# Simultaneous well-balanced orientations of graphs

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**Abstract**

Nash-Williams' well-balanced orientation theorem [11] is extended for orienting several graphs simultaneously.

We prove that if $G_1, ..., G_k$ are pairwise edge-disjoint subgraphs of a graph $G$, then $G$ has a well-balanced orientation $\vec{G}$ such that the inherited orientations $\vec{G}_i$ of $G_i$ are well-balanced for all $1 \leq i \leq k$. We also have new results about simultaneous well-balanced orientations of non-disjoint subgraphs of an Eulerian graph as well as those of different contractions of a graph.

*Key words:* orientation, connectivity

## 1 Introduction

This paper concerns undirected and directed graphs, more precisely we consider orientations of undirected graphs. Multiple edges are allowed, but loops are forbidden. The starting point is Robbins' theorem [13] which states that an undirected graph $G$ has a strongly connected orientation if and only if $G$ is 2-edge-connected. The following generalization was proved by Nash-Williams

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in [11]: an undirected graph $G$ has a $k$-arc-connected orientation if and only if $G$ is $2k$-edge-connected (see Theorem 2 below). This result about global edge-connectivity can be easily proved by applying Lovász' splitting off theorem [8]. Nash-Williams [11] also provided the following extension on local edge-connectivity (for a stronger statement see Theorem 3): any undirected graph $G$ has a **well-balanced** orientation $\vec{G}$, that is for every ordered pair of vertices $(u, v)$, if the maximum number of edge disjoint $(u, v)-$paths was $\lambda_G(u, v)$ in $G$, then the maximum number of arc disjoint directed $(u, v)-$paths is at least $\lfloor \lambda_G(u, v)/2 \rfloor$ in the resulting directed graph $\vec{G}$. The well-balanced orientation may also be required to be **smooth**, that is the in-degree and the out-degree of every vertex differ by at most one. A smooth well-balanced orientation is called **best-balanced**. In fact, Nash-Williams proved an even stronger result in [11] the so-called odd vertex pairing theorem (see Theorem 5).

Nash-Williams [12] formulated an extension of his orientation theorem: for an arbitrary subgraph $H$ of an undirected graph $G$ there exists a best-balanced orientation of $H$ that can be extended to a best-balanced orientation of $G$ (see Theorem 4). He mentioned that *"Given Theorem 5, the proof of Theorem 4 is not unreasonably difficult. At a certain stage in the proof of Theorem 3 ... we had occasion to select an arbitrary di-Eulerian orientation $\Delta$ of the finite Eulerian graph $G + P$. ... the proof of Theorem 4 depends essentially on the idea of modifying this step ... by choosing $\Delta$ to be, not just any di-Eulerian orientation of $G + P$, but one which satisfies certain additional restrictions."*

The main contribution of the present paper is to provide a simple proof for a generalization of this result, namely if $G_1, ..., G_k$ are pairwise edge-disjoint subgraphs of a graph $G$, then $G$ has a best-balanced orientation $\vec{G}$ such that the inherited orientations $\vec{G}_i$ are best-balanced orientations of $G_i$ for all $1 \leq i \leq k$. We also have a new result about simultaneous best-balanced orientations of contractions of $G$.

For an Eulerian graph $G$ we can prove more: there exists simultaneous best-balanced orientations of $G - v$ for all $v \in V$. This solves a conjecture of Frank [2], a special case of an interesting conjecture about $k$-vertex-connected orientations (see Conjecture 1), and generalizes a theorem of Berg and Jordán [1]. We also provide a couple of consequences of the theorems.

## 2 Notation, definitions

We denote a directed graph by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph $\vec{G}$, a set $X \subseteq V$ and $u, v \in V$, let $\boldsymbol{\delta_{\vec{G}}(X)}:= |\{uv \in A : u \in X,\ v \notin X\}|$, $\boldsymbol{\varrho_{\vec{G}}(X)}:= \delta_{\vec{G}}(V - X)$, $\boldsymbol{f_{\vec{G}}(X)}:= \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$, $\boldsymbol{\lambda_{\vec{G}}(u, v)}:= \min\{\delta_{\vec{G}}(Y) : u \in Y,\ v \in V - Y\}$, and $\boldsymbol{\overleftarrow{G}}:= (V, \{vu : uv \in A\})$.

For an undirected graph $G$, a set $X \subseteq V$ and $u, v \in V$, let $\boldsymbol{d_G(X)}$:= $|\{uv \in E : u \in X,\ v \notin X\}|$, $\boldsymbol{T_G}$:= $\{v \in V : d_G(v)$ is odd$\}$, $\boldsymbol{\lambda_G(u,v)}$:= $\min\{d_G(Y) : u \in Y,\ v \in V - Y\}$ , $\boldsymbol{R_G(X)}$:= $\max\{\lambda_G(x,y) : x \in X,\ y \in V - X\}$ (let $R_G(\emptyset) = R_G(V) = 0$), $\boldsymbol{b_G(X)}$:= $d_G(X) - 2\cdot\lfloor R_G(X)/2 \rfloor$, $\boldsymbol{G[X]}$:= $G - (V - X)$.

Observe that for all $X \subseteq V$, we have $0 \leq b_G(X) \leq d_G(X)$ and

$$f_{\vec{G}}(X) = \sum_{v \in X} f_{\vec{G}}(v). \tag{1}$$

Let $G = (V, E)$ be an undirected graph. $G$ is called $k$**-edge-connected** if $G - F$ is connected for all $F \subseteq E$ with $|F| \leq k - 1$. In this paper, if it is not explicitly stated, graphs may be disconnected, and we use the notion **Eulerian graph** for a possibly disconnected graph with all degrees even. Similarly, a directed graph is called Eulerian if at every vertex the in-degree equals the out-degree. Let $D = (V, A)$ be a directed graph. $D$ is **strongly connected** if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed $(u, v)$-path in $D$. $D$ is called $k$**-arc-connected** if $G - F$ is strongly connected for all $F \subseteq A$ with $|F| \leq k - 1$. $D$ is called $k$**-vertex-connected** if $|V| > k$ and $G - X$ is strongly connected for all $X \subseteq V$ with $|X| \leq k - 1$. An orientation $\vec{G}$ of $G$ is called **well-balanced** if

$$\lambda_{\vec{G}}(x, y) \geq \left\lfloor \frac{\lambda_G(x, y)}{2} \right\rfloor \quad \text{for all} \quad (x, y) \in V \times V, \tag{2}$$

and $\vec{G}$ is called **smooth** if

$$|f_{\vec{G}}(v)| \leq 1 \quad \text{for all} \quad v \in V. \tag{3}$$

A smooth well-balanced orientation is called **best-balanced**. Note that if $\vec{G}$ is best-balanced then so is $\overleftarrow{G}$.

A **pairing** $M$ of $G$ is a new graph on vertex set $T_G$ in which each vertex has degree one. Let $M$ be a pairing of $G$. An orientation $\vec{M}$ of $M$ is called **good** if

$$f_{\vec{M}}(X) \leq b_G(X) \qquad \text{for all } X \subseteq V. \tag{4}$$

$M$ is **well-orientable** if there *exists* a good orientation of $M$, $M$ is **strong** if *every* orientation of $M$ is good and $M$ is **feasible** if

$$d_M(X) \leq b_G(X) \qquad \text{for all } X \subseteq V. \tag{5}$$

Clearly an oriented pairing $\vec{M}$ is good if and only if $\overleftarrow{M}$ is good. We say that two arc disjoint directed graphs $\vec{G}$ and $\vec{H}$ on the same vertex set $V$ are **compatible** if

$$f_{\vec{G}}(v) = f_{\vec{H}}(v) \quad \text{for all } v \in V \tag{6}$$

or equivalently if $\vec{G} + \overleftarrow{H}$ is Eulerian.

## 3 Eulerian graphs

The following statements are well-known and/or are easy exercises.

**Proposition 1** *Every undirected Eulerian graph $G$ has an Eulerian orientation and every Eulerian orientation of $G$ is best-balanced.*

**Proposition 2** *If $\vec{G}^1$ and $\vec{G}^2$ are Eulerian orientations of a graph $G$, then $\vec{G}^2$ can be obtained from $\vec{G}^1$ by reversing the orientation of some directed cycles.*

**Proposition 3** *The edge-set of an undirected graph $G$ can be partitioned into some cycles and $|T_G|/2$ paths. Hence, every graph $G$ has a pairing $M$ such that $d_M(X) \leq d_G(X)$ for all $X \subset V$.*

**Proposition 4** *$\vec{G}$ is Eulerian if and only if $f_{\vec{G}}(X) = 0$ for all $X \subseteq V$. Hence if $\vec{G}$ is Eulerian then the contracted graph $\vec{G}/X$ is also Eulerian for all $X \subseteq V$.*

For an Eulerian graph $G$, an **edge-pairing** at vertex $v$ is an arbitrary partition of the edges incident to $v$ into pairs. Suppose that we are given an edge-pairing at each vertex. An Eulerian orientation is called **admissible** if at each vertex every edge-pair becomes a directed path.

**Proposition 5** *We are given an Eulerian graph $G$ and an edge-pairing at every vertex. Then there exists an admissible Eulerian orientation of $G$.*

A nice theorem of Ford and Fulkerson [3] about Eulerian orientations of mixed graphs implies easily the following theorem that plays an important role in this paper.

**Theorem 1** *Let $M$ be a pairing of an undirected graph $G$ and $\vec{M}$ be a good orientation of $M$. Then $G$ has an orientation $\vec{G}$ compatible with $\vec{M}$.*

## 4 Equivalent forms

**Claim 1** *An orientation $\vec{G}$ of an undirected graph $G$ is well-balanced if and only if*

$$f_{\vec{G}}(X) \leq b_G(X) \quad \textit{for all} \quad X \subseteq V. \tag{7}$$

**Proof.** Note that $b_G(X) - f_{\vec{G}}(X) = (d_G(X) - 2\lfloor R_G(X)/2 \rfloor) - (\varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)) = 2(\delta_{\vec{G}}(X) - \lfloor R_G(X)/2 \rfloor)$. If $\vec{G}$ is well-balanced then clearly $\delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor$, i.e. $b_G(X) \geq f_{\vec{G}}(X)$ for all $X$. If $\delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor$ for all $X$, then, by Menger's theorem [10] and the definition of $R$, $\vec{G}$ is well-balanced. □

**Claim 2** *A pairing $M$ of $G$ is strong if and only if $M$ is feasible.*

**Proof.** If $M$ is feasible, then for each orientation $\vec{M}$, $f_{\vec{M}}(X) \leq d_M(X) \leq b_G(X)$ for all $X$ by (5), that is $\vec{M}$ is good so $M$ is strong. If $M$ is not feasible, then let $X \subseteq V$ with $d_M(X) > b_G(X)$. Let $\vec{M}$ be an orientation of $M$ with $\delta_{\vec{M}}(X) = 0$. Then $f_{\vec{M}}(X) = d_M(X) > b_G(X)$, that is $\vec{M}$ is not good so $M$ is not strong. □

**Claim 3** *An undirected graph $G$ has a best-balanced orientation if and only if there exists a well-orientable pairing $M$ of $G$. If $\vec{M}$ is a good orientation of pairing $M$ then there exists an orientation $\vec{G}$ compatible with $\vec{M}$, and every such orientation is best-balanced.*

**Proof.** We start by proving the second statement, in which the first part is the repetition of Theorem 1. As $\vec{G}$ is compatible with the oriented pairing $\vec{M}$, it is clearly smooth. By (1), (6) and (4), $f_{\vec{G}}(X) = f_{\vec{M}}(X) \leq b_G(X)$ so $\vec{G}$ is best-balanced by Claim 1.

To prove the first statement, suppose that $\vec{G}$ is best-balanced. Let $u_1, \ldots, u_t$ be the vertices with $\delta_{\vec{G}}(u_i) = \varrho_{\vec{G}}(u_i) + 1$ and $v_1, \ldots, v_t$ the vertices with $\delta_{\vec{G}}(v_i) = \varrho_{\vec{G}}(v_i) - 1$. Then the oriented pairing consisting of arcs $u_i v_i$ $(1 \leq i \leq t)$ is compatible with $\vec{G}$ and is good by (1) and by Claim 1. The other direction follows from the second statement. □

## 5 Theorems

The following four theorems are due to Nash-Williams [11,12].

**Theorem 2 (Nash-Williams)** *A graph $G$ has a $k$-arc-connected orientation if and only if $G$ is $2k$-edge-connected.*

**Theorem 3 (Nash-Williams)** *Every graph has a best-balanced orientation.*

**Theorem 4 (Nash-Williams)** *Every subgraph $H$ of $G$ has a best-balanced orientation that can be extended to a best-balanced orientation of $G$.*

**Theorem 5 (Nash-Williams)** *Every graph has a feasible pairing.*

We present in the following claim the global case of the above "odd vertex pairing" theorem. A short proof of Claim 4 is given in the next section.

**Claim 4** *Every $2k$-edge-connected graph $G = (V, E)$ has a pairing $M$ so that*

$$d_M(X) \leq d_G(X) - 2k \quad \text{for all } X \subset V,\ X \neq \emptyset. \tag{8}$$

By Claim 2, Theorem 5 is equivalent to Theorem 6.

**Theorem 6** *Every graph has a strong pairing.*

By Theorem 3 and Claim 3, every graph has a well-orientable pairing. In the following theorem we generalize this result.

**Theorem 7** *Every pairing is well-orientable.*

The main results of this paper are the following generalizations of Theorem 4 and Theorem 3.

**Theorem 8** *Let $G = (V, E)$ be a graph, $\{E_1, ..., E_k\}$ be an arbitrary partition of $E$ and let $G_i := (V, E_i)$ $1 \leq i \leq k$. Then $G$ has a best-balanced orientation $\vec{G}$ such that the inherited orientation $\vec{G}_i$ of each $G_i$ is also best-balanced.*

**Theorem 9** *For every partition $\{X_1, ..., X_l\}$ of $V = V(G)$, $G$ has an orientation $\vec{G}$ such that $\vec{G}$ and its contractions $((\vec{G}/X_1)...)/X_l$ and $\vec{G}/(V - X_i)$ for all $1 \leq i \leq l$ are best-balanced orientations of the corresponding graphs.*

For Eulerian graphs we have the following result.

**Theorem 10** *Every Eulerian graph $G = (V, E)$ has a best-balanced orientation $\vec{G}$ such that $\vec{G} - v$ is a best-balanced orientation of $G - v$ for all $v \in V$.*

The statement of Theorem 10 is not necessarily true for non-Eulerian graphs, as the example of $K_4$ shows.

## 6 Proofs

In this section we apply Theorem 6 (or equivalently, Theorem 5) to prove all the other results in the previous section. For a relatively simple proof for Theorem 5 see [4]. A polynomial time algorithm to find a feasible pairing can be found in [6].

First, we mention that Theorems 2, 3 and 4 are easy consequences of Theorems 3, 6 and 8, respectively. We must emphasize that we do not have a new proof neither for Theorem 5 nor for Theorem 3. However, for Claim 4 we have the following simple proof.

**Proof of Claim 4:** If $k = 0$ then the claim is true by Proposition 3. From now on we assume that $k \geq 1$. We prove the statement by induction on $|E|$.

**Case 1** There is $s \in V$ with $d(s)$ even. Then, by Lovász' splitting off theorem [8], there exists an edge-pairing $\{u_i s, sv_i\}_{i=1}^{d(s)/2}$ at $s$ such that replacing each non-parallel pair $u_i s, sv_i$ by a new edge $u_i v_i$ and then deleting the vertex $s$, the new graph $G'$ is $2k$-edge-connected. Note that $T_{G'} = T_G$ and $|E(G')| < |E|$ so by induction there is a pairing $M$ of $G'$ that satisfies (8) for $G'$. Then $M$ is a pairing of $G$ and, since $d_{G'}(X) \leq d_G(X)$ for all $X \subset V$, clearly $M$ satisfies (8) for $G$ as well and we are done.

**Case 2** Otherwise, $T_G = V$. By a result of Mader [9], since there is no vertex $v$ with $d(v) = 2k$, there exists an edge $uv \in E$ such that $G' := G - uv$ is $2k$-edge-connected. Note that $T_{G'} = T_G - \{u, v\}$ and $|E(G')| < |E|$ so by induction there is a pairing $M'$ of $G'$ so that (8) is satisfied for $G'$ and $M'$. Let $M := M' \cup uv$. Then $M$ is a pairing of $G$ and for all $X \subseteq V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (8) is satisfied for $G$ and $M$ and this completes the proof. □

**Proof of Theorem 7:** Let $M_1$ be an arbitrary pairing and $M_2$ be a strong pairing of $G$. $M_2$ exists by Theorem 6. The graph $M_1 \cup M_2$ is Eulerian so it has an Eulerian orientation $\vec{M}_1 \cup \vec{M}_2$. Then $f_{\vec{M}_1}(v) = f_{\overleftarrow{M_2}}(v)$ for all $v \in V$. Thus, by (1) and using that $\overleftarrow{M}_2$ is a good orientation of $M_2$, $f_{\vec{M}_1}(X) = f_{\overleftarrow{M_2}}(X) \leq b_G(X)$ for all $X \subseteq V$, so $\vec{M}_1$ is a good orientation of $M_1$. □

By the above proof, if we know a feasible pairing, then for every pairing we can find a good orientation in polynomial time. Note that if we apply Theorem 4 with $H' = G$ and $G' = G + M$ we get another proof for Theorem 7.

**Proof of Theorem 8:** Let $M_0$ and $M_i$ be strong pairings of $G$ and of $G_i$ for $1 \leq i \leq k$ provided by Theorem 6. Note that for $K := \bigcup_0^k M_i$, $d_K(v) = \sum_0^k d_{M_i}(v) \equiv d_G(v) + \sum_1^k d_{G_i}(v) = 2d_G(v)$ is even for all $v \in V$, so $K$ has an

Eulerian orientation $\vec{K} = \bigcup_0^k \vec{M}_i$ that is $\bigcup_1^k \vec{M}_i$ and $\overleftarrow{M}_0$ are compatible. For $1 \le i \le k$, $\vec{M}_i$ is a good orientation of $M_i$, so, by Claim 3, $G_i$ has a best-balanced orientation $\vec{G}_i$ compatible with $\vec{M}_i$. Let $\vec{G} := \bigcup_1^k \vec{G}_i$. Then $\vec{G}$ and $\bigcup_1^k \vec{M}_i$ are compatible hence so are $\vec{G}$ and $\overleftarrow{M}_0$. Since the orientation $\overleftarrow{M}_0$ is good, $\vec{G}$ is a best-balanced orientation of $G$ by Claim 3. □

**Proof of Theorem 9:** Let $G_0 := (((G/X_1)/X_2)/...)/X_l$ and $G_i := G/(V - X_i)$ for $1 \le i \le l$. Let $M_i$ be a strong pairing of $G_i$ $(0 \le i \le l)$ provided by Theorem 6. It is easy to see that $G$ has a unique pairing $M$ whose restriction in $G_i$ is $M_i$ for all $0 \le i \le l$. By Theorem 7, $M$ has a good orientation $\vec{M}$. By Claim 3, $G$ has a best-balanced orientation $\vec{G}$ compatible with $\vec{M}$. $\vec{G}$ and $\vec{M}$ define the orientations $\vec{G}_i$ of $G_i$ and $\vec{M}_i$ of $M_i$ for $0 \le i \le l$. Then, by Proposition 4, $\vec{G}_i$ and $\vec{M}_i$ are compatible. Since $\vec{M}_i$ is a good orientation of $M_i$, $\vec{G}_i$ is a best-balanced orientation of $G_i$ by Claim 3. □

**Proof of Theorem 10:** We define an edge-pairing for all $v \in V$ as follows. Take a maximum number of disjoint pairs of parallel edges incident to $v$. Since $G$ is Eulerian, the other edges from $v$ go to $T_{G-v}$. These edges can be naturally paired, defined by a strong pairing $M_v$ of $G - v$, where $M_v$ exists by Theorem 6. By Proposition 5 there is an admissible Eulerian orientation $\vec{G}$ of $G$. Let $\vec{M}_v$ be the natural orientation of $M_v$ (for all $v \in V$) defined by $\vec{G}$; as $M_v$ is strong, $\vec{M}_v$ is good. Now $\vec{G} - v + \vec{M}_v$ is an Eulerian orientation of $G - v + M_v$, so by Claim 3, $\vec{G} - v$ is a best-balanced orientation of $G - v$ for all $v \in V$. □

## 7 Corollaries

Theorem 4 implies the following result for global edge-connectivity.

**Corollary 1** *For a subgraph $H$ of $G$, $H$ has an $l$-arc-connected orientation that can be extended to a $k$-arc-connected orientation of $G$ if and only if $H$ is $2l$-edge-connected and $G$ is $2k$-edge-connected.*

Note that the simple proof given for Claim 4, together with the short proof of Theorem 8 gives a direct proof for Corollary 1.

**Corollary 2** *If $H$ is an Eulerian subgraph of $G$, then any Eulerian orientation of $H$ can be extended to a best-balanced orientation of $G$.*

**Proof.** By Theorem 4, $H$ has a best-balanced orientation $\vec{H}$ that can be extended to a best-balanced orientation of $G$. Since $\vec{H}$ is smooth and $H$ is Eulerian, $\vec{H}$ is an Eulerian orientation. By Proposition 2, any other Eulerian orientation of $H$ can be reached by reversing directed cycles, and this operation cannot make the best-balanced orientation of $G$ wrong by Claim 1. □

More generally, we may consider the following problem: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $E_1 \cap E_2 \neq \emptyset$, decide whether there exist simultaneous best-balanced orientations of $G_1$ and $G_2$. This problem is $NP$-complete even if both $G_1$ and $G_2$ are restricted to be Eulerian [7]. By Corollary 2, if $E_1 \cap E_2$ defines an Eulerian graph then such orientations always exist.

**Corollary 3** *Let $x, y \in V(G)$ with $\lambda_G(x, y) = 2k + 1$. Then $G$ has a best-balanced orientation $\vec{G}$ such that $\lambda_{\vec{G}}(x, y) = k + 1$.*

**Proof.** Let $G' = G + xy$ and $H' = G$. Note that $\lambda_{G'}(x, y) = 2k + 2$. By applying Theorem 4 for $G'$ and $H'$ the corollary follows (either $\vec{G}$ or $\overleftarrow{G}$ is appropriate). □

By Proposition 3 the edge-set of any undirected graph $G$ can be decomposed into cycles and $|T_G|/2$ paths. Theorem 8 easily implies the following.

**Corollary 4** *Let us fix a decomposition of the edge-set of an undirected graph $G$ into cycles and paths. There exists a best-balanced orientation of $G$ where all the prescribed cycles and paths become directed cycles and paths.*

As a counterpart to Theorem 9 we have the following result by Theorem 8.

**Corollary 5** *For every partition $\{X_1, ..., X_l\}$ of $V(G)$, $G$ has an orientation $\vec{G}$ such that $\vec{G}$ and $\vec{G}[X_i]$ for all $1 \leq i \leq l$ are best-balanced orientations of the corresponding graphs.*

Finally we mention a conjecture on vertex-connectivity orientation (see in [5]), and prove a special case of it and some related statements.

**Conjecture 1** *Let $G = (V, E)$ be an undirected graph with $|V| > k$. Then $G$ has a $k$-vertex-connected orientation if and only if for all $X \subseteq V$ with $|X| < k$, $G - X$ is $(2k - 2|X|)$-edge-connected.*

Corollary 1 implies at once the following.

**Corollary 6** *Let $G = (V, E)$ be an undirected graph and $v \in V$. Then $G$ has a $k$-arc-connected orientation $\vec{G}$ such that $\vec{G} - v$ is $(k-1)$-arc-connected if and only if $G$ is $2k$-edge-connected and $G - v$ is $(2k - 2)$-edge-connected.*

Concerning global edge-connectivity we can replace Theorem 6 by Claim 4 in the proof of Theorem 10 and hence we have short simple proofs for the following corollaries of Theorem 10.

**Corollary 7** *An Eulerian graph $G = (V, E)$ has a $k$-arc-connected orientation $\vec{G}$ such that $\vec{G} - v$ is $(k - 1)$-arc-connected for all $v \in V$ if and only if $\vec{G}$ is $2k$-edge-connected and $G - v$ is $(2k - 2)$-edge-connected for all $v \in V$.*

The statement of Corollary 7 is not necessarily true for non-Eulerian graphs, as an example, consider the graph obtained from $K_4$ by replacing each edge by three parallel edges.

The following result was conjectured by Frank in [2].

**Corollary 8** *An Eulerian graph $G = (V, E)$ has an Eulerian orientation $\vec{G}$ such that $\vec{G} - v$ is $k$-arc-connected for all $v \in V$ if and only if $G - v$ is $2k$-edge-connected for all $v \in V$.*

For the special case of Conjecture 1 when the graph is Eulerian and $k = 2$, Berg and Jordán [1] provided a sophisticated proof. Their result below follows immediately from Corollary 8.

**Corollary 9 (Berg-Jordán)** *Let $G = (V, E)$ be a 4-edge-connected Eulerian graph such that $|V| \geq 3$ and $G - v$ is 2-edge-connected for all $v \in V$. Then $G$ has a 2-vertex-connected Eulerian orientation.*

The interested readers may find many counter-examples for problems related to well-balanced orientations in [7].

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