Eulerian orientations and vertex-connectivity

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Abstract

It is well-known that *every* Eulerian orientation of an Eulerian $2k$-edge-connected undirected graph is k-arc-connected. A long-standing goal in the area has been to obtain analogous results for vertex-connectivity. Levit, Chandran and Cheriyan recently proved in [8] that every Eulerian orientation of a hypercube of dimension $2k$ is k-vertex-connected. Here we provide an elementary proof for this result.

We also show other families of $2k$-regular graphs for which every Eulerian orientation is k-vertex-connected, namely the even regular complete bipartite graphs, the incidence graphs of projective planes of odd order, the line graphs of regular complete bipartite graphs and the line graphs of complete graphs.

Furthermore, we provide a *simple* graph counterexample for a conjecture of Frank attempting to characterize graphs admitting at least one k-vertex-connected orientation.

1. Introduction

This paper concerns ways of orienting undirected graphs so that certain connectivity requirements are satisfied. The case of edge-connectivity is already well-understood [9, 5, 6]. Here we contribute to the development of the theory of highly vertex-connected orientations.

Let $G = (V, E)$ be an undirected graph. For $X, Y \subseteq V$, we use $\boldsymbol{\delta_G(X, Y)}$ to denote the set of edges between $X \setminus Y$ and $Y \setminus X$. We use $\boldsymbol{\delta_G(X)}$ for $\delta_G(X, V \setminus X)$, $\boldsymbol{d_G(X)}$ for $|\delta_G(X)|$ and $\boldsymbol{d_G(v)}$ for $d_G(\{v\})$. The subgraph induced by X is denoted by $\boldsymbol{G[X]}$ and the number of edges of $G[X]$ is denoted by $\boldsymbol{i_G(X)}$. The graph G is called *k-regular* if $d_G(v) = k$ for all $v \in V$. We denote by $\boldsymbol{N_G(X)}$ the set of *neighbors* of X, that is, the set of vertices in $V \setminus X$ which are adjacent to a vertex in X. We say that G is *k-edge-connected* if $d_G(X) \geq k$ for all $\emptyset \neq X \subsetneq V$. We call G *Eulerian* if every vertex of G is of even degree. An *orientation* of G is a directed graph obtained from G by replacing each edge uv by exactly one of the arcs uv or vu.

Let $D = (V, A)$ be a directed graph. For $X \subseteq V$, we use $\boldsymbol{\delta^-_D(X)}$ for the set of arcs from $V \setminus X$ to X, $\boldsymbol{\delta^+_D(X)}$ for $\delta^-_D(V \setminus X)$, $\boldsymbol{d^-_D(X)}$= $|\delta^-_D(X)|$ for the *in-degree* of X and $\boldsymbol{d^+_D(X)}$= $d^-_D(V \setminus X)$ for the *out-degree* of X. As before, $\boldsymbol{d^-_D(v)}$ (resp. $\boldsymbol{d^+_D(v)}$) is used for $d^-_D(\{v\})$ (resp. $d^+_D(\{v\})$). The subgraph induced by X is denoted by $\boldsymbol{D[X]}$. We say that D is *k-arc-connected* if $d^+_D(X) \geq k$ for all $\emptyset \neq X \subsetneq V$. We say that D is *k-vertex-connected* if $|V| \geq k + 1$ and after deleting any vertex set of size $k - 1$ the remaining graph is 1-arc-connected. We call D *Eulerian* if $d^-_D(v) = d^+_D(v)$ for all $v \in V$.

It is well-known that if D is Eulerian, then we have $d^-_D(X) = d^+_D(X)$ for all $X \subseteq V$. Therefore, every Eulerian orientation of a $2k$-edge-connected Eulerian graph results in a directed graph that is k-arc-connected. A fundamental result of Nash-Williams [9] states that a $2k$-edge-connected undirected graph can be oriented such that the resulting directed graph is k-edge-connected. A long-standing goal in the area is to extend this to obtain an analogous result for vertex-connectivity [6]. Frank [4] conjectured a characterization of graphs admitting a k-vertex-connected orientation which was proved by Thomassen [10] for $k = 2$ and disproved by

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Durand de Gevigney [2] for $k \geq 3$. In Section 4 we provide a counterexample to Frank's conjecture for $k = 3$ that is smaller than that in [2]. We also provide a simple graph counterexample.

The *hypercube* $\boldsymbol{Q_k}$ of dimension k is the graph whose vertex set is the set of all subsets of $\{1, \ldots, k\}$ and two vertices are connected by an edge if the two corresponding subsets differ in exactly one element. It is well-known that Q_{k+1} can be obtained from two disjoint copies of Q_k by adding an edge between the corresponding vertices of the two copies. Using this construction it is easy to prove that Q_{2k} has an Eulerian orientation that is k-vertex-connected. Recently, Levit, Chandran and Cheriyan proved in [8] the following surprising result on hypercubes.

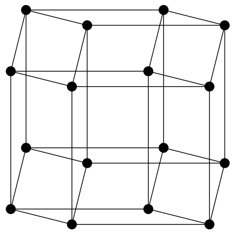

Figure 1: The hypercube Q_4.

Theorem 1 ([8]). *Every Eulerian orientation of a hypercube Q_{2k} is k-vertex-connected.*

One of the contributions of the present paper is to provide a concise proof for Theorem 1, see Subsection 3.5.

Cheriyan [1] posed the question whether there exist other classes of graphs satisfying the following definition.

Definition 2. *A $2k$-regular undirected graph is* good *if every of its Eulerian orientations is k-vertex-connected,* bad *otherwise.*

In Section 2 we provide a characterization of bad graphs and show that almost all complete graphs are bad. In Section 3 we present some classes of good graphs, namely the even regular complete bipartite graphs, the incidence graphs of projective planes of odd order, the line graphs of regular complete bipartite graphs and the line graphs of complete graphs.

2. Bad graphs

We will frequently use the easy characterization of bad graphs presented in Proposition 3 and the further properties given in Proposition 4.

Proposition 3. *A $2k$-regular simple graph $G = (V, E)$ is bad if and only if there exist an orientation D of G and a partition of V into non-empty sets Z, S and T such that*

$$d^-_D(v) = d^+_D(v) = k \text{ for all } v \in V, \tag{1}$$

$$|Z| = k - 1, \tag{2}$$

$$\text{every edge of } \delta_G(S,T) \text{ is oriented from } S \text{ to } T \text{ in } D. \tag{3}$$

Moreover, S can be chosen so that

$$|S| \leq |T|, \tag{4}$$

$$\text{every vertex } s \text{ of } S \text{ has an out-neighbor in } S \text{ in } D, \tag{5}$$

$$G[S] \text{ contains a cycle.} \tag{6}$$

Proof: The first part of the proposition follows by the definition of bad graphs.

In order to show (4) – (6), let us choose the partition and the orientation so that $|S|$ is minimum.

Since the orientation obtained from D by reversing all arcs and the partition Z, T and S of V satisfy (1) – (3), $|T| < |S|$ would contradict the choice of S, hence (4) follows.

By (1) – (3) and the fact the G is simple, we have $|S| \geq 2$. Suppose that there exists a vertex v in S that has no out-neighbor in S. Let $S' := S \setminus \{s\}, T' := T \cup \{s\}$. By $|S| \geq 2$, $S' \neq \emptyset$. Then the orientation D and the partition Z, S' and T' of V satisfy (1) – (3), hence $|S'| < |S|$ contradicts the minimality of S, so (5) follows.

By (5), $D[S]$ contains a circuit and so (6) follows. ■

Proposition 4. *Let D be an orientation of a simple graph G and $\{Z, S, T\}$ a partition of $V(G)$ into non-empty sets satisfying* (1), (2) *and* (3)*. Then the following hold.*

$$d_D^-(S) \leq k\min\{|Z|, |S|\}, \tag{7}$$
$$|\delta_G(S, T)| \leq k^2 - k - i_G(Z). \tag{8}$$

Proof: Since, by (1), $d_D^-(v) = k$ for all $v \in S$, it follows that $d_D^-(S) \leq k|S|$. Moreover, by (3), all arcs entering S come from Z. As, by (1), $d_D^+(v) = k$ for all $v \in Z$, it follows that $d_D^-(S) \leq d_D^+(Z) \leq k|Z|$. These inequalities imply (7).

By (3), (1), and (2), we have (8) : $|\delta_G(S,T)| \leq d_D^+(S) = d_D^-(S) \leq d_D^+(Z) = \sum_{z\in Z} d_D^+(z) - i_G(Z) = k|Z| - i_G(Z) = k^2 - k - i_G(Z)$. ■

It is easy to see that the complete graphs K_{2k+1} are good for $k \leq 3$. We show that these are the only good complete graphs.

Theorem 5. *The complete graphs K_{2k+1} are bad for all $k \geq 4$.*

Proof: Let $k \geq 4$ be an integer and $G = (V, E)$ the complete graph K_{2k+1}. Let S, T and Z' be three disjoint sets in V such that $|S| = \lfloor \frac{k}{2} \rfloor + 1$ and $|T| = |Z'| = \lceil \frac{k}{2} \rceil + 1$. By $k \geq 4$, $\lfloor \frac{k}{2} \rfloor + 1 + 2(\lceil \frac{k}{2} \rceil + 1) \leq 2k + 1$, so such sets exist. Let $Z := V \setminus (S \cup T)$. Note that $|Z| = k - 1$ and $Z \supseteq Z'$. Let M be the empty set if k is even and a perfect matching of the graph $G' = (T \cup Z', \delta_G(T, Z'))$ if k is odd. Since $|T| = |Z'|$ and G is a complete graph, G' is a regular complete bipartite graph, so M exists. Let us orient all edges in $\delta_G(S,T)$ from S to T, all edges in $\delta_G(T, Z') \setminus M$ from T to Z' and all edges in $\delta_G(Z', S)$ from Z' to S. Note that the set of arcs already defined induces an Eulerian directed graph. Hence the corresponding set F of edges induces an Eulerian subgraph of G. Since G is Eulerian, $G - F$ is also Eulerian. Combining the orientation of F with an arbitrary Eulerian orientation of $G - F$, we have an orientation D of G and a partition $\{Z, S, T\}$ of V that satisfy (1), (2) and (3). Thus, by Proposition 3, $G = K_{2k+1}$ is bad. ■

3. Good graphs

In this section, we show that the following graph families are good: the complete bipartite graphs $K_{2k,2k}$, the incidence graphs of projective planes of even degree, the line graphs of regular complete bipartite graphs, the line graphs of complete graphs and the hypercubes Q_{2k}.

We will apply the following easy observation: for all triples of reals (a, b, c) with $a, b \geq c$, since $(a-c)(b-c) \geq 0$, we have

$$ab \geq c(a + b - c). \tag{9}$$

Let a be a non-negative integer. We use the notation $\binom{a}{2}$ for $\frac{a(a-1)}{2}$ and we apply the following inequality:

$$\binom{a}{2} \geq \max\{a - 1, 2a - 3\}. \tag{10}$$

3.1. Complete bipartite graphs

Let us first consider even regular complete bipartite graphs.

Theorem 6. *The complete bipartite graphs $K_{2k,2k}$ are good for all $k \geq 1$.*

Proof: We assume for a contradiction that the bipartite graph $G = (V_1, V_2; E) = K_{2k,2k}$ is bad. By Proposition 3, there exist an orientation D of G and a partition of $V_1 \cup V_2$ into non-empty sets Z, S and T such that (1), (2) and (3) are satisfied. Let $\boldsymbol{z_i}:= |Z \cap V_i|$, $\boldsymbol{s_i}:= |S \cap V_i|$ and $\boldsymbol{t_i}:= |T \cap V_i|$. Note that, by (2), $z_1, z_2 \geq 0, z_1 + z_2 = |Z| = k - 1$.

Claim 7. *The following hold:*

$$s_1 + s_2 + t_1 + t_2 = 3k + 1, \tag{11}$$
$$1 \leq s_1, s_2, t_1, t_2 \leq k, \tag{12}$$
$$s_1, s_2, t_1, t_2 \in \mathbb{Z}. \tag{13}$$

Proof: By $|V(G)| = 4k$ and $|Z| = k - 1$, we have $s_1 + s_2 + t_1 + t_2 = |V(G)| - |Z| = 4k - (k - 1) = 3k + 1$, so (11) holds. By $S \neq \emptyset$, without loss of generality we may assume that there exists $v \in S \cap V_1$. Then, by (3), (1), $z_1 + z_2 = k - 1$ and $z_1 \geq 0$, we have $s_2 \geq d^-_D(v) - z_2 \geq k - (k - 1) = 1$, so $s_1, s_2 \geq 1$ and similarly $t_1, t_2 \geq 1$. Moreover, by (1) and (3), we have $k = d^+_D(v) \geq |\delta_G(v, T \cap V_2)| = t_2$ and similarly $s_1, s_2, t_1 \leq k$, so (12) holds. By definition, (13) obviously holds. ■

Claim 8. *The minimum of $s_1t_2 + s_2t_1$ subject to* (11)*,* (12) *and* (13) *is $k^2 + k$.*

Proof: Let the minimum be attained at $(\overline{s}_1, \overline{s}_2, \overline{t}_1, \overline{t}_2)$. If $k > \overline{s}_1 \geq \overline{t}_2 > 1$, then, by (13), $(\overline{s}'_1, \overline{s}'_2, \overline{t}'_1, \overline{t}'_2) := (\overline{s}_1 + 1, \overline{s}_2, \overline{t}_1, \overline{t}_2 - 1)$ satisfies (11), (12) and (13) and we have $\overline{s}'_1\overline{t}'_2 + \overline{s}'_2\overline{t}'_1 = \overline{s}_1\overline{t}_2 + \overline{t}_2 - \overline{s}_1 - 1 + \overline{s}_2\overline{t}_1 < \overline{s}_1\overline{t}_2 + \overline{s}_2\overline{t}_1$, a contradiction. So either $\max\{\overline{s}_1, \overline{t}_2\} = k$ or $\min\{\overline{s}_1, \overline{t}_2\} = 1$. Similarly, either $\max\{\overline{s}_2, \overline{t}_1\} = k$ or $\min\{\overline{s}_2, \overline{t}_1\} = 1$. If one of $\overline{s}_1, \overline{s}_2, \overline{t}_1, \overline{t}_2$ equals 1, then, by (11) and (12), the others equal k and we have $\overline{s}_1\overline{t}_2 + \overline{s}_2\overline{t}_1 = k^2 + k$. Otherwise, by (11), we have $\overline{s}_1\overline{t}_2 + \overline{s}_2\overline{t}_1 = k(\min\{\overline{s}_1, \overline{t}_2\} + \min\{\overline{s}_2, \overline{t}_1\}) = k(3k + 1 - 2k) = k^2 + k$. ■

By Claims 7 and 8 and (8), we have $k^2 + k \leq s_1t_2 + s_2t_1 = |\delta_G(S, T)| \leq k^2 - k$. Then, by $k \geq 1$, we have a contradiction that finishes the proof of Theorem 6. ■

We mention that the previous proof can be easily modified to show that the bipartite graphs obtained from $K_{2k+1,2k+1}$ by deleting a perfect matching are good for all $k \geq 1$.

3.2. Incidence graphs of projective planes

Let G be the incidence graph of a projective plane of order $2k - 1$. It is well-known that G is a simple connected $2k$-regular bipartite graph with unique color classes V_1 and V_2 both being of size $(2k - 1)^2 + (2k - 1) + 1 = 4k^2 - 2k + 1$. The main property of G is the following:

$$\text{any two vertices in } V_i \text{ have exactly one common neighbor for } i \in \{1, 2\}. \tag{14}$$

Theorem 9. *The incidence graph $G = (V_1, V_2; E)$ of a projective plane of order $2k - 1$ is good for all $k \geq 1$.*

Proof: We assume for a contradiction that G is bad. Then, by Proposition 3, there exist an orientation D of G and a partition of $V_1 \cup V_2$ into non-empty sets Z, S and T such that (1) – (6) are satisfied.

For $i = 1, 2$, let $\boldsymbol{S_i}$, $\boldsymbol{T_i}$, $\boldsymbol{Z_i}$ be $V_i \cap S, V_i \cap T$ and $V_i \cap Z$, respectively, and let $\boldsymbol{s_i} := |S_i|$, $\boldsymbol{t_i} := |T_i|$ and $\boldsymbol{z_i} := |Z_i|$. By (4), we have either $s_1 \leq t_1$ or $s_2 \leq t_2$, say $s_1 \leq t_1$. We define a function $\boldsymbol{f}$: $S_1 \times T_1 \to Z_2 \cup \delta_G(S, T)$ as follows: for $s \in S_1$ and $t \in T_1$, by (14), exactly one common neighbor $v \in V_2$ exists, let $f(s, t) := v$ if $v \in Z_2$, vt if $v \in S_2$ and sv if $v \in T_2$.

Claim 10. $|f^{-1}(Z_2 \cup \delta_G(S,T))| \leq z_2k^2 + |\delta_G(S,T)|(2k-1)$.

Proof: For a vertex $z \in Z_2$, $d_G(z,S_1) + d_G(z,T_1) \leq d_G(z) = 2k$. Then $d_G(z,S_1)d_G(z,T_1) \leq k^2$, so we have $|f^{-1}(z)| \leq k^2$. For an edge $e = uv \in \delta_G(S,T)$ with $u \in V_1$, $f(x,y) = uv$ implies that either $x = u$ and $y \in N_G(v) \setminus \{u\}$ or $y = u$ and $x \in N_G(v) \setminus \{u\}$, so in both cases we have $|f^{-1}(e)| \leq d_G(v) - 1 = 2k-1$. The last two inequalities imply the claim. ■

Since G is bipartite, (6) implies that $s_2 \geq 2$, and hence, by (1), (3), and (14), S_2 has at least $k+k-1$ neighbors in $S_1 \cup Z_1$. Then, by $z_1 \leq k-1$, we have $s_1 \geq 2k-1-z_1 \geq k$. By $s_1 \leq t_1$, (9) applied to (s_1,t_1,k), $|V_1| = 4k^2-2k+1$, by definition of f, Claim 10, $s_1 \leq \frac{1}{2}|V_1| \leq 2k^2$, (2), (8) and $k \geq 1$, we have $k(4k^2-3k+1) \leq s_1(4k^2-2k+1-s_1) = s_1(t_1+z_1) = |f^{-1}(Z_2 \cup \delta_G(S,T))| + s_1z_1 \leq z_2k^2 + |\delta_G(S,T)|(2k-1) + 2k^2z_1 \leq 2k^2(k-1) + (k^2-k)(2k-1) = k(4k^2-5k+1)$, a contradiction that finishes the proof of Theorem 9. ■

3.3. Line graphs of regular complete bipartite graphs

Let us consider the regular complete bipartite graph $K_{k+1,k+1}$ and denote its bipartition classes by $\{x_1,\ldots,x_{k+1}\}$ and $\{y_1,\ldots,y_{k+1}\}$. We denote by $\boldsymbol{G}$ its line graph: the vertex set of G is the set $\{(x_i,y_j) : 1 \leq i,j \leq k+1\}$ and two vertices (x_i,y_j) and $(x_{i'},y_{j'})$ are connected by an edge if $i = i'$ or $j = j'$. We mention that G is also called *Rook graph*. The graph G for $k=2$ is given in Figure 2. Note that G is $2k$-regular.

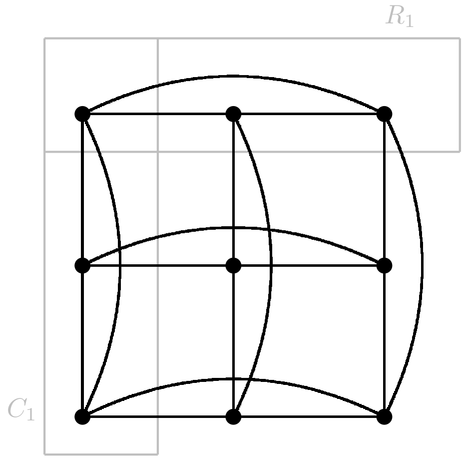

Figure 2: The line graph of $K_{3,3}$, the row R_1 and the column C_1.

By a *row* R_i (resp. *column* C_j) of G we denote the vertex set $\{(x_i,y_j) : 1 \leq j \leq k+1\}$ (resp. $\{(x_i,y_j) : 1 \leq i \leq k+1\}$). The set of rows (resp. columns) is denoted by $\boldsymbol{\mathcal{R}}$ (resp. $\boldsymbol{\mathcal{C}}$). By a *line* we mean a row or a column. The set of lines is denoted by $\boldsymbol{\mathcal{L}}$. Observe that $\mathcal{R}$ contains $k+1$ rows, $\mathcal{C}$ contains $k+1$ columns, $\mathcal{L}$ contains $2k+2$ lines and every line contains $k+1$ vertices. Note that, by construction, it follows that

$$\text{each line of } G \text{ is a clique of } G, \tag{15}$$

$$\text{a line and a stable set of } G \text{ have at most one vertex in common.} \tag{16}$$

It is well-known (and can easily be derived from Kőnig's theorem [7] on edge-colorings of bipartite graphs) that G is a perfect graph. This means that every induced subgraph H of G has a vertex coloring with $\omega(H)$ colors, where $\boldsymbol{\omega}(H)$ denotes the size of a maximum clique of H. Our proof will use the perfectness of G.

Theorem 11. *The line graph G of the complete bipartite graph $K_{k+1,k+1}$ is good for all $k \geq 1$.*

Proof: We assume for a contradiction that G is bad. Then, by Proposition 3, there exist an orientation D of G and a partition of $V(G)$ into non-empty sets Z, S and T such that (1), (2) and (3) are satisfied. For a line $L_i \in \mathcal{L}$, let $\boldsymbol{s_i}$, $\boldsymbol{t_i}$ and $\boldsymbol{z_i}$ denote $|L_i \cap S|, |L_i \cap T|$ and $|L_i \cap Z|$, respectively. Since $|L_i| = k+1$, the following holds:

$$s_i + t_i + z_i = k + 1. \tag{17}$$

Let $\boldsymbol{\mathcal{R}_S}$ (resp. $\boldsymbol{\mathcal{R}_T}$) be the set of rows that are disjoint from T (resp. S). The column classes $\boldsymbol{\mathcal{C}_S}$ and $\boldsymbol{\mathcal{C}_T}$ are similarly defined. Let $\boldsymbol{\mathcal{L}_S} := \mathcal{R}_S \cup \mathcal{C}_S$, $\boldsymbol{\mathcal{L}_T} := \mathcal{R}_T \cup \mathcal{C}_T$ and $\boldsymbol{\mathcal{L}'}$ the rest of the lines.

Note that, by definition, we have

$$\text{the intersection of a line of } \mathcal{L}_S \text{ and a line of } \mathcal{L}_T \text{ is in } Z. \tag{18}$$

In the first part of the proof we show that $\mathcal{L}_S$ or $\mathcal{L}_T$ contains at least half of the lines. We first provide a lower bound on the number of lines in $\mathcal{L}_S \cup \mathcal{L}_T$.

Claim 12. *$\mathcal{L}_S \cup \mathcal{L}_T$ contains at least $k+2$ lines.*

Proof: Since each line L_i in $\mathcal{L}'$ intersects both S and T, we may apply (9) to $(s_i, t_i, 1)$ and we get, by (15) and (17), that L_i contains at least $s_i + t_i - 1 = k - z_i$ edges between S and T. Then, by (8), since the $G[L_i]$'s are edge-disjoint, since a vertex belongs to two lines and by (2), we have $(k-1)k \geq |\delta_G(S,T)| \geq \sum_{L_i \in \mathcal{L}'} (k - z_i) \geq |\mathcal{L}'|k - 2|Z| > (|\mathcal{L}'| - 2)k$, thus $|\mathcal{L}'| \leq k$. Hence, by $|\mathcal{L}| = 2k+2$, we have $|\mathcal{L}_S| + |\mathcal{L}_T| = |\mathcal{L}| - |\mathcal{L}'| \geq (2k+2) - k = k+2$. ∎

Now we show in several steps that one of $\mathcal{L}_S$ and $\mathcal{L}_T$ is almost empty.

Claim 13. *One of $\mathcal{R}_S, \mathcal{R}_T, \mathcal{C}_S$ and $\mathcal{C}_T$ is empty.*

Proof: Suppose for a contradiction that none of $\mathcal{R}_S, \mathcal{R}_T, \mathcal{C}_S$ and $\mathcal{C}_T$ are empty. Then, by (9) applied to $(|\mathcal{R}_S|, |\mathcal{C}_T|, 1)$ and to $(|\mathcal{R}_T|, |\mathcal{C}_S|, 1)$, Claim 12, (2) and (18), we have $|\mathcal{R}_S||\mathcal{C}_T| + |\mathcal{R}_T||\mathcal{C}_S| \geq (|\mathcal{R}_S| + |\mathcal{C}_T| - 1) + (|\mathcal{R}_T| + |\mathcal{C}_S| - 1) = |\mathcal{L}_S| + |\mathcal{L}_T| - 2 \geq (k+2) - 2 > |Z| \geq |\mathcal{R}_S||\mathcal{C}_T| + |\mathcal{R}_T||\mathcal{C}_S|$, a contradiction. ∎

By Claim 13, we may suppose that $\mathcal{C}_S$ is empty. Indeed, by symmetry of G, we can exchange the rows and columns of G if needed, we may hence suppose that one of $\mathcal{C}_S$ and $\mathcal{C}_T$ is empty. Eventually exchanging the role of S and T and reversing the arcs of D, we may suppose that $\mathcal{C}_S$ is empty.

Claim 14. *At most one column contains at least k vertices of S.*

Proof: Suppose there exist two columns C_i and C_j such that $s_i, s_j \geq k$. By $\mathcal{C}_S = \emptyset$, we have $t_i, t_j \geq 1$. Then, by (17) and $z_i \geq 0$, we have $s_i, s_j = k$ and $t_i, t_j = 1$. Let $X := T \cap (C_i \cup C_j)$. Note that $|X| = 2$, $X \subseteq T$ and $(C_i \cup C_j) \setminus X \subseteq S$. So, by (3), all the neighbors of X in C_i and C_j are in-neighbors of X, and hence all the arcs leaving X enter columns different from C_i and C_j. Then, by $s_i = s_j = k$, (15), (1), $|\mathcal{C}| = k+1$ and since there exists exactly one edge between any vertex u and any column not containing u, we have $2k \leq d_D^-(X) = d_D^+(X) \leq 2(k-1)$, a contradiction. ∎

Claim 15. *$\mathcal{L}_S$ contains at most one line.*

Proof: Suppose for a contradiction that $|\mathcal{L}_S| \geq 2$. Since $\mathcal{C}_S$ is empty, we have $|\mathcal{R}_S| \geq 2$. Then, for every column C_j, we have $s_j + z_j \geq |\mathcal{R}_S| \geq 2$. By Claim 14, at most one column C_i satisfies $s_i \geq k$. Thus, by (17), we have $t_j + z_j = (k+1) - s_j \geq (k+1) - (k-1) = 2$ for every column $C_j \neq C_i$. So we may apply (9) to $(s_j, t_j, 2 - z_j)$ and, by (15) and (17), we get that every column $C_j \in \mathcal{C}' := \mathcal{C} \setminus (\mathcal{C}_T \cup \{C_i\})$ contains at least $(2 - z_j)(k-1)$ edges between S and T. By (18), the columns in $\mathcal{C}_T$ contain at least $|\mathcal{R}_S||\mathcal{C}_T|$ vertices of Z. Then, by (8), since the $G[C_j]$'s are edge-disjoint, $|\mathcal{C}| = k+1$, by (2) and $|\mathcal{R}_S| \geq 2$, we have $(k-1)k \geq |\delta_G(S,T)| \geq \sum_{C_j \in \mathcal{C}'} |\delta_{G[C_j]}(S,T)| \geq \sum_{C_j \in \mathcal{C}'} (2 - z_j)(k-1) \geq (k-1)\big(2(k - |\mathcal{C}_T|) - ((k-1) - |\mathcal{R}_S||\mathcal{C}_T|)\big) > (k-1)(k + (|\mathcal{R}_S| - 2)|\mathcal{C}_T|) \geq (k-1)k$, a contradiction. ∎

We can now see that $\mathcal{L}_T$ contains at least half of the lines. Indeed, Claims 12 and 15 imply that

Claim 16. *$\mathcal{L}_T$ contains at least $k+1$ lines.*

In the second part of the proof our goal is to give an upper bound on the size of S. In order to do that we consider a particular vertex-coloring of $\boldsymbol{H} := G[S]$ and show that each color class is small. Since G is a perfect graph, the vertices of H can be colored by $\omega(H)$ colors. Let us choose a vertex-coloring $\boldsymbol{\mathcal{I}}$ with $\omega(H)$ colors that minimizes the number of color classes of size 1.

For $X \subseteq S$, let $\boldsymbol{p(X)} := |\delta_D^-(S) \cap \delta_D^-(X)| - |\delta_D^+(S) \cap \delta_D^+(X)|$. Since $\mathcal{I}$ is a partition of S and by (1), we have $\sum_{I\in\mathcal{I}} p(I) = p(S) = |\delta_D^-(S)| - |\delta_D^+(S)| = 0$, and hence there exists a color class $\boldsymbol{I} \in \mathcal{I}$ for which $p(I) \geq 0$. Now we are ready to provide upper bounds on the sizes of the color classes.

Claim 17. *The color class I contains exactly one vertex.*

Proof: Let $\boldsymbol{Z'}$ be the set of vertices in Z contained in a line of $\mathcal{L}_T$ and $\boldsymbol{Z''}= Z \setminus Z'$. Since I is a stable set, by (16), each vertex that belongs to a line in $\mathcal{L}_T$ has at most one neighbor in I and each vertex of Z'' has at most two neighbors in I. Let v be a vertex of I. By $I \subseteq S$, v is not in a line of $\mathcal{L}_T$. It follows, by (15) and Claim 16, that v has at least $|\mathcal{L}_T| \geq k+1$ neighbors in the lines of $\mathcal{L}_T$. Hence I has at least $|I|(k+1)$ neighbors in the lines of $\mathcal{L}_T$. By (3), each of these neighbors is either a vertex in Z' or an out-neighbor of v in D. Hence $|\delta_D^-(S) \cap \delta_D^-(I)| \leq |Z'| + 2|Z''|$ and $|\delta_D^+(S) \cap \delta_D^+(I)| \geq |I|(k+1) - |Z'|$. Then, the choice of I, $Z' \cup Z'' = Z$ and (2) yield, that $0 \leq p(I) \leq (|Z'| + 2|Z''|) - (|I|(k+1) - |Z'|) = (2 - |I|)(k+1) - 4$, so $|I| = 1$. ■

Claim 18. *Every color class of $\mathcal{I}$ contains at most two vertices.*

Proof: By Claim 17, $I = \{u\}$ for some vertex $\boldsymbol{u}$. Let $\boldsymbol{I'} \in \mathcal{I} \setminus \{I\}$. If there exists a vertex w in I' that is not adjacent to u, then replacing $\{u\}$ and I' by $\{u, w\}$ and $I' \setminus \{w\}$ in $\mathcal{I}$ yields a vertex-coloring of H with $\omega(H)$ colors, so the choice of $\mathcal{I}$ implies that $|I'| \leq 2$. If u is adjacent to each vertex of the stable set I', then, by (16), we have $|I'| \leq 2$ again. ■

Claim 19. *S contains at most $2\omega(H) - 1$ vertices.*

Proof: Since $\mathcal{I}$ is a partition of S, by Claims 17 and 18 and $|I| = \omega(H)$, we have $|S| = |I| + \sum_{I'\in\mathcal{I}\setminus\{I\}} |I'| \leq 1 + (|\mathcal{I}| - 1)2 = 2\omega(H) - 1$. ■

Since each clique of G is contained in a line, we can choose a line $\boldsymbol{L_i}$ that contains $\omega(H)$ vertices of S. Note that $s_i \geq 1$. Let $\boldsymbol{S_i} := L_i \cap S$, $\boldsymbol{S_i'} := N_G(S_i) \cap S$ and $\boldsymbol{S_i''} := S \setminus (S_i \cup S_i')$.

Finally, in order to derive a contradiction, we provide bounds for $\delta_G(S, T)$ and $\delta_G(S, Z)$.

Claim 20. $s_it_i + s_i(k+1) - (|Z| - z_i) - s_i - |S_i'| + |S_i'| + |S_i''| \leq d_G(S, T)$.

Proof: By (15), we have $s_it_i = |\delta_G(S_i, T \cap L_i)|$. Since there are $s_i(k+1)$ elements in the lines that intersect L_i in an element of S and at most $|Z \setminus L_i| + s_i + |S_i'|$ of them belong to $Z \cup S$, by the fact that Z, S and T is a partition of $V(G)$ and (15), we have $s_i(k+1) - (|Z| - z_i) - s_i - |S_i'| \leq |\delta_G(S_i, T \setminus L_i)|$. By Claim 15 and (15), we have at least one edge from each vertex of $S_i' \cup S_i''$ to T, that is $|S_i'| + |S_i''| \leq |\delta_G(S_i' \cup S_i'', T)|$. By $|\delta_G(S_i, T \cap L_i)| + |\delta_G(S_i, T \setminus L_i)| + |\delta_G(S_i' \cup S_i'', T)| = d_G(S, T)$, the claim follows. ■

Claim 21. $d_G(S, Z) \leq s_iz_i + (|Z| - z_i) + (|Z| - z_i)(|S_i'| + |S_i''|) + |S_i''|$.

Proof: By (15), we have $|\delta_G(S_i, Z \cap L_i)| = s_iz_i$. Since $(Z \setminus L_i) \cap L_i = \emptyset$ and $S_i \subseteq L_i$, every element of $Z \setminus L_i$ has at most one neighbor in S_i and hence $|\delta_G(S_i, Z \setminus L_i)| \leq |Z \setminus L_i|$. Since $N_G(S_i') \cap L_i \subseteq S_i$ and $(L_i \cap Z) \cap S_i = \emptyset$, there is no edge between $L_i \cap Z$ and S_i', that is $|\delta_G(S_i', Z \cap L_i)| = 0$. Clearly, we have $|\delta_G(S_i' \cup S_i'', Z \setminus L_i)| \leq (|Z| - z_i)(|S_i'| + |S_i''|)$. Since $S_i'' \cap L_i = \emptyset$ and $L_i \cap Z \subseteq L_i$, every element of S_i'' has at most one neighbor in $L_i \cap Z$ and hence $|\delta_G(S_i'', Z \cap L_i)| \leq |S_i''|$. By $d_G(S, Z) = |\delta_G(S_i, Z \cap L_i)| + |\delta_G(S_i, Z \setminus L_i)| + |\delta_G(S_i', Z \cap L_i)| + |\delta_G(S_i' \cup S_i'', Z \setminus L_i)| + |\delta_G(S_i'', Z \cap L_i)|$, the claim follows. ■

Now we are ready to finish the proof of Theorem 11. Claims 20 and 21, (3) and (1) yield that $s_it_i + s_i(k+1) - (|Z| - z_i) - s_i - |S_i'| + |S_i'| + |S_i''| \le d_G(S,T) \le d_D^+(S) = d_D^-(S) \le d_G(S,Z) \le s_iz_i + (|Z| - z_i) + (|Z| - z_i)(|S_i'| + |S_i''|) + |S_i''|$. Then, by (17), (2), $|S_i'| + |S_i''| = |S| - s_i$, Claim 19, $\omega(H) = s_i$, $t_i \ge 0$ and $s_i \ge 1$, we have $0 \ge s_i(t_i + k - z_i) - (|Z| - z_i)(|S| - s_i + 2) \ge s_i(2t_i + s_i - 1) - (s_i + t_i - 2)(s_i + 1) = t_i(s_i - 1) + 2 \ge 2$, a contradiction. This finishes the proof of Theorem 11. ■

3.4. Line graphs of complete graphs

Let us consider the complete graph K_{k+2} and denote its vertex set by $\boldsymbol{U}$ and its line graph by $\boldsymbol{G}$. Note that a pair of adjacent (resp. non-adjacent) edges in K_{k+2} corresponds to a pair of adjacent (resp. non-adjacent) vertices in G. Since each edge of K_{k+2} is adjacent to exactly $2k$ other edges, G is $2k$-regular.

Theorem 22. *The line graph G of K_{k+2} is good for all $k \ge 1$.*

Proof: We assume for a contradiction that G is bad. Clearly, $k \ge 2$. Then, by Proposition 3, there exist an orientation D of G and a partition of $V(G)$ into non-empty sets Z, S and T such that (1) – (6) are satisfied.

For a vertex set X of G, we denote by $\boldsymbol{E_X}$ the corresponding edge set of K_{k+2}. For a vertex $v \in U$, let $\boldsymbol{s_v}$, $\boldsymbol{t_v}$ and $\boldsymbol{z_v}$ be the number of edges incident to v that are in E_S, E_T and E_Z, respectively. We call an ordered pair (e, f) of edges of K_{k+2} an (S,T)-*pair* if $e \in E_S$ and $f \in E_T$. The sets of adjacent and non-adjacent (S,T)-pairs are denoted by $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, respectively. Observe that $|P_1| = d_G(S,T)$ and $|S||T| = |P_1| + |P_2|$.

First we provide an upper bound on $|P_1|$.

Claim 23. $|P_1| \le k^2 - k - \max\{0, k-4\}$.

Proof: Note that a vertex $v \in U$ provides exactly $\binom{z_v}{2}$ edges in $G[Z]$. Then, by (10) and (2), we have $i_G(Z) = \sum_{v\in U} \binom{z_v}{2} \ge \sum_{v \in U}(z_v - 1) = 2|E_Z| - |U| = 2(k-1) - (k+2) = k - 4$. Thus, by (8), we have $|P_1| = |\delta_G(S,T)| \le k^2 - k - i_G(Z) \le k^2 - k - \max\{0, k-4\}$. ■

We next prove an upper bound on $|P_2|$.

Claim 24. $2|P_2| \le (k-1)|P_1| + k^2 - 3k + 2$.

Proof: A 4-cycle of K_{k+2} is called *special* if it contains a non-adjacent (S,T)-pair. Let $\mathcal{C}$ be the set of special cycles. A special cycle is said to be of *type* i if it contains i edges of E_Z for $i = 0, 1, 2$. Let $\boldsymbol{n_i}$ denote the number of special cycles of type i for $i = 0, 1, 2$.

Note that every special cycle of type 1 or 2 contains exactly one non-adjacent (S,T)-pair, every special cycle of type 0 contains at most 2 non-adjacent (S,T)-pairs and every non-adjacent (S,T)-pair is part of exactly 2 special cycles. It follows that $2|P_2| = \sum_{p \in P_2} \sum_{p \subsetneq C \in \mathcal{C}} 1 = \sum_{C \in \mathcal{C}} \sum_{p \in P_2 \cap C} 1 \le 2n_0 + n_1 + n_2$.

Since every special cycle of type i contains $2 - i$ adjacent (S,T)-pairs for $i = 0, 1, 2$ and every adjacent (S,T)-pair is contained in exactly $(k-1)$ 4-cycles, we have $2n_0 + n_1 = \sum_{C \in \mathcal{C}} \sum_{p \in P_1 \cap C} 1 = \sum_{p \in P_1} \sum_{p \subsetneq C \in \mathcal{C}} 1 \le \sum_{p \in P_1}(k-1) = (k-1)|P_1|$.

Observe that every special cycle of type 2 contains 2 non-adjacent edges of E_Z, every pair of non-adjacent edges is contained in exactly two 4-cycles and there are at most $\binom{k-1}{2}$ pairs of non-adjacent edges of E_Z. This implies that $n_2 \le 2\binom{k-1}{2} = k^2 - 3k + 2$.

The above inequalities imply the claim. ■

We use the previous results to show an upper bound on $|S|$.

Claim 25. *$|S| \le k$.*

Proof: Otherwise, by (4), we have $|T| \geq |S| \geq k+1$. By (2), we have $|S|+|T| = \binom{k+2}{2} - (k-1)$. Then, by (9) applied to $(|S|, |T|, k+1)$, we have $|S||T| \geq (k+1)(\binom{k+2}{2} - 2k) = \frac{k^3+k+2}{2}$. Then Claims 24 and 23 and $k \geq 1$ yield $k^3+k \leq 2|S||T|-2 = 2|P_2|+2|P_1|-2 \leq (k+1)|P_1|+k^2-3k \leq (k+1)(k^2-k-\max\{0, k-4\})+k^2-3k = k^3+k-(5k-k^2)-\max\{0, k^2-3k-4\} = k^3+k-\max\{k(5-k), 2(k-2)\} < k^3+k$, a contradiction. ■

The following result shows that the edges of E_S are adjacent to many edges of $E_{S\cup Z}$.

Claim 26. *For every $uv \in E_S$, $s_u + z_u + s_v + z_v \geq k+3$.*

Proof: By (1), (3) and (5), the vertex of D that corresponds to uv has k in-neighbors in $S \cup Z$ and at least one out-neighbor in S in D and their corresponding edges in K_{k+2} are incident to u or v. As uv is counted in s_u and s_v, we obtain that $s_u + z_u + s_v + z_v \geq k+3$. ■

The next result shows that S forms a clique in G.

Claim 27. *The edges of E_S are pairwise adjacent.*

Proof: Suppose that E_S contains two non-adjacent edges v_1v_2 and v_3v_4. Note that K_{k+2} has 6 edges having both ends in $\{v_1, v_2, v_3, v_4\}$. Applying Claim 26 to both v_1v_2 and v_3v_4 and using Claim 25 and (2), we obtain $2(k+3) \leq \sum_{i=1}^{4}(s_{v_i} + z_{v_i}) \leq |E_S| + |E_Z| + 6 \leq 2k+5$, a contradiction. ■

Claim 28. *The edges of E_S do not form a triangle in K_{k+2}.*

Proof: Suppose that E_S forms a triangle on v_1, v_2, v_3 in K_{k+2}. Applying Claim 26 to all 3 edges of E_S, we count the edges of E_S twice and the edges of E_Z at most once, so we get $3(k+3) \leq 2\sum_{i=1}^{3}(s_{v_i} + z_{v_i}) \leq 2(2|E_S| + |E_Z|) \leq 2(6 + (k-1))$, that contradicts $k \geq 2$. ■

By Claims 27 and 28, the edges of E_S are all incident to a vertex $\boldsymbol{v}$ in K_{k+2}. Let $\boldsymbol{Q}$ be the clique of size $k+1$ in G that corresponds to the set of edges incident to v in K_{k+2}. Note that $|S| = |Q \cap S| = s_v$, $|Q \cap T| = t_v$ and $|Q \cap Z| = z_v$. Since every edge of E_Z that is not incident to v is adjacent to at most 2 edges of E_S in K_{k+2}, each vertex of $Z \setminus Q$ is adjacent to at most 2 vertices of S in G. This implies, by (3), that $d_D^-(S) \leq 2|Z \setminus Q| + s_v z_v$. Then, by (6), (1), $s_v = |S| \geq 2$, (2), $|Q| = k+1$ and (10), we have $0 = \sum_{u\in S}(d_D^-(u) - k) = d_D^-(S) + \binom{|S|}{2} - |S|k \leq 2|Z \setminus Q| + s_v z_v + \binom{s_v}{2} - s_v(s_v - 1 + t_v + z_v) \leq 2(k-1-z_v) - 2t_v - \binom{s_v}{2} = 2(s_v - 2) - \binom{s_v}{2} < 0$, a contradiction. This finishes the proof of Theorem 22. ■

3.5. Hypercubes

In this subsection we provide a short self-contained proof for Theorem 1 that is restated below. Let us recall that Q_k has 2^k vertices and Q_k is k-regular.

Theorem 29. *The hypercube Q_{2k} is good for all $k \geq 1$.*

The key ingredient of the proof of Theorem 29 in [8] is a lemma proved by the authors of [8] stating that $|N_{Q_{2k}}(X)| \geq k \min\{k, |X|+1\}$ for all $X \subseteq V(Q_{2k})$ with $1 \leq |X| \leq 2^{2k-1}$. The following lemma extends this for dimension of arbitrary parity. Our contribution is an elementary proof of Lemma 30.

Lemma 30. *For all $X \subseteq V(Q_k)$,*

(a) $|N_{Q_k}(X)| \geq \lfloor \frac{k}{2} \rfloor (|X|+1)$ *if* $1 \leq |X| \leq \lfloor \frac{k}{2} \rfloor$,

(b) $|N_{Q_k}(X)| \geq \lfloor \frac{k}{2} \rfloor \lceil \frac{k}{2} \rceil$ *if* $\lfloor \frac{k}{2} \rfloor \leq |X| \leq 2^{k-1}$.

First we show how to prove Theorem 29 using Lemma 30 as pointed out in [8].

Proof: (of Theorem 29) We assume for a contradiction that Q_{2k} is bad. Then, by Proposition 3, there exist an orientation D of Q_{2k} and a partition of $V(Q_{2k})$ into non-empty sets Z, S and T such that (1), (2) and (3) are satisfied. Then, by (7), (1), (3), Lemma 30 and (2), we have $k \min\{|Z|, |S|\} \geq d_D^-(S) = d_D^+(S) \geq |N_{2k}(S)| - |Z| \geq k \min\{k, |S|+1\} - k + 1 = k \min\{|Z|, |S|\} + 1$, a contradiction. ■

It is easy to verify that for all positive integers k, the following holds:

$$\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil+\lfloor\frac{k+1}{2}\rfloor=\lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil. \tag{19}$$

We introduce two functions $f,g:\mathbb{Z}_+\to\mathbb{Z}_+$: let $\boldsymbol{f(k)}:=\lfloor\frac{k}{2}\rfloor(\lceil\frac{k}{2}\rceil+1)-1$ and $\boldsymbol{g(k)}:=2^k-f(k)$. We need the following inequality for $g(k)$.

Proposition 31. *For $k\geq 1$, $2g(k)+2-2^k\geq\lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil$.*

Proof: We first show by induction that $2^k\geq 4\lceil\frac{k}{2}\rceil-\lfloor\frac{k}{2}\rfloor+1$ for all $k\geq 2$. For $k=2$ it is true. If it is true for some $k\geq 2$, then, by the induction hypothesis, it is true for $k+1: 2^{k+1}=2^k+2^k\geq 4+4\lceil\frac{k}{2}\rceil-\lfloor\frac{k}{2}\rfloor+1\geq 4\lceil\frac{k+1}{2}\rceil-\lfloor\frac{k+1}{2}\rfloor+1$.

By (19), the inequality of the claim is equivalent to $2^k+4\geq 3\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil+2\lfloor\frac{k}{2}\rfloor+\lceil\frac{k}{2}\rceil$ for $k\geq 1$. For $k=1,2$ it is true. If it is true for some $k\geq 2$, then, by the above inequality, the induction hypothesis and (19), it is true for $k+1: 2^{k+1}+4=2^k+2^k+4\geq 4\lceil\frac{k}{2}\rceil-\lfloor\frac{k}{2}\rfloor+1+3\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil+2\lfloor\frac{k}{2}\rfloor+\lceil\frac{k}{2}\rceil=3\lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil+2\lfloor\frac{k+1}{2}\rfloor+\lceil\frac{k+1}{2}\rceil$.
■

Proof: (of Lemma 30) (a) First we prove a lower bound on the number of neighbors of an arbitrary vertex set X of Q_k and then we show how this yields (a).

Claim 32. $|N_{Q_k}(X)|\geq\sum_{v\in X}d_{Q_k}(v)-2\binom{|X|}{2}$ *for all $X\subseteq V(Q_k)$.*

Proof: Let $H:=Q_k[X]$ and $A_v:=N_{Q_k}(v)\setminus X$ for all $v\in X$. It is known by the sieve formula that $|\cup_{v\in X}A_v|-\sum_{v\in X}|A_v|+\sum_{u,v\in X}|A_u\cap A_v|\geq 0$. Note that $|\cup_{v\in X}A_v|=|N_{Q_k}(X)|$, $\sum_{v\in X}|A_v|=\sum_{v\in X}(d_{Q_k}(v)-d_H(v))=\sum_{v\in X}d_{Q_k}(v)-2|E(H)|$. Since $|N_{Q_k}(\{u\})\cap N_{Q_k}(\{v\})|=0$ if $uv\in E(Q_k)$ and ≤ 2 if $uv\in E(\overline{Q}_k)$, we have $\sum_{u,v\in X}|A_u\cap A_v|\leq\sum_{uv\in E(H)}0+\sum_{uv\in E(\overline{H})}2=2|E(\overline{H})|$. By $|E(H)|+|E(\overline{H})|=\binom{|X|}{2}$, the claim follows.
■

Let $X\subseteq V(Q_k)$ with $1\leq|X|\leq\lfloor\frac{k}{2}\rfloor$. By Claim 32 and the k-regularity of Q_k, we have $|N_{Q_k}(X)|\geq\sum_{v\in X}d_{Q_k}(v)-2\binom{|X|}{2}=|X|(k+1-|X|)\geq\lfloor\frac{k}{2}\rfloor(|X|+1)+(\lfloor\frac{k}{2}\rfloor-|X|)(|X|-1)\geq\lfloor\frac{k}{2}\rfloor(|X|+1)$.

(b) We prove this case by induction on k. For $k=1$, it is trivial. For $k=2$, it follows since Q_2 is connected. Suppose that the lemma is true for some $k\geq 2$. We use that Q_{k+1} can be obtained from two disjoint copies $\boldsymbol{Q^1}$ and $\boldsymbol{Q^2}$ of Q_k by adding an edge between the corresponding vertices of Q^1 and Q^2. Let $\boldsymbol{X}\subsetneq V(Q_{k+1})$ with $\lfloor\frac{k+1}{2}\rfloor\leq|X|\leq 2^k$, $\boldsymbol{X_i}:=X\cap V(Q^i)$, $\boldsymbol{X_i^c}:=V(Q^i)\setminus X_i$, $\boldsymbol{X_i^*}:=X_i^c\setminus N_{Q^i}(X_i)$. By the construction of Q_{k+1} from Q^1 and Q^2, we have, for $i\in\{1,2\}$,

$$|N_{Q_{k+1}}(X)\cap V(Q^i)|\geq\max\{|X_{3-i}|-|X_i|,|N_{Q^i}(X_i)|\}. \tag{20}$$

The following claim strengthens the induction hypothesis by relaxing the condition on the size of X_i.

Claim 33. $|N_{Q^i}(X_i)|\geq\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil$ *if* $\lfloor\frac{k}{2}\rfloor\leq|X_i|\leq g(k)$.

Proof: For $|X_i|\leq 2^{k-1}$, by the induction hypothesis, we are done. Otherwise, $|X_i^*|\leq|X_i^c|<2^{k-1}$. For $|X_i^*|\geq\lfloor\frac{k}{2}\rfloor$, by $N_{Q^i}(X_i)\supseteq N_{Q^i}(X_i^*)$ and the induction hypothesis, we have $|N_{Q^i}(X_i)|\geq|N_{Q^i}(X_i^*)|\geq\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil$. For $|X_i^*|\leq\lfloor\frac{k}{2}\rfloor-1$, by $2^k-|X_i^c|=|X_i|\leq g(k)=2^k-f(k)$, we have $|N_{Q^i}(X_i)|=|X_i^c|-|X_i^*|\geq f(k)-(\lfloor\frac{k}{2}\rfloor-1)=\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil$. ■

We finish the proof by distinguishing several cases.

Case 1. $1\leq|X_i|\leq\lfloor\frac{k}{2}\rfloor$ for $i=1,2$. By (20), Lemma 30(a), $|X_i|\leq\lfloor\frac{k}{2}\rfloor$ and $|X|\geq\lfloor\frac{k+1}{2}\rfloor$, we have $|N_{Q_{k+1}}(X)|\geq\sum_{i=1}^2|N_{Q^i}(X_i)|\geq\sum_{i=1}^2\lfloor\frac{k}{2}\rfloor(|X_i|+1)\geq\sum_{i=1}^2|X_i|(\lfloor\frac{k}{2}\rfloor+1)=|X|(\lfloor\frac{k}{2}\rfloor+1)\geq\lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil$.

Case 2. $|X_1|\geq g(k)+1$. By (20), $|X|\leq 2^k$ and Proposition 31, we have

$|N_{Q_{k+1}}(X)| \geq |N_{Q_{k+1}}(X) \cap V(Q^2)| \geq |X_1| - |X_2| = 2|X_1| - |X| \geq 2g(k) + 2 - 2^k \geq \lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil$.

Case 3. $\lfloor\frac{k}{2}\rfloor \leq |X_2| \leq |X_1| \leq g(k)$. By (20), Claim 33 and $k \geq 2$, we have
$|N_{Q_{k+1}}(X)| \geq \sum_{i=1}^{2} |N_{Q^i}(X_i)| \geq 2\lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil \geq \lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil$.

Case 4. $1 \leq |X_2| \leq \lfloor\frac{k}{2}\rfloor \leq |X_1| \leq g(k)$. By (20), Claim 33, Lemma 30(a), $k \geq 2$ and (19), we have
$|N_{Q_{k+1}}(X)| \geq \sum_{i=1}^{2} |N_{Q^i}(X_i)| \geq \lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil + \lfloor\frac{k}{2}\rfloor(|X_2|+1) \geq \lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil + \lfloor\frac{k+1}{2}\rfloor = \lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil$.

Case 5. $X_2 = \emptyset$ and $\lfloor\frac{k}{2}\rfloor \leq |X_1| \leq g(k)$. By (20), Claim 33, $|X| \geq \lfloor\frac{k+1}{2}\rfloor$ and (19), we have
$|N_{Q_{k+1}}(X)| \geq |N_{Q^1}(X)| + |N_{Q_{k+1}}(X) \cap V(Q^2)| = |N_{Q^1}(X)| + |X| \geq \lfloor\frac{k}{2}\rfloor\lceil\frac{k}{2}\rceil + \lfloor\frac{k+1}{2}\rfloor = \lfloor\frac{k+1}{2}\rfloor\lceil\frac{k+1}{2}\rceil$.

Up to symmetry of X_1 and X_2, this case distinction is complete. Thus Lemma 30(b) is true for $k+1$. ■

4. Counterexamples for Frank's conjecture

We now come back to the question of characterizing graphs admitting at least one k-vertex-connected orientation. Frank [4] conjectured that an undirected graph $G = (V, E)$ with $|V| > k$ has a k-vertex-connected orientation if and only if for all $X \subseteq V$ with $|X| < k$, $G - X$ is $(2k - 2|X|)$-edge-connected. Durand de Gevigney [2] provided a counterexample to this conjecture on 10 vertices. Here we present a counterexample on 6 vertices. Starting from our example we also present a simple graph counterexample. The idea of the constructions comes from [2, 3].

Let G_1 be the first graph in Figure 3. It is easy to check that for $k = 3$, G_1 satisfies the condition of Frank's conjecture. Suppose now that G_1 has a 3-vertex-connected orientation D_1. Then for any i, $D_1 - v_i - v_{i+2}$ is 1-arc-connected, so v_{i+1} has one grey arc entering and one grey arc leaving. Hence, the grey cycle is oriented as a circuit in D_1. It follows that in $D_1 - v_1 - v_4$ the two arcs between $\{v_2, v_3\}$ and $\{v_5, v_6\}$ form a directed cut and hence D_1 is not 3-vertex-connected. Thus G_1 is a counterexample to Frank's conjecture. Note that since G_1 is 6-regular and has no 3-vertex-connected orientation, G_1 is bad.

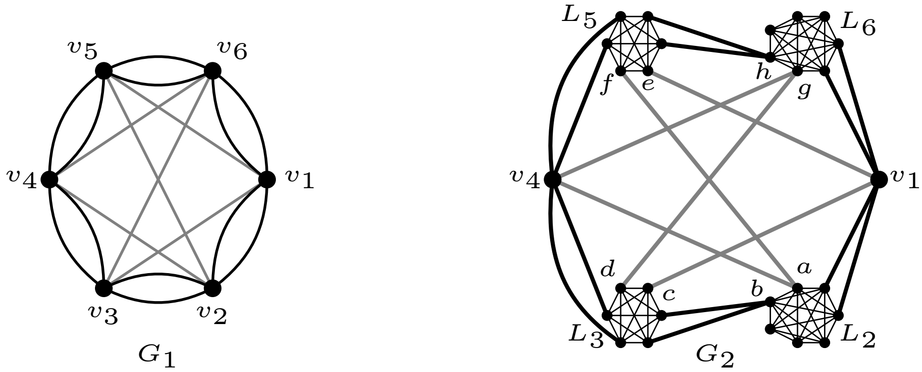

Figure 3: Counterexamples to Frank's conjecture.

We now construct a simple graph G_2 which is a counterexample to Frank's conjecture. We replace the vertices v_2, v_3, v_5 and v_6 in G_1 by appropriate cliques, see Figure 3. Note that G_2 is a simple graph. It is easy to check that for $k = 3$, G_2 satisfies the condition of Frank's conjecture. Suppose now that G_2 has a 3-vertex-connected orientation $D_2 = (V, A)$. By reversing all arcs if necessary, we may suppose that $gd \in A$. Since $D_2 - b - v_4$ is 1-arc-connected, $cv_1 \in A$. Since $D_2 - a - b$ (resp. $D_2 - g - h$) is 1-arc-connected, one of the two arcs between v_1 and L_2 (resp. L_6) goes from v_1 to L_2 (resp. L_6) and the other one goes from L_2 (resp. L_6) to v_1. Then, since $d^-_{D_2}(v_1) = 3 = d^+_{D_2}(v_1)$, $v_1 e \in A$. Finally, since $D_2 - h - v_4$ is 1-arc-connected, $fa \in A$. It follows that in $D_2 - v_1 - v_4$ the two arcs gd and fa between $L_2 \cup L_3$ and $L_5 \cup L_6$ form a directed cut and hence D_2 is not 3-vertex-connected. Thus the simple graph G_2 is a counterexample to Frank's conjecture.

5. Conclusion

We provided five classes of good graphs in this paper. Further investigations could allow the identification of more classes of good graphs. We are particularly interested in the graph class described below which extends two of the classes of good graphs dealt with in this paper.

Let W be a set of size w. The Hamming graph $H(d,w)$ is the graph with vertex set W^d, where two vertices are adjacent if they differ in exactly one coordinate. Note that $H(1,w)$ is the complete graph K_w, $H(d,2)$ is the hypercube of dimension d and $H(2,w)$ is the line graph of $K_{w,w}$. It is easy to see that $H(d,w)$ is $d(w-1)$-regular. We conjecture that $H(d,w)$ is a good graph whenever $d(w-1)$ is even and $d \geq 2$. This would generalize Theorems 11 and 29.

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