Matroid-rooted packing of arborescences

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Abstract

The problem of matroid-based packing of arborescences was introduced and solved in [3]. Frank [10] reformulated the problem in an extended framework. We proved in [6] that the problem of matroid-based packing of spanning arborescences is NP-complete in the extended framework. Here we show a characterization of the existence of a matroid-based packing of spanning arborescences in the original framework. This leads us to the introduction of a new problem on packing of arborescences with a new matroid constraint. We solve two problems: on the one hand on mixed graphs having a packing of mixed arborescences, on the other hand on dypergraphs having a packing of dyperarborescences such that their roots form a basis in a given matroid, each vertex belongs to exactly k of them and each vertex v is the root of least $f(v)$ and at most $g(v)$ of them.

1 Introduction

Packing arborescences in directed graphs is a fundamental and well-studied problem in graph theory. We introduce in this article a new problem on packing arborescences, called matroid-rooted. It is closely related to the problem of matroid-based packing of arborescences that was earlier introduced in [3].

The basic problem of packing of spanning arborescences with fixed roots is due to Edmonds [4]. Later Frank [7] solved the problem of packing of spanning arborescences with flexible roots. In fact, Frank [7] (and independently Cai [2]) provided a result on (f, g)-bounded packings of spanning arborescences where $f(v)$ is a lower bound and $g(v)$ is an upper bound on the number of v-arborescences in the packing for every vertex v. Durand de Gevigney, Nguyen, Szigeti [3] considered the problem of matroid-based packing of arborescences. In this problem we are given a digraph $D = (V, A)$, a multiset S of vertices in V and a matroid M on S, and we want to have a packing $\mathcal{B}$ of (not necessarily spanning) arborescences such that for every $v \in V$, the set of roots of the arborescences in $\mathcal{B}$ that contain v must form a basis of M. We gave in [3] a characterization of the existence of a matroid-based packing of arborescences.

The above problems were generalized for mixed graphs by Frank [7], Gao,Yang [12], and Fortier et al. [5], also for directed hypergraphs by Frank, Király, Király [11], Hörsch, Szigeti [14], and [5], and even for mixed hypergraphs in [5], [14], and [5], respectively.

Frank [10] reformulated the problem of matroid-based packing of arborescences in the extended framework, where the extended digraph can be obtained from D by adding a new vertex and a new arc from this vertex to each element of S and the matroid is considered not on S but on the corresponding new arcs. We proved in [6] that the problem of matroid-based packing of $\mathsf{spanning}$ arborescences is NP-complete in the extended framework.

Here we show a characterization of the existence of a matroid-based packing of spanning arborescences in the original framework. This leads us to the introduction of a new problem on packing of arborescences with a new matroid constraint. Given a digraph $D = (V, A)$, a multiset S of vertices in V and a matroid M on S, a packing $\mathcal{B}$ of (not necessarily spanning) arborescences is called M-*rooted* if the set of roots of the arborescences in $\mathcal{B}$ is a basis of M. Note that if each arborescence in $\mathcal{B}$ is spanning then the condition of M-based packing coincides with the condition of M-rooted packing. We provide a characterization of the existence of an M-rooted k-regular packing of arborescences. Here k-*regular* means that each vertex must belong to exactly k of the arborescences. Note that if k is equal to the rank of the matroid then the problem is equivalent to an M-rooted packing of spanning arborescences. We will consider two generalization of our problem. The first contribution of this article solves the problem of M-rooted (f, g)-bounded k-regular packing of hyperarborescences in dypergraphs. This result will be obtained from the theory of generalized polymatroids. The second contribution is the solution of the problem of M-rooted (f, g)-bounded k-regular packing of mixed arborescences in mixed graphs. This result will be derived from its directed version, which is the graphic case of the previous result, and a new orientation theorem. Finally, we will propose a conjecture that would give a common generalization of our two results.

The organization of this article is as follows. In Section 2 we provide all the definitions we need. In Section 3 we give the list of known results that are important for this article. In Section 4 we present our new results. In Section 5 we recall the main properties of generalized polymatroids that will be applied to obtain one of our main results. Section 6 contains the proofs of the main results. Finally, our conjecture can be found in Section 7.

2 Definitions

Let $\boldsymbol{V}$ be a finite set. The set of subsets of V is denoted by $\boldsymbol{2^V}$. A *set function* on V is a function defined on 2^V. For an element v of V, the set $\{v\}$ will sometimes be shortened to v. A *subset* of V may contain each element of V at most once. For a subset X of V, $\overline{\boldsymbol{X}}$ denotes $V - X$, the *complement* of X. Two subsets of V are called *intersecting* if their intersection is non-empty. By a *partition* of V we mean a set of disjoint subsets of V whose union is V. More generally, a set of disjoint subsets of V is called a *subpartition* of V. For a subpartition $\mathcal{P}$ of V, we denote by $\cup\boldsymbol{\mathcal{P}}$ the vertex set which is the union of the members of $\mathcal{P}$. A *multiset* of V may contain multiple occurrences of elements. For a multiset S of V and a subset X of V, $\boldsymbol{S_X}$ denotes the multiset consisting of the elements of X with the same multiplicities as in S.

The sets of reals, integers and non-negative integers are denoted by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively. A real vector m on V will be denoted by $m \in \mathbb{R}^V$. The set function $\boldsymbol{\infty_0}$ has the value ∞ everywhere except for the empty set where it has value 0. For $k \in \mathbb{Z}$ and $g \in \mathbb{Z}^V$, we define $\boldsymbol{g_k} \in \mathbb{Z}^V$ as $g_k(v) = \min\{g(v), k\}$ for all $v \in V$.

A set function b on V is called *non-decreasing* if (1) holds and *subcardinal* if (2) holds.

$$b(X) \leq b(Y) \text{ for all } X \subseteq Y \subseteq V, \tag{1}$$

$$b(X) \leq |X| \text{ for all } X \subseteq V. \tag{2}$$

Set functions m, b and p on V are called *modular, submodular* and *supermodular* if for all $X, Y \subseteq V$, (3), (4) and (5) hold, respectively. We say that p is *intersecting supermodular* if

(5) holds for all intersecting subsets X and Y of V.

$$
\begin{aligned}
m(X)+m(Y) &= m(X\cap Y)+(X\cup Y), &(3)\\
b(X)+b(Y) &\geq b(X\cap Y)+b(X\cup Y), &(4)\\
p(X)+p(Y) &\leq p(X\cap Y)+p(X\cup Y). &(5)
\end{aligned}
$$

We use usual notions from matroid theory. Let S be a finite ground set and r a non-negative integer valued function on S such that $r(\emptyset)=0$, r is subcardinal, non-decreasing and submodular. Then $\mathsf{M}=(S,r)$ is called a *matroid*. The function r is called the *rank function* of the matroid M. For a matroid M, its rank function will be denoted by $\boldsymbol{r_{\mathsf{M}}}$. An *independent set* of M is a subset X of S such that $r_{\mathsf{M}}(X)=|X|$. Any subset of an independent set is also independent. The set of independent sets of M is denoted by $\boldsymbol{\mathcal{I}_{\mathsf{M}}}$. A maximal independent set of M is called a *basis* of M. Every basis of a matroid has the same size, namely $r_{\mathsf{M}}(S)$.

Let $\boldsymbol{D}=(V,A)$ be a directed graph, shortly *digraph*, where the non-empty finite set $\boldsymbol{V}$ is the set of *vertices* of D and the finite set $\boldsymbol{A}$ is the set of *arcs* of D. An arc $e=uv$ is an ordered pair of different vertices, where u is the *tail* and v is the *head* of e. For a subset X of V, we say that an arc uv *enters* X if $v\in X$ and $u\in\overline{X}$. The set of arcs in A *entering* X is denoted by $\boldsymbol{\rho_A(X)}$ and the *in-degree* of X is $\boldsymbol{d_A^-(X)}=|\rho_A(X)|$. Similarly, uv *leaves* X if $u\in X$ and $v\in\overline{X}$, the set of arcs in A *leaving* X is denoted by $\boldsymbol{\delta_A(X)}=\rho_A(\overline{X})$ and the *out-degree* of X is $\boldsymbol{d_A^+(X)}=|\delta_A(X)|$. A *subgraph* of D is a digraph obtained from D by deleting some vertices in V and then some arcs in A. A subgraph of D that contains all the vertices of D is called a *spanning subgraph* of D. We call a digraph (U,F) an *arborescence with root* s, shortly s-*arborescence*, if there exists a unique path from s to every $v\in U$.

A set of arc-disjoint subgraphs of D is called a *packing* of subgraphs. Let $\mathcal{B}$ be a packing of arborescences in D. By the *root set* $\boldsymbol{R^{\mathcal{B}}}$ of $\mathcal{B}$ we mean the multiset of the roots of the arborescences in $\mathcal{B}$. The vector $m^{\mathcal{B}}\in\mathbb{Z}_+^V$ such that $m^{\mathcal{B}}(v)=|R_v^{\mathcal{B}}|$ for all $v\in V$ is called the *root vector* of $\mathcal{B}$. We say that $\mathcal{B}$ is k-*regular* if each vertex is contained in exactly k arborescences in $\mathcal{B}$. For $f,g\in\mathbb{Z}_+^V$, $\mathcal{B}$ is called (f,g)-*bounded* if the number of v-arborescences in $\mathcal{B}$ is at least $f(v)$ and at most $g(v)$, that is $f(v)\leq m^{\mathcal{B}}(v)\leq g(v)$ for all $v\in V$. Let S be a multiset of V and M a matroid on S. The function $\boldsymbol{b_{\mathsf{M}}}$ is defined as follows: $b_{\mathsf{M}}(X)=r_{\mathsf{M}}(S_X)$ for all $X\subseteq V$. The packing $\mathcal{B}$ is called M-*rooted* if $R^{\mathcal{B}}$ is a basis of M. We say that the packing $\mathcal{B}$ is M-*based* if for every $v\in V$, the multiset of roots of the arborescences containing v in $\mathcal{B}$ forms a basis of M. Note that if $\mathcal{B}$ is a packing of spanning arborescences then $\mathcal{B}$ is M-rooted if and only if $\mathcal{B}$ is M-based.

Let $\boldsymbol{G}=(V,E)$ be an undirected graph where the non-empty finite set $\boldsymbol{V}$ is the set of *vertices* of G and the finite set $\boldsymbol{E}$ is the set of *edges* of G. An edge e is a pair of different vertices. For a subset X of V, we say that an edge uv *enters* X if $v\in X$ and $u\in\overline{X}$. The number of edges in E entering X, denoted by $\boldsymbol{d_E(X)}$, is called the *degree* of X. We say that G is *simple* if no two edges have the same end-vertices. A graph G is called *bipartite* if there exists a bipartition $\{A,B\}$ of its vertex set such that every edge of G connects a vertex of A to a vertex of B. A bipartite graph G is denoted by $\boldsymbol{(A,B;E)}$, where $\{A,B\}$ is the bipartition of the vertex set of G and E is the edge set of G. For bipartite graph $G=(A,B;E)$, $X\subseteq A$, and $F\subseteq E$, we denote by $\boldsymbol{\Gamma_F(X)}$ the set of vertices that are connected by an edge of F to at least one vertex in X.

Let $\boldsymbol{F}=(V,E\cup A)$ be a *mixed graph*, where V is the set of vertices, E is the set of edges and A is the set of arcs. By *orienting* an edge $uv\in E$, we mean the operation that replaces the edge uv by one of the arcs uv and vu. A mixed graph that has an orientation that is an s-arborescence is called a *mixed s-arborescence*. We say that F has an M-*rooted*/(f,g)-*bounded*/k-*regular* packing of mixed arborescences if E has an orientation $\vec{E}$ such that the digraph $(V,\vec{E}\cup A)$ has an M-rooted/(f,g)-bounded/k-regular packing of arborescences.

Let $\boldsymbol{\mathcal{D}} = (V, \mathcal{A})$ be a directed hypergraph, shortly *dypergraph*, where V is the set of vertices and $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{D}$. A *dyperedge* is an ordered pair (Z, z) such that z is a vertex in V, called the *head*, and Z is a non-empty subset of $V - z$, called the set of *tails*. For $X \subseteq V$, we say that a dyperedge $(Z, z) \in \mathcal{A}$ *enters* X if $z \in X$ and $Z \cap \overline{X} \neq \emptyset$. The set of dyperedges in $\mathcal{A}$ *entering* X is denoted by $\boldsymbol{\rho_{\mathcal{A}}(X)}$ and the *in-degree* of X is $\boldsymbol{d^-_{\mathcal{A}}(X)}$ $= |\rho_{\mathcal{A}}(X)|$. For $k \in \mathbb{Z}_+$, we introduce the function $\boldsymbol{p_{\mathcal{A},k}}$ as follows: $p_{\mathcal{A},k}(X) = k - d^-_{\mathcal{A}}(X)$ for all $\emptyset \neq X \subseteq V$ and 0 for $X = \emptyset$. The operation that replaces a dyperedge (Z, z) by an arc yz where $y \in Z$ is called *trimming*. We say that $\mathcal{D}$ is an *s-hyperarborescence*, if $\mathcal{D}$ can be trimmed to an s-arborescence. We say that $\mathcal{D}$ has an M*-rooted*/(f, g)*-bounded*/k*-regular* packing of hyperarborescences if $\mathcal{D}$ can be trimmed to a digraph that has an M-rooted/(f, g)-bounded/k-regular packing of arborescences.

Let $\boldsymbol{\mathcal{F}} = (V, \mathcal{E} \cup \mathcal{A})$ be a *mixed hypergraph*, where V is the set of vertices, $\boldsymbol{\mathcal{E}}$ is the set of hyperedges and $\boldsymbol{\mathcal{A}}$ is the set of dyperedges of $\mathcal{F}$. A *hyperedge* is a subset of V of size at least two. A hyperedge X *enters* a subset Y of V if $X \cap Y \neq \emptyset \neq \overline{X} \cap Y$. By *orienting* a hyperedge X, we mean the operation that replaces the hyperedge X by a dyperedge $(X - x, x)$ for some $x \in X$. A mixed hypergraph that has an orientation that is an s-hyperarborescence is called a *mixed s-hyperarborescence*. By a *packing* of mixed subhypergraphs in $\mathcal{F}$ we mean a set of mixed subhypergraphs that are hyperedge- and dyperedge-disjoint. For a subpartition $\mathcal{P}$ of subsets of V, we denote by $\boldsymbol{e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ and dyperedges in $\mathcal{A}$ that enter some member of $\mathcal{P}$.

3 Known results

We start by a fundamental result on packing spanning arborescences due to Edmonds [4].

Theorem 1 (Edmonds [4])**.** *Let $D = (V, A)$ be a digraph, $s \in V$, and $k \in \mathbb{Z}_+$. There exists a packing of k spanning s-arborescences in D if and only if*

$$d^-_A(X) \quad \geq \quad k \qquad \text{for every } \emptyset \neq X \subseteq V - s.$$

Edmonds [4] also presented a seemingly more general form of Theorem 1.

Theorem 2 (Edmonds [4])**.** *Let $D = (V, A)$ be a digraph and S a multiset of vertices in V. There exists a packing of spanning s-arborescences $(s \in S)$ in D if and only if*

$$d^-_A(X) \quad \geq \quad |S_{V-X}| \qquad \text{for every } \emptyset \neq X \subseteq V.$$

Theorem 1 implies the following extension as well.

Theorem 3 (Edmonds [4])**.** *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $k \in \mathbb{Z}_+$. There exists a k-regular packing of s-arborescences $(s \in S)$ in D if and only if*

$$k \quad \geq \quad |S_v| \qquad \text{for every } v \in V, \tag{6}$$

$$|S_X| + d^-_A(X) \quad \geq \quad k \qquad \text{for every } \emptyset \neq X \subseteq V. \tag{7}$$

Frank [7] considered the problem of packing arborescences whose roots are not fixed in advance and proved the following result.

Theorem 4 (Frank [7])**.** *Let $D = (V, A)$ be a digraph and $k \in \mathbb{Z}_+$. There exists a packing of k spanning arborescences in D if and only if*

$$e_A(\mathcal{P}) \quad \geq \quad k(|\mathcal{P}| - 1) \qquad \text{for every subpartition } \mathcal{P} \text{ of } V.$$

It is not difficult to see that Theorems 1, 2, 3 and 4 are equivalent.

Theorem 4 was generalized for (f, g)-bounded packings as follows.

Theorem 5 (Frank [7], Cai [2]). *Let $D = (V, A)$ be a digraph, $f, g \in \mathbb{Z}_+^V$, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning arborescences in D if and only if*

$$g(v) \geq f(v) \qquad \textit{for every } v \in V, \tag{8}$$
$$e_A(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{9}$$

Theorem 2 can be generalized by adding a matroid constraint as follows.

Theorem 6 (Durand de Gevigney, Nguyen, Szigeti [3]). *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_\mathsf{M})$ a matroid. There exists an M-based packing of arborescences in D if and only if*

$$r_\mathsf{M}(S_X) + d_A^-(X) \geq r_\mathsf{M}(S) \qquad \textit{for every } \emptyset \neq X \subseteq V.$$

The following reformulation of the problem of matroid-based packing of arborescences was proposed by Frank [10]. We define the *extended digraph* $\boldsymbol{D'} = (V \cup s, A')$ which is obtained from $(D = (V, A), S)$ by adding a new vertex s and a new arc ss' for every $s' \in S$ and we consider the matroid $\boldsymbol{\mathsf{M}'}$ on $\delta_{A'}(s)$ obtained from M by replacing every element $s' \in S$ by the arc ss'. A packing of s-arborescences in D' is called M'*-based* if for every vertex $v \in V$, the set of arcs in $\delta_{A'}(s)$ that belong to the unique (s, v)-paths of s-arborescences containing v in the packing forms a basis of M'. We refer to this version of the problem as the extended framework.

The matroid-based packing of spanning arborescences problem in the extended framework can not probably be solved by the following result.

Theorem 7 (Fortier, Király, Szigeti, Tanigawa [6]). *It is NP-complete to decide whether there exists an M'-based packing of spanning s-arborescences in the extended framework (D', M').*

We will later see that surprisingly the same problem can be solved in the original framework, see Theorem 12. In fact this was the motivation for the introduction of matroid-rooted packings of arborescences.

We also present the following extensions of Theorems 1 and 5 to mixed graphs.

Theorem 8 (Frank [7]). *Let $F = (V, E \cup A)$ be a mixed graph, $s \in V$, and $k \in \mathbb{Z}_+$. There exists a packing of k spanning mixed s-arborescences in F if and only if*

$$e_{E\cup A}(\mathcal{P}) \geq k|\mathcal{P}| \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V - s.$$

Theorem 9 (Gao,Yang [12]). *Let $F = (V, E \cup A)$ be a mixed graph, $f, g \in \mathbb{Z}_+^V$, and $k \in \mathbb{Z}_+$. There exists an (f, g)-bounded packing of k spanning mixed arborescences in F if and only if (8) holds and*

$$e_{E\cup A}(\mathcal{P}) \geq k|\mathcal{P}| - \min\{k - f(\overline{\cup\mathcal{P}}), g(\cup\mathcal{P})\} \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V.$$

Theorem 9 easily implies Theorems 5 and 8.

We will need the following simple extension of Theorem 3 to dypergraphs from [5].

Theorem 10 (Fortier, Király, Léonard, Szigeti, Talon [5]). *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, S a multiset of vertices in V, and $k \in \mathbb{Z}_+$. There exists a k-regular packing of s-hyperarborescences $(s \in S)$ in $\mathcal{D}$ if and only if (6) holds and*

$$|S_X| + d_\mathcal{A}^-(X) \geq k \qquad \textit{for every } \emptyset \neq X \subseteq V. \tag{10}$$

In order to prove one of our main results we need the following theorem.

Theorem 11 (Theorem 13.1.2 in [9])**.** *Let $G = (S, V; E)$ be a simple bipartite graph, $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid with independent sets $\mathcal{I}_{\mathsf{M}}$, and $m \in \mathbb{Z}_+^V$. There exists $F \subseteq E$ such that*

$$\begin{aligned} d_F(s) &\leq 1 && \text{for every } s \in S, && (11)\\ d_F(v) &= m(v) && \text{for every } v \in V, && (12)\\ \Gamma_F(V) &\in \mathcal{I}_{\mathsf{M}} && && (13)\end{aligned}$$

if and only if

$$r_{\mathsf{M}}(\Gamma_E(X)) \geq m(X) \quad \text{for every } X \subseteq V. \tag{14}$$

4 New results

Our simplest new result is about matroid-rooted packing of spanning arborescences. It obviously implies Theorem 1.

Theorem 12. *Let $D = (V, A)$ be a digraph, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-rooted packing of spanning arborescences in D if and only if*

$$r_{\mathsf{M}}(S_{\cup\mathcal{P}}) + e_A(\mathcal{P}) \geq r_{\mathsf{M}}(S)|\mathcal{P}| \quad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{15}$$

Note that for packings of spanning arborescences, the notion of matroid-rooted and matroid-based coincide. Hence Theorem 12 provides a characterization for the existence of matroid-based packings of spanning arborescences. Theorem 12 can easily be obtained from either of Theorems 13 and 14.

Theorem 12 can be generalized in many directions: for k-regular packings, for (f, g)-bounded packings, for mixed graphs, and for dypergraphs. The following result shows a generalization that contains three of these directions.

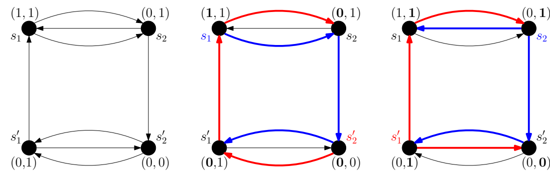
Theorem 13. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $k \in \mathbb{Z}_+$, $f, g \in \mathbb{Z}_+^V$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-rooted (f, g)-bounded k-regular packing of hyperarborescences in $\mathcal{D}$ if and only if for all $U, W \subseteq V$ and all subpartitions $\mathcal{P}$ of W,*

$$\begin{aligned} g_k(v) &\geq f(v) \quad \text{for every } v \in V, && (16)\\ r_{\mathsf{M}}(S_U) + g_k(V - U) &\geq r_{\mathsf{M}}(S), && (17)\\ e_{\mathcal{A}}(\mathcal{P}) + r_{\mathsf{M}}(S_U) + g_k(W - U) &\geq k|\mathcal{P}| + f(U - W). && (18)\end{aligned}$$

Theorem 13 extends Theorems 5, 10, 12 and follows from Theorems 16(a) and 17.

Theorem 13 provides a characterization of the existence of matroid-rooted (f, g)-bounded packing of spanning arborescences in a digraph. Nonetheless, the problem of matroid-rooted packing of spanning arborescences does not have the linking property. To see this let us consider the instance of the problem of Figure 1(a). Figure 1(b) and (c) show the existence of an M-rooted packing of spanning arborescences satisfying f and one satisfying g. Suppose that there exists an M-rooted packing of spanning arborescences satisfying both f and g with root set B. Since $g(s_2') = 0$, we get $s_2' \notin B$. Since $f(s_1) = 1$, we get $s_1 \in B$. Then, since B is a basis of M, we get $s_1' \notin B$. On the other hand, since only one arc enters $\{s_1', s_2'\}$, one of the roots in B belongs to $\{s_1', s_2'\}$. This contradiction shows that the linking property does not hold.

We propose another generalization of Theorem 12 that contains three other directions among the above mentioned four.



(a) (b) (c)

Figure 1: (a) The digraph $D = (V, A)$, with $V = \{s_1, s'_1, s_2, s'_2\}$, the matroid M with ground set V and with bases $\{\{s_1, s_2\}, \{s_1, s'_2\}, \{s'_1, s_2\}, \{s'_1, s'_2\}\}$ and the ordered pairs $(f(v), g(v))$ on each vertex v. (b) An M-rooted packing of spanning arborescences satisfying f, (c) An M-rooted packing of spanning arborescences satisfying g.

Theorem 14. *Let $F = (V, E \cup A)$ be a mixed graph, $k \in \mathbb{Z}_+$, $f, g \in \mathbb{Z}_+^V$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-rooted (f, g)-bounded k-regular packing of mixed arborescences in F if and only if* (16) *and* (17) *hold and for all $W, U \subseteq V$ and all subpartitions $\mathcal{P}$ of W,*

$$e_{E\cup A}(\mathcal{P}) + r_{\mathsf{M}}(S_U) + g_k(W - U) \quad \geq \quad k|\mathcal{P}| + f(U - W). \tag{19}$$

Theorem 14 extends Theorems 9 and 12 and its proof can be found in Subsection 6.3. In order to prove Theorem 14 we need a new orientation theorem that extends Theorem 15.4.13 in [9].

Theorem 15. *Let $F = (V, E \cup A)$ be a mixed graph, h an integer-valued intersecting supermodular function on V, and b an integer-valued submodular function on V. There exists an orientation $\vec{E}$ of E such that*

$$e_{\vec{E}\cup A}(\mathcal{P}) \quad \geq \quad \sum_{X \in \mathcal{P}} h(X) - b(\cup\mathcal{P}) \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V \tag{20}$$

if and only if

$$e_{E\cup A}(\mathcal{P}) \quad \geq \quad \sum_{X \in \mathcal{P}} h(X) - b(\cup\mathcal{P}) \qquad \textit{for every subpartition } \mathcal{P} \textit{ of } V. \tag{21}$$

The proof of Theorem 15 can be found in Subsection 6.2.

5 Generalized polymatroids

Theorem 12 will be obtained applying the theory of generalized polymatroids. Generalized polymatroids were introduced by Hassin [13] and independently by Frank [8]. For a pair (p, b) of set functions on S, $\alpha \in \mathbb{R}$, and $f, g \in \mathbb{R}^S$, let us introduce the following polyhedra

$$\begin{aligned}
\boldsymbol{Q(p,b)} &= \{x \in \mathbb{R}^S : p(Z) \leq x(Z) \leq b(Z) \ \text{ for all } Z \subseteq S\},\\
\boldsymbol{B(b)} &= \{x \in \mathbb{R}^S : x(Z) \leq b(Z) \ \text{ for all } Z \subseteq S, x(S) = b(S)\},\\
\boldsymbol{C(p)} &= \{x \in \mathbb{R}^S : p(Z) \leq x(Z) \ \text{ for all } Z \subseteq S, x \geq 0\},\\
\boldsymbol{K(\alpha)} &= \{x \in \mathbb{R}^S : x(S) = \alpha\},\\
\boldsymbol{T(f,g)} &= \{x \in \mathbb{R}^S : f \leq x \leq g\}.
\end{aligned}$$

If $p(\emptyset) = b(\emptyset) = 0$, p is supermodular, b is submodular and $b(X) - p(Y) \geq b(X - Y) - p(Y - X)$ for all $X, Y \subseteq S$, the polyhedron $Q(p, b)$ is called a *generalized polymatroid*, shortly *g-polymatroid*. The polyhedron $B(b)$ is called *base-polyhedra* and the polyhedron $C(p)$ is called a *contra-polymatroid*, even if p is only intersecting supermodular.

We summarize in the following theorem all the properties of generalized polymatroids we need.

Theorem 16 ([9]). *The following hold.*

(a) If p and b are integral, then $Q(p,b)$ contains an integral element.

(b) $C(p_{\mathcal{A},k}) = Q(p^*_{\mathcal{A},k}, \infty_0)$, where

$$\boldsymbol{p^*_{\mathcal{A},k}(X)} = \max\{\sum_{X' \in \mathcal{P}} p_{\mathcal{A},k}(X') : \mathcal{P} \text{ subpartition of } X\},$$

(c) $Q(p,b) = B(b)$ if and only if $p(S) = b(S)$.

(d) Let $M = Q(p,b) \cap K(\alpha)$.

(i) $M \neq \emptyset$ if and only if $p \leq b$, $p(S) \leq \alpha \leq b(S)$.

(ii) M is a g-polymatroid.

(iii) If $M \neq \emptyset$, then $M = Q(p^\alpha, b^\alpha)$ with

$$\boldsymbol{p^\alpha(Z)} = \max\{p(Z), \alpha - b(S-Z)\}, \tag{22}$$
$$\boldsymbol{b^\alpha(Z)} = \min\{b(Z), \alpha - p(S-Z)\}. \tag{23}$$

(e) Let $M = Q(p,b) \cap T(f,g)$.

(i) $M \neq \emptyset$ if and only if $\max\{p,f\} \leq \min\{b,g\}$.

(ii) M is a g-polymatroid.

(iii) If $M \neq \emptyset$, then $M = Q(p^g_f, b^g_f)$ with

$$\boldsymbol{p^g_f(Z)} = \max\{p(X) - g(X-Z) + f(Z-X) : X \subseteq S\}, \tag{24}$$
$$\boldsymbol{b^g_f(Z)} = \min\{b(X) - f(X-Z) + g(Z-X) : X \subseteq S\}. \tag{25}$$

(f) $B(b_1) \cap B(b_2) \neq \emptyset$ if and only if $b_1(X) + b_2(S-X) \geq b_1(S) = b_2(S)$ for every $X \subseteq S$.

To obtain Theorem 13 we have to prove the following result.

Theorem 17. *Let $\mathcal{D} = (V, \mathcal{A})$ be a dypergraph, $k, \ell \in \mathbb{Z}_+$, $f, g \in \mathbb{Z}^V_+$, S a multiset of vertices in V, $\mathsf{M} = (S, r_\mathsf{M})$ a matroid of rank ℓ,*

$$\boldsymbol{N^{f,g}_{k,\ell}} = C(p_{\mathcal{A},k}) \cap T(f, g_k) \cap K(\ell),$$
$$\boldsymbol{N^{f,g}_{k,\mathsf{M}}} = N^{f,g}_{k,\ell} \cap B(b_\mathsf{M}).$$

(a) The root vectors of the M-rooted (f,g)-bounded k-regular packings of hyperarborescences in $\mathcal{D}$ are exactly the integer points of $N^{f,g}_{k,\mathsf{M}}$.

(b) $N^{f,g}_{k,\ell} \neq \emptyset$ if and only if (16) holds and for every $X \subseteq V$ and subpartition $\mathcal{P}$ of X,

$$g_k(V) \geq \ell, \tag{26}$$
$$e_\mathcal{A}(\mathcal{P}) \geq k|\mathcal{P}| - \ell + f(\overline{X}), \tag{27}$$
$$e_\mathcal{A}(\mathcal{P}) \geq k|\mathcal{P}| - g_k(X). \tag{28}$$

(c) If $N^{f,g}_{k,\ell} \neq \emptyset$*, then* $N^{f,g}_{k,\ell} = Q(p^{f,g}_{k,\ell}, b^{f,g}_{k,\ell})$ *with*

$$\boldsymbol{p^{f,g}_{k,\ell}(Z)} = \max\{p^*_{\mathcal{A},k}(X) - g_k(X - Z) + f(Z - X), \ell - g_k(\overline{Z}) : X \subseteq V\}, \quad (29)$$

$$\boldsymbol{b^{f,g}_{k,\ell}(Z)} = \min\{g_k(Z), \ell - p^*_{\mathcal{A},k}(X) + g_k(X - \overline{Z}) - f(\overline{Z} - X) : X \subseteq V\}. \quad (30)$$

(d) $N^{f,g}_{k,\mathsf{M}} \neq \emptyset$ *if and only if* (16), (17) *and* (18) *hold.*

The proof of Theorem 17 can be found in Subsection 6.1. We mention that the intersection with $K(\ell)$ is not really needed because, by $r_{\mathsf{M}}(S) = \ell$, the same condition is contained in $B(b_{\mathsf{M}})$. Our choice is justified by the fact that the calculations became simpler.

6 Proofs

In this section we prove our results, Theorems 17, 15 and 14.

6.1 Proof of Theorem 17

Proof. (a) First, let $\boldsymbol{m}$ be the root vector of an M-rooted (f,g)-bounded k-regular packing of hyperarborescences in $\mathcal{D}$, with root set $\boldsymbol{S'}$. Then S' is a basis of M and

$$g_k(v) \geq m(v) \geq f(v) \quad \text{for every } v \in V, \quad (31)$$

$$m(V) = \ell, \quad (32)$$

$$m(X) + d^-_{\mathcal{A}}(X) \geq k \quad \text{for every } \emptyset \neq X \subseteq V. \quad (33)$$

Thus, by (31) and (32), m is an integer point in $T(f, g_k) \cap K(\ell)$. Further, m is in $C(p_{\mathcal{A},k})$ because $m(v) \geq 0$ for every $v \in V$, $m(\emptyset) = 0 = p_{\mathcal{A},k}(\emptyset)$ and, by (33), we have $m(X) \geq k - d^-_{\mathcal{A}}(X) = p_{\mathcal{A},k}(X)$ for every $\emptyset \neq X \subseteq V$. We get that $m \in N^{f,g}_{k,\ell}$. Since, by $S'_X \subseteq S' \cap S_X$, S'_X is an independent set in M contained in S_X, we have

$$b_{\mathsf{M}}(X) = r_{\mathsf{M}}(S_X) \geq |S'_X| = m(X) \text{ for every } X \subseteq V,$$

with equality for V. Hence $m \in B(b_{\mathsf{M}})$. It follows that m is an integer point of $N^{f,g}_{k,\mathsf{M}}$.

Let now $\boldsymbol{m}$ be an integer point of $N^{f,g}_{k,\mathsf{M}} = N^{f,g}_{k,\ell} \cap B(b_{\mathsf{M}})$. Since m is an integer point in $C(p_{\mathcal{A},k})$, we have $m \in \mathbb{Z}_+$. Let $\boldsymbol{S'}$ be the multiset of vertices such that $|S'_v| = m(v)$ for every $v \in V$. Since $m \in T(f, g_k)$, we have

$$|S'_v| = m(v) \leq g_k(v) \leq k \text{ for every } v \in V,$$

so (6) holds for S'. Since $m \in C(p_{\mathcal{A},k})$, we have

$$|S'_X| + d^-_{\mathcal{A}}(X) = m(X) + k - p_{\mathcal{A},k}(X) \geq k \text{ for every } \emptyset \neq X \subseteq V,$$

so (10) holds for S'. Then, by Theorem 10, there exists a k-regular packing $\{\boldsymbol{B_i}\}^{\ell}_{i=1}$ of s-hyperarborescences ($s \in S'$) in $\mathcal{D}$ with root set S' that is with root vector m. Since $m \in K(\ell)$, the packing contains ℓ hyperarborescences. Since $m \in T(f, g_k)$, we have

$$f(v) \leq m(v) = |S'_v| = m(v) \leq g_k(v) \leq g(v) \text{ for every } v \in V,$$

so the packing is (f,g)-bounded.

Since $m \in B(b_{\mathsf{M}})$, we have

$$m(X) \leq b_{\mathsf{M}}(X) = r_{\mathsf{M}}(S_X) \quad \text{for every } X \subseteq V, \quad (34)$$

with equality holds for V. Let $\boldsymbol{G}=(S,V;E)$ be the bipartite graph such that for $s\in S$ and $v\in V$, $sv\in E$ if and only if $s\in S_v$. By (34), we get that (14) holds. Then, by Theorem 11, there exists an independent set $\boldsymbol{S^*}$ in M such that for every $v\in V$, $|S^*_v|=m(v)=|S'_v|$ and hence the $m(v)$ v-hyperarborescences in $\{B_i\}$ can be rooted at S^*_v. Then, by $|S^*|=m(V)=r_{\mathsf{M}}(S)$, we obtain that S^* is a basis of M. It follows that the packing is M-rooted.

(b) By Theorem 16(b) and 16(e)(i), $C(p_{\mathcal{A},k})\cap T(f,g_k)=Q(p^*_{\mathcal{A},k},\infty_0)\cap T(f,g_k)\neq\emptyset$ if and only if $p^*_{\mathcal{A},k}\leq\infty_0, f\leq g_k, p^*_{\mathcal{A},k}\leq g_k$ and $f\leq\infty_0$ if and only if (16) and (28) hold. By Theorem 16(e)(iii), if the intersection is non-empty, then $C(p_{\mathcal{A},k})\cap T(f,g_k)=Q(p^{f,g}_{\mathcal{A},k},b^{f,g}_{\mathcal{A},k})$ where

$$p^{f,g}_{\mathcal{A},k}(Z) \;=\; \max\{p^*_{\mathcal{A},k}(X)-g_k(X-Z)+f(Z-X): X\subseteq V\}, \tag{35}$$

$$b^{f,g}_{\mathcal{A},k}(Z) \;=\; \min\{\infty_0(X)-f(X-Z)+g_k(Z-X): X\subseteq V\}=g_k(Z). \tag{36}$$

By Theorem 16 (d)(i), $C(p_{\mathcal{A},k})\cap T(f,g_k)\cap K(\ell)=Q(p^{f,g}_{\mathcal{A},k},b^{f,g}_{\mathcal{A},k})\cap K(\ell)\neq\emptyset$ if and only if (16) and (28) hold and $p^{f,g}_{\mathcal{A},k}(V)\leq\ell\leq b^{f,g}_{\mathcal{A},k}(V)$. These conditions are equivalent to (16), (28), (27) and (26).

(c) By Theorem 16(d)(iii), if the intersection is non-empty, then $N^{f,g}_{k,\ell}=Q(p^{f,g}_{k,\ell},b^{f,g}_{k,\ell})$ with

$$p^{f,g}_{k,\ell}(Z) \;=\; \max\{p^{f,g}_{\mathcal{A},k}(Z),\ell-b^{f,g}_{\mathcal{A},k}(\overline{Z}): X\subseteq V\}, \tag{37}$$

$$b^{f,g}_{k,\ell}(Z) \;=\; \min\{b^{f,g}_{\mathcal{A},k}(Z),\ell-p^{f,g}_{\mathcal{A},k}(\overline{Z}): X\subseteq V\}. \tag{38}$$

By combining (35) and (36) with (37) and (38), we get (29) and (30).

(d) First we show that in both directions, we have $N^{f,g}_{k,\ell}\neq\emptyset$ and (16) holds. If $N^{f,g}_{k,\mathsf{M}}\neq\emptyset$, then it is evident. Suppose now that (16), (17) and (18) hold. Applying (17) for $U=\emptyset$, we obtain (26). Applying (18) for $U=V$ and $W=X$, we obtain (27). Applying (18) for $U=\emptyset$ and $W=X$, we obtain (28). Furthermore, (16) also holds. Then, by (b), we get $N^{f,g}_{k,\ell}\neq\emptyset$.

Thus, by (c), $\ell\leq p^{f,g}_{k,\ell}(V)\leq b^{f,g}_{k,\ell}(V)\leq\ell$. It follows that $p^{f,g}_{k,\ell}(V)=b^{f,g}_{k,\ell}(V)=\ell$. Hence, by Theorem 16(c), $N^{f,g}_{k,\ell}=B(b^{f,g}_{k,\ell})$. By Theorem 16(f), $\emptyset\neq N^{f,g}_{k,\mathsf{M}}=B(b^{f,g}_{k,\ell})\cap B(b_{\mathsf{M}})$ if and only if $b^{f,g}_{k,\ell}(V)=b_{\mathsf{M}}(V)$ (that holds since both are equal to ℓ) and

$$b^{f,g}_{k,\ell}(\overline{U})\geq\ell-b_{\mathsf{M}}(U)\text{ for every }U\subseteq V. \tag{39}$$

By (30), we get that (39) is equivalent to for all $U,W\subseteq V$,

$$\ell-g_k(\overline{U}) \;\leq\; r_{\mathsf{M}}(S_U), \tag{40}$$

$$p^*_{\mathcal{A},k}(W)-g_k(W-U)+f(U-W) \;\leq\; r_{\mathsf{M}}(S_U), \tag{41}$$

which are equivalent to (17) and (18). □

6.2 Proof of Theorem 15

Proof. The proof follows the proof ideas of Theorem 9.5.1 of [9]. The **necessity** is obtained from the fact that for every orientation $\vec{E}$ of E and every subpartition $\mathcal{P}$ of V, we have $e_{E\cup A}(\mathcal{P})\geq e_{\vec{E}\cup A}(\mathcal{P})$.

To see the **sufficiency**, let $(F=(V,E\cup A),h,b)$ be a counterexample that minimizes $|E|$. Note that $E\neq\emptyset$, otherwise (20) and (21) coincide. Let $\boldsymbol{e=uv}$ be an arbitrary edge in E, $\boldsymbol{F_1}=(V,E_1\cup A_1)$ and $\boldsymbol{F_2}=(V,E_2\cup A_2)$ the mixed graphs obtained from F by orienting e as $\vec{\boldsymbol{a}}_{\boldsymbol{1}}=uv$ and $\vec{\boldsymbol{a}}_{\boldsymbol{2}}=vu$, respectively. If for some $i=1,2$, F_i satisfies (21), then, by the minimality of E, there exists an orientation $\vec{E}_i$ of E_i such that $(V, \vec{E}_i \cup A_i) = (V, \vec{E}_i \cup \{\vec{a}_i\} \cup A)$ satisfies (20) and hence (F, h, b) is not a counterexample. It follows that, for $i = 1, 2$, there exists a subpartition $\boldsymbol{\mathcal{P}_i}$ of V such that

$$e_E(\mathcal{P}_i) = \sum_{X \in \mathcal{P}_i} h'(X) - b(P_i) \tag{42}$$

and uv is between $P_1 - P_2$ and $P_2 - P_1$, where $\boldsymbol{h'(X)} = h(X) - d_A^-(X)$ for every $X \subseteq V$ and $\boldsymbol{P_i} = \cup \mathcal{P}_i$. Let $\boldsymbol{\mathcal{P}} = \mathcal{P}_1 \cup \mathcal{P}_2$. Note that $\mathcal{P}$ covers each vertex in $P_1 \cap P_2$ twice and each vertex in $(P_1 \cup P_2) - (P_1 \cap P_2)$ once. Using the usual uncrossing technique for $\mathcal{P}$, we obtain a laminar family $\boldsymbol{\mathcal{P}'}$ that covers each vertex in $P_1 \cap P_2$ twice and each vertex in $(P_1 \cup P_2) - (P_1 \cap P_2)$ once. Then $\mathcal{P}'$ can be decomposed into a partition $\boldsymbol{\mathcal{P}'_1}$ of $P_1 \cap P_2$ and a partition $\boldsymbol{\mathcal{P}'_2}$ of $P_1 \cup P_2$. Since h is intersecting supermodular and d_A^- is submodular, h' is intersecting supermodular. Thus

$$\sum_{X \in \mathcal{P}'_1 \cup \mathcal{P}'_2} h'(X) \geq \sum_{X \in \mathcal{P}_1 \cup \mathcal{P}_2} h'(X). \tag{43}$$

Since d_E is submodular,

$$\sum_{X \in \mathcal{P}_1 \cup \mathcal{P}_2} d_E(X) \geq \sum_{X \in \mathcal{P}'_1 \cup \mathcal{P}'_2} d_E(X). \tag{44}$$

Further, since uv is between $P_1 - P_2$ and $P_2 - P_1$, we have

$$d_E(\overline{P_1}) + d_E(\overline{P_2}) > d_E(\overline{P_1} \cap \overline{P_2}) + d_E(\overline{P_1} \cup \overline{P_2}). \tag{45}$$

It follows, by (44) and (45), that we have

$$\begin{aligned}
2(e_E(\mathcal{P}_1) + e_E(\mathcal{P}_2)) &= (d_E(\overline{P_1}) + \sum_{X \in \mathcal{P}_1} d_E(X)) + (d_E(\overline{P_2}) + \sum_{X \in \mathcal{P}_2} d_E(X)) \\
&> (d_E(\overline{P_1} \cup \overline{P_2}) + \sum_{X \in \mathcal{P}'_1} d_E(X)) + (d_E(\overline{P_1} \cap \overline{P_2}) + \sum_{X \in \mathcal{P}'_2} d_E(X)) \\
&= 2(e_E(\mathcal{P}'_1) + e_E(\mathcal{P}'_2)).
\end{aligned} \tag{46}$$

Hence, by (43), (42), (46), the submodularity of b and (21) applied for $\mathcal{P}'_1$ and $\mathcal{P}'_2$, we have

$$\begin{aligned}
\sum_{X \in \mathcal{P}'_1 \cup \mathcal{P}'_2} h'(X) &\geq \sum_{X \in \mathcal{P}_1 \cup \mathcal{P}_2} h'(X) \\
&= (e_E(\mathcal{P}_1) + b(P_1)) + (e_E(\mathcal{P}_2) + b(P_2)) \\
&> (e_E(\mathcal{P}'_1) + b(P_1 \cap P_2)) + (e_E(\mathcal{P}'_2) + b(P_1 \cup P_2)) \\
&\geq \sum_{X \in \mathcal{P}'_1 \cup \mathcal{P}'_2} h'(X),
\end{aligned}$$

a contradiction. □

6.3 Proof of Theorem 14

Proof. We show that Theorem 14 follows from Theorems 15 and 13. Let us define two functions as follows. For all $X \subseteq V$,

$$\begin{aligned}
h(X) &= k \text{ if } X \neq \emptyset \text{ and } 0 \text{ if } X = \emptyset, \qquad (47) \\
b(X) &= \min\{-f(U - W) + g_k(W - U) + r_{\mathsf{M}}(S_U) : X \subseteq W \subseteq V, U \subseteq V\}. \qquad (48)
\end{aligned}$$

Clearly, h is intersecting supermodular. In order to be able to apply Theorem 15 we show that b is submodular.

Claim 1. *The function b is submodular.*

Proof. For $i = 1, 2$, let $\boldsymbol{X_i} \subseteq V$, and $X_i \subseteq \boldsymbol{W_i} \subseteq V$, $\boldsymbol{U_i} \subseteq V$ such that

$$b(X_i) = -f(U_i - W_i) + g_k(W_i - U_i) + r_{\mathsf{M}}(S_{U_i}). \tag{49}$$

Let us introduce the following sets:

$$\begin{aligned}
\boldsymbol{U_3} &= U_1 \cap U_2, \quad \boldsymbol{U_4} = U_1 \cup U_2,\\
\boldsymbol{W_3} &= W_1 \cap W_2, \quad \boldsymbol{W_4} = W_1 \cup W_2,\\
\boldsymbol{Z} &= ((W_1 - U_1) \cap (U_2 - W_2)) \cup ((W_2 - U_2) \cap (U_1 - W_1)).
\end{aligned}$$

The following can be easily checked.

$$\begin{aligned}
X_1 \cap X_2 &\subseteq W_3, & (50)\\
X_1 \cup X_2 &\subseteq W_4, & (51)\\
\text{multiset } (U_3 - W_3) \cup (U_4 - W_4) \cup Z &= \text{multiset } (U_1 - W_1) \cup (U_2 - W_2), & (52)\\
\text{multiset } (W_3 - U_3) \cup (W_4 - U_4) \cup Z &= \text{multiset } (W_1 - U_1) \cup (W_2 - U_2). & (53)
\end{aligned}$$

Then, by the modularity of f and g_k, (16) and the submodularity of r_{M}, we have

$$\begin{aligned}
b(X_1) + b(X_2) &= -f(U_1 - W_1) + g_k(W_1 - U_1) + r_{\mathsf{M}}(S_{U_1})\\
&\quad -f(U_2 - W_2) + g_k(W_2 - U_2) + r_{\mathsf{M}}(S_{U_2})\\
&\geq -f(U_3 - W_3) + g_k(W_3 - U_3) + r_{\mathsf{M}}(S_{U_3})\\
&\quad -f(U_4 - W_4) + g_k(W_4 - U_4) + r_{\mathsf{M}}(S_{U_4}) - f(Z) + g_k(Z)\\
&\geq b(X_1 \cap X_2) + b(X_1 \cup X_2),
\end{aligned}$$

and the claim is proved. □

By Claim 1, b is submodular. Then Theorem 15 can be applied for h and b. Since in this case (21) and (19) are equivalent, (21) holds, and hence it follows that there exists an orientation $\vec{E}$ of E such that $D = (V, \vec{E} \cup A)$ satisfies (20) which is equivalent to (18) for $\mathcal{D} = D$. Note that (16) and (17) also hold by assumption. Then, by Theorem 13, there exists an M-rooted (f, g)-bounded k-regular packing of arborescences in D that provides, by replacing the arcs in $\vec{E}$ by the edges in E, an M-rooted (f, g)-bounded k-regular packing of mixed arborescences in F. □

7 Conclusion

In this paper we introduced a new problem on packings of arborescences, namely matroid-rooted packings. We provided characterizations for matroid-rooted (f, g)-bounded k-regular packings of hyperarborescences and for matroid-rooted (f, g)-bounded k-regular packings of mixed arborescences. The problem of matroid-rooted (f, g)-bounded k-regular packings of mixed hyperarborescences remains open. The natural conjecture about this problem is the following.

Conjecture 1. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, $k \in \mathbb{Z}_+$, $f, g \in \mathbb{Z}_+^V$, S a multiset of vertices in V, and $\mathsf{M} = (S, r_{\mathsf{M}})$ a matroid. There exists an M-rooted (f, g)-bounded k-regular packing of mixed hyperarborescences in $\mathcal{F}$ if and only if (16) and (17) hold and for all $U, W \subseteq V$ and subpartition $\mathcal{P}$ of W,*

$$e_{\mathcal{E} \cup \mathcal{A}}(\mathcal{P}) + r_{\mathsf{M}}(S_U) + g_k(W - U) \geq k|\mathcal{P}| + f(U - W). \tag{54}$$

Theorem 13 and the following conjecture, a possible extension of Theorem 15 to mixed hypergraphs, would imply Conjecture 1.

Conjecture 2. *Let $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ be a mixed hypergraph, h an integer-valued, intersecting supermodular function on V and b a submodular function on V. There exists an orientation $\vec{\mathcal{E}}$ of $\mathcal{E}$ such that*

$$e_{\vec{\mathcal{E}}\cup\mathcal{A}}(\mathcal{P}) \quad \geq \quad \sum_{X\in\mathcal{P}} h(X) - b(\cup\mathcal{P}) \qquad \text{for every subpartition } \mathcal{P} \text{ of } V \tag{55}$$

if and only if

$$e_{\mathcal{E}\cup\mathcal{A}}(\mathcal{P}) \quad \geq \quad \sum_{X\in\mathcal{P}} h(X) - b(\cup\mathcal{P}) \qquad \text{for every subpartition } \mathcal{P} \text{ of } V. \tag{56}$$

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