörsch-Szigeti

matroid-based, (f,g)-bounded, mixed — Szigeti

mixed, matroid-based, reachability — Gao-Yang

matroid-based, reachability, hypergraph — Fortier-Cs.Király-Léonard-Szigeti-Talon

$M_\oplus$-restricted, matroid-based, reachability — Cs.Király-Szigeti-Tanigawa

(f,g)-bounded, spanning, mixed — Gao-Yang

mixed, reachability — Matsuoka-Tanigawa

reachability, hypergraph — Bérczi-Frank

matroid-based, reachability — Cs.Király

matroid-rooted, spanning — Szigeti

bounded number

regular

(f,g)-bounded, spanning — Frank, Cai

mixed, spanning — Frank, Gao-Yang

$M_\oplus$-restricted, spanning — Frank, Bernáth-T.Király

matroid-based — Durand de Gevegny, Nguyen-Szigeti

hypergraph, spanning — Frank-T.Király-Z.Király

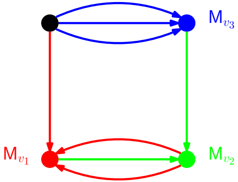
reachability — Kamiyama-Katoh-Takizawa (Hörsch-Szigeti)

spanning — Edmonds, Frank

Extensions

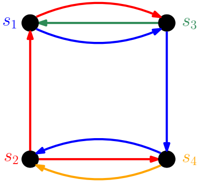
Definition

$M_\oplus$-restricted arc set A':
$\rho_{A'}(v)$ is independent in a given matroid M_v on $\rho_A(v)$ for all $v \in V$.



Definition

matroid-based packing of arborescences:
for every $v \in V$, roots of arborescences containing v in the packing form a basis of a given matroid M.

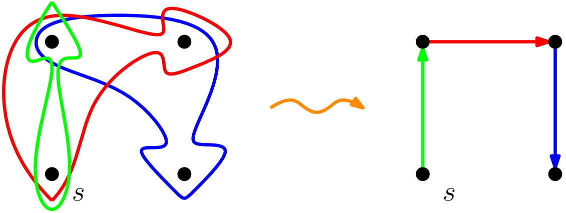


$$\mathcal{B}_{\mathsf{M}} = \{\{s_1, s_2, s_3\}, \{s_1, s_2, s_4\}\}$$

Extensions

Definition

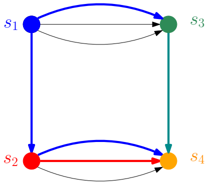
1. hyperarborescence: directed hypergraph that can be trimmed to an arborescence.
2. spanning s-hyperarborescence: directed hypergraph that can be trimmed to a spanning s-arborescence.



Extensions

Definition

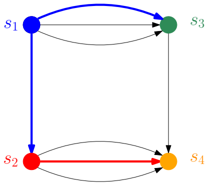
reachability s-arborescence of D:
s-arborescence containing all vertices reachable from s in D.



Extensions

Definition

matroid-reachability-based packing of arborescences:
for every $v \in V$, roots of arborescences containing v in the packing form a basis of the set of elements from which v is reachable in D.



M $= U_{4,2}$

Negative results

Theorem

The following problems are NP-complete:

1. packing of a spanning s-arborescence and a spanning s-in-arborescence (Thomassen),
2. f-factor packing of s-arborescences (Szigeti),
3. packing of k s-arborescences, each of size ℓ (Hörsch),
4. (f, g)-bounded packing of k reachability arborescences (Bérczi, Frank),
5. matroid-based packing of spanning s-arborescences in the extended framework (Fortier, Cs. Király, Szigeti, Tanigawa),
    - in the framework of this talk it is equivalent to matroid-rooted packing of spanning arborescences that can be solved,
6. packing of 2 Steiner s-arborescences (Cheriyan, Salavatipour).

Open problems

Open problems

1. Matroid-based packing of ℓ arborescences.
   - for $\ell = |S|$, it is equivalent to the original matroid-based packing of arborescences that can be solved,
   - for $\ell = r_{\mathsf{M}}(S)$, it is equivalent to matroid-rooted packing of spanning arborescences that can be solved,
2. (f, g)-bounded, matroid-based packing of mixed hyperarborescences.
3. (f, g)-bounded, matroid-rooted, regular packing of mixed hyperarborescences.