Edge-connectivity augmentation of graphs over symmetric parity families

Zoltán Szigeti *

November 26, 2007

Abstract

In this note we solve the edge-connectivity augmentation problem over symmetric parity families. It provides a solution for the minimum T-cut augmentation problem. We also extend a recent result of C. Q. Zhang [8].

1 Introduction

Parity families were introduced in Goemans, Ramakrishnan [2]. In this note we consider only *symmetric* parity families. The definition and some examples for such families can be found in Section 2. Our definition is equivalent to that of Goemans, Ramakrishnan by Lemma 9 in [2].

The main purpose of this paper is to solve the minimum T-cut augmentation problem, namely for a given connected undirected graph G, a subset T of vertices of G of even cardinality and an integer k, what is the minimum number of new edges whose addition results in a graph where the minimum cardinality of a T-cut is at least k. In fact we will solve a more general problem, namely the minimum $\mathcal{F}$-cut augmentation problem where $\mathcal{F}$ is a symmetric parity family. Our main result (Theorem 5) also contains as a special case the min-max theorem of Watanabe, Nakamura [7] for the global edge-connectivity augmentation problem.

This paper is organized as follows. After the necessary definitions we give some easy properties of symmetric parity families and $\mathcal{F}$-joins. Then, in Section 4, we present a min-max theorem on the minimum value of a symmetric submodular function over a symmetric parity family. We need this result to prove, in Section 5, the existence of a splitting off that maintains the minimum $\mathcal{F}$-cut in a graph G for a symmetric parity family $\mathcal{F}$. This splitting off theorem will be applied, in Section 6, to solve the global edge-connectivity augmentation problem over a symmetric parity family. In Section 5 we also provide a common generalization of the weak orientation theorem of Nash-Williams [4] and a result of Rizzi [5]. In Section 7, we generalize a splitting off result of C. Q. Zhang [8].

2 Definitions

Let $G = (V, E)$ be an undirected graph. For a vertex s of G we denote by $\boldsymbol{\Gamma_G(s)}$ the set of neighbours of s. For $X, Y \subseteq V$, $\boldsymbol{\delta_G(X, Y)}$ denotes the set of edges between $X - Y$ and $Y - X$, $\boldsymbol{\delta_G(X)}$= $\delta_G(X, V - X)$, $\boldsymbol{d_G(X)}$= $|\delta_G(X)|$, $\boldsymbol{d_G(X, Y)}$= $|\delta_G(X, Y)|$ and $\boldsymbol{\overline{d}_G(X, Y)}$= $d_G(X \cap Y, V - (X \cup Y))$. The subgraph of G induced by X is denoted by $\boldsymbol{G[X]}$. For all $X, Y \subseteq V$,

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y), \quad (1)$$

$$d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2\overline{d}_G(X, Y). \quad (2)$$

*G-SCOP, INPGrenoble, UJF, CNRS, 46 avenue Félix Viallet, 38031 Grenoble, France. This research was partially done while the author was visiting the Research Institute for Discrete Mathematics, Lennéstrasse 2, 53113 Bonn, Germany by an Alexander von Humboldt fellowship.

For two vertices u and v of G, the **local edge-connectivity** between u and v is defined $\boldsymbol{\lambda_G(u,v)}= \min\{d_G(X) : X \subset V, u \in X, v \in V - X\}$. Let $D = (V, A)$ be a directed graph. For a set $X \subseteq V$, the number of arcs leaving X is denoted by $\boldsymbol{d^+_D(X)}$. For two vertices u and v of D, the **local edge-connectivity** from u to v is defined $\boldsymbol{\lambda_D(u,v)}= \min\{d^+_D(X) : X \subset V, u \in X, v \in V - X\}$. More generally, for a function f on V and for $u, v \in V$, we define the **local f-connectivity** as $\boldsymbol{\lambda_f(x,y)}:= \min\{f(X) : X \subset V, x \in X, y \in V - X\}$.

Suppose G is connected and let $T \subseteq V$ with $|T|$ even. The pair (G, T) is called **graft**. A subset X of V is called **T-odd** if $|X \cap T|$ is odd and then the cut $\delta(X)$ is called a **T-cut**. An edge set F is called a **T-join** if $T = \{v \in V : d_F(v) \text{ is odd}\}$. A **T-pairing** is a perfect matching of the complete graph on T.

A family $\mathcal{F}$ of subsets of V is called **symmetric parity family** if (i),(ii),(iii) are satisfied: (i) $\emptyset, V \notin \mathcal{F}$, (ii) if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$, (iii) if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. The most important examples for symmetric parity families are the following: $\mathcal{F} := 2^V - \{\emptyset, V\}$ and $\mathcal{F} := \{X \subset V : |X \cap T| \text{ is odd}\}$ where $T \subseteq V$ with $|T|$ even; for others see [2].

Let $\mathcal{F}$ be a symmetric parity family on V. Let $H = (V, E')$ be a tree. For each edge e of H, let $V_e \subset V$ so that $\delta_H(V_e, V - V_e) = \{e\}$. Let $\boldsymbol{J_{\mathcal{F}}(H)}:= \{e \in E' : V_e \in \mathcal{F}\}$. An edge set $F \subseteq V \times V$ is called **$\mathcal{F}$-join** if there exists a tree H on V so that $F = J_{\mathcal{F}}(H)$. For a symmetric function f on V, let us define the value of an $\mathcal{F}$-join F by $\boldsymbol{val_f(F)}:= \min\{\lambda_f(x, y) : xy \in F\}$. If $G = (V, E)$ is a graph and $X \in \mathcal{F}$, then $\delta_G(X)$ is called an **$\mathcal{F}$-cut**. Let $\boldsymbol{\lambda^G_{\mathcal{F}}}$ denote the minimum size of an $\mathcal{F}$-cut, that is $\lambda^G_{\mathcal{F}} = \min\{d_G(X) : X \in \mathcal{F}\}$.

Let $G = (U, E)$ be a graph and $s \in U$. For two edges sr, st, the graph obtained from G by **splitting off** sr, st is denoted by $\boldsymbol{G_{r,t}}:= G - \{sr, st\} + rt$. Let $\mathcal{F}$ be a symmetric parity family on $V \subseteq U$. The pair $\{sr, st\}$ is called **$\boldsymbol{\lambda_{\mathcal{F}}}$-admissible** if after splitting off this pair, the minimum $\mathcal{F}$-cut does not decrease that is if $\lambda^{G_{r,t}}_{\mathcal{F}} \geq \lambda^G_{\mathcal{F}}$.

A function f on 2^V is called **symmetric** and **submodular** if for all $X, Y \subseteq V$, $f(X) = f(V - X)$ and $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$. Note that the degree function of an undirected graph is symmetric and, by (1), it is submodular. A function p on 2^V is called **skew-supermodular** if at least one of (3) and (4) hold for all $X, Y \subseteq V$.

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{3}$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{4}$$

3 Preliminaries

An easy observation on T-odd sets, namely if we have two T-odd sets then either their intersection and union or their differences are T-odd sets, can be generalized for symmetric parity families as follows.

Claim 1 *Let $\mathcal{F}$ be a symmetric parity family. Then, for all $X, Y \in \mathcal{F}$, either $X \cap Y, X \cup Y \in \mathcal{F}$ or $X - Y, Y - X \in \mathcal{F}$.*

Proof. Suppose $X \cap Y \notin \mathcal{F}$. Then, by $X \in \mathcal{F}$ and (iii), $X - Y \in \mathcal{F}$. Similarly, by $Y \in \mathcal{F}$ and (iii), $Y - X \in \mathcal{F}$. Now suppose that $X \cup Y \notin \mathcal{F}$, so by (ii), $V - (X \cup Y) \notin \mathcal{F}$. Since $X, Y \in \mathcal{F}$, we have, by (ii), $V - X, V - Y \in \mathcal{F}$. Then, by $V - X \in \mathcal{F}$ and (iii), $Y - X \in \mathcal{F}$. Similarly, by $V - Y \in \mathcal{F}$ and (iii), $X - Y \in \mathcal{F}$. □

Now we generalize the fact that a T-join and a T-cut always have an edge in common.

Lemma 1 *Let $\mathcal{F}$ be a symmetric parity family. If F is an $\mathcal{F}$-join and $A \in \mathcal{F}$, then $\delta_F(A) \neq \emptyset$.*

Proof. By definition there exists a tree H on V so that $F = J_{\mathcal{F}}(H)$. Let us denote by $A_1, ..., A_k$ the connected components of $H[A]$. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, there exists an index i so that $A_i \in \mathcal{F}$ by (iii). Let us denote by $B_1, ..., B_l$ the connected components of $H - A_i$. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, there exists an index j so that $B_j \in \mathcal{F}$ by (iii). H

is a tree, $H[A_i]$ and $H[B_j]$ are connected and B_j is a connected component of $H - A_i$, so there exists exactly one edge $e \in E(H)$ between $H[A_i]$ and $H[B_j]$. It follows that $e \in J_{\mathcal{F}}(H) = F$ and e enters A which was to be proved. □

If $\mathcal{F} = \{\text{T-odd sets}\}$, then $\mathcal{F}$-joins can be characterized as follows.

Claim 2 *If $\mathcal{F} := \{X \subset V : |X \cap T| \text{ is odd}\}$ where $T \subseteq V$ with $|T|$ even, then the $\mathcal{F}$-joins are exactly the cycle free T-joins.*

Proof. Note that any tree H on V contains a T-join F and a set X belongs to $\mathcal{F}$ if and only if $d_F(X)$ is odd. Thus if F is a T-join and H is a tree on V containing F, then $J_{\mathcal{F}}(H) = F$ and the claim follows. □

4 Min-max theorems

Let f be a symmetric submodular function on 2^V and $\mathcal{F}$ a symmetric parity family on V. It is mentioned in Goemans, Ramakrishnan [2] that there exists a cut equivalent tree H_f for f and f can be minimized over $\mathcal{F}$ using H_f, namely $\min\{f(X) : X \in \mathcal{F}\} = val_f(J_{\mathcal{F}}(H_f))$. This can be presented as a min-max result.

Theorem 1 *Let f be a symmetric submodular function on 2^V and $\mathcal{F}$ a symmetric parity family on V. Then*

$$\min\{f(X) : X \in \mathcal{F}\} = \max\{val_f(F) : F \text{ is an } \mathcal{F}\text{-join}\}.$$

Proof. To prove max ≤ min, let $X' \in \mathcal{F}$ and let F' be an $\mathcal{F}$-join. By Lemma 1, there exists an edge $x'y' \in \delta_{F'}(X')$. Then $val_f(F') = \min\{\lambda_f(x, y) : xy \in F'\} \leq \lambda_f(x', y') = \min\{f(Y) : Y \subset V, x' \in Y, y' \in V - Y\} \leq f(X')$ and the inequality follows. As we mentioned above, $J_{\mathcal{F}}(H_f)$ provides equality, thus min = max. □

If $\mathcal{F} := \{\text{T-odd sets}\}$ then the dual objects in Theorem 1 can be simplified. For $f = d_G$, the following theorem gives a result of R. Rizzi [5] on T-cuts.

Theorem 2 *Let f be a symmetric submodular function on 2^V and $\mathcal{F} := \{X \subset V : |X \cap T| \text{ is odd}\}$ where $T \subseteq V$ with $|T|$ even. Then*

$$\min\{f(X) : X \in \mathcal{F}\} = \max\{val_f(P) : P \text{ is a } T\text{-pairing}\}.$$

Proof. Let $\alpha := \max\{val_f(F) : F \text{ is an } \mathcal{F}\text{-join}\}$ and $\beta := \max\{val_f(P) : P \text{ is a } T\text{-pairing}\}$. We show that $\alpha = \beta$ and then Theorem 1 implies Theorem 2. Since a perfect matching P on T is a T-join and, by Claim 2, is an $\mathcal{F}$-join, we have $\alpha \geq \beta$. Now, let F be an $\mathcal{F}$-join of value α for which $|F|$ is minimum. We prove that F is a T-pairing implying $\alpha \leq \beta$. Otherwise, F contains two adjacent edges uv and vw. Let F' be obtained from F by splitting off these edges. By Claim 2, F' is an $\mathcal{F}$-join. Since a set separating u and w separates either u and v or v and w, $val_f(F') \geq val_f(F) = \alpha$. Then F' is an optimum $\mathcal{F}$-join and $|F'| < |F|$, a contradiction. □

We note that Theorem 2 is true for symmetric parity families that satisfy the following condition: if $X, Y \in \mathcal{F}$ and $X \cap Y = \emptyset$, then $X \cup Y \notin \mathcal{F}$. However, it can be shown that such a family can be defined as the T-odd sets for some $T \subseteq V$ with $|T|$ even.

5 Local edge-connectivity

Lemma 2 *Let $G = (U, E)$ be a graph, $s \in U, U - s \subseteq V \subseteq U$ and let $\mathcal{F}$ be a symmetric parity family on V. Then for all $A \in \mathcal{F}$ there exist $x \in A$ and $y \in V - A$ so that $\lambda_G(x, y) \geq \lambda^G_{\mathcal{F}}$.*

Proof. Let $f(X) := \min\{d_G(X), d_G(V-X)\}$ for all $X \subseteq V$. It is easy to check, by (1) and (2), that f is a symmetric submodular function on 2^V. Then, by Theorem 1, there exists an $\mathcal{F}$-join F on V with $\min\{\lambda_f(a,b) : ab \in F\} = \min\{f(X) : X \in \mathcal{F}\}$. By Lemma 1, there exists an edge $xy \in F$ with $x \in A$ and $y \in V - A$. Let $\delta_G(Y)$ be a minimum cut in G separating x and y so that $s \notin Y$. Then $\lambda_G(x,y) = d_G(Y)$ and $f(Y) \geq \lambda_f(x,y)$. Since $V = U - s$ or U, we have $Y \subseteq V$, thus $d_G(Y) \geq f(Y)$. It follows that $\lambda_G(x,y) = d_G(Y) \geq f(Y) \geq \lambda_f(x,y) \geq \min\{\lambda_f(a,b) : ab \in F\} = \min\{f(X) : X \in \mathcal{F}\} = \min\{\min\{d_G(X), d_G(V-X)\} : X \in \mathcal{F}\} \geq \lambda_{\mathcal{F}}^G$ and we are done. □

We show two applications of the above lemma. The first result is on splitting off and it will be applied in Section 6 to prove the augmentation result.

Lemma 3 *Let $G = (V+s, E)$ be a graph so that $d(s) \neq 3$ and no cut edge is incident to s. Let $\mathcal{F}$ be a symmetric parity family on V. Then there exists a $\lambda_{\mathcal{F}}$-admissible pair.*

Proof. By Mader's local splitting off theorem [3], there exists a pair of edges $\{sr, st\}$ so that $\lambda_{G_{r,t}}(x,y) = \lambda_G(x,y)$ for all $x, y \in V$. By Lemma 2, applied for $U = V + s$ and V, for all $X \in \mathcal{F}$, there exist $x \in X$ and $y \in V - X$ so that $\lambda_G(x,y) \geq \lambda_{\mathcal{F}}^G$. Then $d_{G_{r,t}}(X) \geq \lambda_{G_{r,t}}(x,y) = \lambda_G(x,y) \geq \lambda_{\mathcal{F}}^G$ so $\lambda_{\mathcal{F}}^{G_{r,t}} \geq \lambda_{\mathcal{F}}^G$ that is $\{sr, st\}$ is a $\lambda_{\mathcal{F}}$-admissible pair. □

The second application of Lemma 2 is the following orientation result. It is a common generalization of the weak orientation theorem of Nash-Williams [4] (if $\mathcal{F} = 2^V - \{\emptyset, V\}$) and an orientation theorem on T-cuts of Rizzi [5] (if $\mathcal{F} = \{\text{T-odd sets}\}$).

Theorem 3 *Let $G = (V, E)$ be an undirected graph and $\mathcal{F}$ a symmetric parity family on V. Then G has an orientation $\vec{G}$ so that $d^+_{\vec{G}}(X) \geq k$ for all $X \in \mathcal{F}$ if and only if $d_G(X) \geq 2k$ for all $X \in \mathcal{F}$.*

Proof. We prove only the non-trivial if part. By Nash-Williams' well-balanced orientation theorem [4], there exists an orientation $\vec{G}$ of G such that $\lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor$ for all $(x,y) \in V^2$. We show that $\vec{G}$ will do. By Lemma 2, applied for $U = V$, for every $X \in \mathcal{F}$, there exist $x \in X$ and $y \in V - X$ so that $\lambda_G(x,y) \geq \lambda_{\mathcal{F}}^G$. Then, since, by the condition, $\lambda_{\mathcal{F}}^G \geq 2k$, we have $d^+_{\vec{G}}(X) \geq \lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor \geq \lfloor \lambda_{\mathcal{F}}^G/2 \rfloor \geq k$. □

6 Augmentation

In this section we solve the following augmentation problem: Given a graph $G = (V, E)$, a symmetric parity family $\mathcal{F}$ on V and an integer k, what is the minimum number of edges whose addition results in a graph in which each $\mathcal{F}$-cut contains at least k edges. As a special case it solves the minimum T-cut augmentation problem: how many new edges must be added to a graft so that the minimum T-cut contains at least k edges. It also contains as a special case the global edge-connectivity augmentation problem, namely how many new edges must be added to a graph to make it k-edge-connected. A general approach to solve edge-connectivity augmentation problems is summarized in the following theorem.

Theorem 4 (Frank [1]) *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular function. Then there exists a graph $(V+s, K)$ with 2γ edges, all incident to s so that $d_K(X) \geq p(X)$ for all $X \subset V$ if and only if for each subpartition $\{X_1, ..., X_l\}$ of V, $\sum_{i=1}^{l} p(X_i) \leq 2\gamma$.*

Theorem 4 and Lemma 3 will provide the main result of this paper. We mention that it provides a new special case of the NP-hard problem of covering symmetric skew-supermodular functions (see in [6]) that can be polynomially solved.

Theorem 5 *For a connected graph $G = (V, E)$, a symmetric parity family $\mathcal{F}$ on V and an integer $k \geq 2$, the minimum cardinality of an $\mathcal{F}$-cut can be augmented to k by adding at most γ edges if and only if for each subpartition $\{X_1, ..., X_l\}$ of V with $X_i \in \mathcal{F}$, $\sum_{i=1}^{l}(k - d_G(X_i)) \leq 2\gamma$.*

Proof. For $X \subset V$, let $p(X) := k - d_G(X)$ if $X \in \mathcal{F}$, and $-\infty$ otherwise. Since $\mathcal{F}$ and d are symmetric so is p. We show that p is skew-supermodular. Let $X, Y \subseteq V$. If $X \notin \mathcal{F}$ or $Y \notin \mathcal{F}$ then (3) and (4) are satisfied. If $X, Y \in \mathcal{F}$, then, by Claim 1, either $X \cap Y, X \cup Y \in \mathcal{F}$ and then (3) is satisfied by (1) or $X - Y, Y - X \in \mathcal{F}$ and then (4) is satisfied by (2).

The subpartition condition of Theorem 5 implies that the subpartition condition of Theorem 4 is satisfied, thus, by Theorem 4, there exists a graph $(V+s, K)$ with 2γ edges, all incident to s so that $d_K(X) \geq p(X)$ for all $X \subset V$. Let $L := (V+s, E \cup K)$. Then $d_L(X) = d_G(X) + d_K(X) \geq d_G(X) + p(X) = k$ for all $X \in \mathcal{F}$ that is $\lambda^L_{\mathcal{F}} \geq k$. Note that $d_L(s) = 2\gamma \neq 3$ and, since G is connected, no cut edge of L is incident to s. Then, by Lemma 3, we can split off all the edges of L incident to s by preserving $\lambda_{\mathcal{F}}$. Thus the resulting graph G' is obtained from G by adding γ edges and $\lambda^{G'}_{\mathcal{F}} = \lambda^L_{\mathcal{F}} \geq k$. □

By applying Theorem 5, for $\mathcal{F} = 2^V - \{\emptyset, V\}$, we get the theorem of Watanabe, Nakamura [7], and for $\mathcal{F} = \{\text{T-odd sets}\}$, we get the following theorem on T-cuts.

Theorem 6 *For a graft (G, T), the minimum cardinality of a T-cut can be augmented to k ($k \geq 2$) by adding at most γ edges if and only if $\sum_1^l (k - d(X_i)) \leq 2\gamma$ for each subpartition $\{X_1, ..., X_l\}$ of $V(G)$ into T-odd sets.* □

7 Splitting off

In this section we present a generalization of a result of Zhang [8].

For a graph $G = (V, E)$, $\boldsymbol{\mathcal{F}_G}$:= $\{X \subset V : d_G(X) \text{ is odd}\}$ is a symmetric parity family such that each $\mathcal{F}_G$-cut is odd. The odd-edge-connectivity λ_o of G is defined as $\lambda^G_{\mathcal{F}_G}$.

Theorem 7 (Zhang [8]) *Let G be a graph with odd-edge-connectivity λ_o. Let s be a vertex of G such that $d(s) \neq \lambda_o$ and $\neq 2$. Arbitrarily label the edges of G incident with s as $\{e_1, ..., e_{d(s)}\}$. Then there is an integer $i \in \{1, ..., d(s)\}$ such that the new graph obtained from G by splitting e_i and e_{i+1} (mod $d(s)$) off at s remains of odd-edge-connectivity λ_o.*

Let s be a vertex of a graph $G = (V, E)$ and let N be a graph on vertex set $\Gamma_G(s)$. A pair $\{sr, st\}$ of edges of G is called **N-allowed** if $rt \in E(N)$. This definition is motivated by Theorem 7 in which we are only allowed to split off consecutive pairs of edges, $e_i = sv_i$ and $e_{i+1} = sv_{i+1}$ for some $1 \leq i \leq d(s)$, that is if $N_Z = (\Gamma_G(s), \{v_i v_{i+1} : 1 \leq i \leq d(s)\})$, then we can only split off N_Z-allowed pairs. Note that $v_1v_2, v_2v_3, ..., v_{d(s)}v_1$ provides an eulerian walk of N_Z, and hence N_Z is connected.

In the following we generalize Theorem 7 and provide a proof that is much shorter than that of [8].

Theorem 8 *Let $G = (V, E)$ be a graph, $s \in V$. Let $\mathcal{F}$ be a symmetric parity family on V such that $d_G(X) \equiv \lambda^G_{\mathcal{F}}$ (mod 2) for all $X \in \mathcal{F}$. Let $2 \leq d(s) \neq \lambda^G_{\mathcal{F}}$. Let $N = (\Gamma_G(s), M)$ be a connected graph with $M \neq \emptyset$. Then there exists an N-allowed $\lambda_{\mathcal{F}}$-admissible pair.*

Proof. We call a set $X \subseteq V - s$ **tight** if $X \in \mathcal{F}$ and $d_G(X) = \lambda^G_{\mathcal{F}}$. Since each element of $\mathcal{F}$ has the same parity as $\lambda^G_{\mathcal{F}}$, a pair $\{sr, st\}$ is not $\lambda_{\mathcal{F}}$-admissible if and only if there exists a tight set containing r and t. Let $t \in \Gamma_G(s)$. If t belongs to no tight set, then for an edge $rt \in M$ (r exists since N is connected and $M \neq \emptyset$) $\{sr, st\}$ is an N-allowed $\lambda_{\mathcal{F}}$-admissible pair. Otherwise, let Q be a maximal tight set containing t. Suppose $\Gamma_G(s) - Q = \emptyset$. If $\{s\} \in \mathcal{F}$, then $\lambda^G_{\mathcal{F}} \leq d(s)$. Since $\delta(s) \subseteq \delta(Q)$ and Q is tight, $d(s) \leq d(Q) = \lambda^G_{\mathcal{F}}$. These inequalities provides $\lambda^G_{\mathcal{F}} = d(s)$, a contradiction. If $\{s\} \notin \mathcal{F}$, then, since $V - Q \in \mathcal{F}$, $V - Q - s \in \mathcal{F}$ and then, by $d(s) \geq 2$, $\lambda^G_{\mathcal{F}} \leq d(V - Q - s) = d(Q) - d(s) < d(Q) = \lambda^G_{\mathcal{F}}$, contradiction. Thus $\Gamma_G(s) - Q \neq \emptyset$, $t \in \Gamma_G(s) \cap Q$ and N is connected so there exists an edge $qr \in M$ such that $r \in Q, q \notin Q$. Then $\{sq, sr\}$ is N-allowed. The following claim completes the proof.

Claim 3 *$\{sq, sr\}$ is $\lambda_{\mathcal{F}}$-admissible.*

Proof. Suppose that $\{sq, sr\}$ is not $\lambda_{\mathcal{F}}$-admissible that is there is a tight set R containing q and r. By Claim 1, either $R-Q, Q-R \in \mathcal{F}$ and then, by (2) and the existence of sr, $\lambda^G_{\mathcal{F}}+\lambda^G_{\mathcal{F}} = d(R)+d(Q) = d(R-Q)+d(Q-R)+2\bar{d}(Q,R) \geq \lambda^G_{\mathcal{F}}+\lambda^G_{\mathcal{F}}+2$, contradiction or $Q \cap R, Q \cup R \in \mathcal{F}$ and then, by (1) and the maximality of Q, $\lambda^G_{\mathcal{F}}+\lambda^G_{\mathcal{F}} = d(Q)+d(R) \geq d(Q \cap R)+d(Q \cup R) > \lambda^G_{\mathcal{F}}+\lambda^G_{\mathcal{F}}$, contradiction. □

Example: The following example shows that the condition that each element of $\mathcal{F}$ is of the same parity can not be omitted from Theorem 8. Let $G := (\{q,r,s,t\}, \{qr, qs, qs, rs, rt, st, st\})$. Let $\mathcal{F} := \{X : X \cap \{q,t\} = 1\}$. Then $\lambda^G_{\mathcal{F}} = 3$ and $d_G(s) = 5$. Let $N := (\{q,r,t\}, \{qr, rt\})$. Then for the N-allowed splittings, $\lambda^{G'}_{\mathcal{F}} = 2 < 3 = \lambda^G_{\mathcal{F}}$.

Acknowledgement I thank A. Frank for calling my attention to the paper of Goemans and Ramakrishnan.

References

[1] A. Frank, Augmenting graphs to meet edge-connectivity requirements, *SIAM J. on Disc. Math.* Vol. 5, No. 1 (1992) 22-53.

[2] M. X. Goemans, V. S. Ramakrishnan, Minimizing submodular functions over families of sets, *Combinatorica* **15** (4) (1995) 499-513.

[3] W. Mader, A reduction method for edge-connectivity in graphs, *Ann. Discrete Math.* **3** (1978) 145-164.

[4] C. St. J. A. Nash-Williams, On orientations, connectivity and odd vertex pairing in finite graphs, *Canad. J. Math.* **12** (1960) 555-567.

[5] R. Rizzi, Minimum T-cuts and optimal T-pairings, *Discrete Math.* **257** (2002) 177-181.

[6] Z. Szigeti, Edge-connectivity augmentations of graphs and hypergraphs, Technical Report No. 04941 of the Research Institute for Discrete Mathematics, Bonn, 2004.

[7] T. Watanabe, A. Nakamura, Edge-connectivity augmentation problems, *Comp. System Sci.* **35** (1987) 96-144.

[8] C. Q. Zhang, Circular flows of nearly eulerian graphs and vertex splitting, *J. Graph Theory* **40** (2002) 147-161.