Augmenting the edge-connectivity of a hypergraph by adding a multipartite graph

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Abstract

Given a hypergraph, a partition of its vertex set and a non-negative integer k, find a minimum number of graph edges to be added between different members of the partition in order to make the hypergraph k-edge-connected. This problem is a common generalization of the following two problems: edge-connectivity augmentation of graphs with partition constraints (J. Bang-Jensen, H. Gabow, T. Jordán, Z. Szigeti, Edge-connectivity augmentation with partition constraints, *SIAM J. Discrete Math.* Vol. 12, No. 2 (1999) 160-207) and edge-connectivity augmentation of hypergraphs by adding graph edges (J. Bang-Jensen, B. Jackson, Augmenting hypergraphs by edges of size two, *Math. Program.* Vol. 84, No. 3 (1999) 467-481). We give a min-max theorem for this problem, that implies the corresponding results on the above mentioned problems, and our proof yields a polynomial algorithm to find the desired set of edges.

1 Introduction

The theory of edge-connectivity augmentation starts with the seminal paper of Watanabe and Nakamura [20], in which they solved the problem of global edge-connectivity augmentation of a graph, that is, given a graph $G = (V, E)$ and a non-negative integer k, find a minimum set of edges whose addition makes the graph k-edge-connected. To present their result we introduce a lower bound : $\boldsymbol{\alpha(G,k)}=\max\{\lceil \frac{1}{2}\sum_{X\in\mathcal{X}}(k-d_G(X))\rceil : \mathcal{X}$ subpartition of $V\}$. Indeed, let F be an edge set such that the new graph $G+F$ is k-edge-connected and let $\mathcal{X}$ be a subpartition of V. Then, $2|F| \geq \sum_{X\in\mathcal{X}} d_F(X) = \sum_{X\in\mathcal{X}}(d_{G+F}(X) - d_G(X)) \geq \sum_{X\in\mathcal{X}}(k - d_G(X))$, hence $\alpha(G, k)$ is a lower bound for the optimum value. The following theorem says that this can always be achieved for $k \geq 2$.

Theorem 1 (Watanabe and Nakamura [20])**.** *Let $G = (V, E)$ be a graph and $k \geq 2$ an integer. Then the minimum number of edges to be added to G in order to make it k-edge-connected is equal to $\alpha(G, k)$.*

An important breakthrough in the area of connectivity augmentation came with Frank's algorithm [9]. It led to an efficient approach to solve this kind of problems. It consists of two steps. In the first step we extend minimally the graph, that is, we add a special vertex s to the starting graph and a minimum set of edges between s and the graph in order to satisfy the desired connectivity property. This step works for general connectivity functions, that is for **symmetric** ($p(X) = p(V\setminus X)$ for all $X \subseteq V$), **skew-supermodular** ($p(X)+p(Y) \leq \max\{p(X\cap Y)+p(X\cup Y), p(X\setminus Y)+p(Y\setminus X)\}$ for all $X, Y \subseteq V$) functions, in which case we want to find a graph $H = (V+s, F)$ that **covers** the function p ($d_H(X) \geq p(X)$ for all $X \subseteq V$) and that minimizes the number of edges in F. Frank [9] showed how to find such an edge set and proved the following min-max theorem.

Theorem 2 (Frank [9], [1])**.** *Let $p : 2^V \to \mathbb{Z}\cup\{-\infty\}$ be a symmetric skew-supermodular function. Then the minimum number of edges to be added to the edgeless graph on $V+s$ between s and V to get a graph H that covers p is equal to* $\max\{\sum_{X\in\mathcal{X}} p(X) : \mathcal{X}$ *subpartition of* $V\}$.

The second step consists of the application of the splitting off technique, that is replace two edges incident to s by an edge between the corresponding vertices of the original graph if the desired connectivity property remains valid. For k-edge-connectivity, this can be done by the following result.

Theorem 3 (Lovász [16])**.** *Let $G = (V+s, E)$ be a k-edge-connected graph in V with $k \geq 2$ and $d(s)$ is even. Then each edge incident to s belongs to a pair whose splitting off at s results in a k-edge-connected graph in V.*

Repeat this operation in order to get rid of the edges incident to s and finally delete the isolated vertex s. The set of new edges obtained provides an optimal solution of the problem by Theorem 2 applied for $p(X) = k - d_G(X)$ (which is symmetric and skew-supermodular). This way we proved Theorem 1.

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In [3], the authors are given not only a graph $G = (V, E)$ and a non-negative integer k, but also a partition $\mathcal{P}$ of the vertex set and they ask for the new edges, whose addition results in a k-edge-connected graph, to connect distinct members of this partition. We may see that it contains the first problem by chosing the partition composed of singletons. Besides $\alpha(G, k)$, we have another lower bound : $\boldsymbol{\beta(G, \mathcal{P}, k)}= \max\{\sum_{Y\in\mathcal{Y}}(k - d_G(Y)) : \mathcal{Y}$ subpartition of $P, P \in \mathcal{P}\}$. Indeed, let F be an edge set such that the new graph $G+F$ is k-edge-connected, P a member of $\mathcal{P}$ and $\mathcal{Y}$ a subpartition of P. Then, since we may not add edges within P, $|F| \geq \sum_{Y\in\mathcal{Y}} d_F(Y) = \sum_{Y\in\mathcal{Y}}(d_{G+F}(Y) - d_G(Y)) \geq \sum_{Y\in\mathcal{Y}}(k - d_G(Y))$, hence $\beta(G, \mathcal{P}, k)$ is a lower bound for the optimum value. The authors of [3] efficiently solve this problem, and show that the lower bound $\boldsymbol{\Phi(G, \mathcal{P}, k)}$=max$\{\alpha(G, k), \beta(G, \mathcal{P}, k)\}$ is almost always the correct answer. Here is the smallest example when this lower bound can not be achieved: let G be the cycle C_4 on four vertices, $\mathcal{P}$ the bipartion of the bipartite graph C_4 and $k = 3$. Then $\alpha(G, k) = 2, \beta(G, \mathcal{P}, k) = 2$ but we need three edges in an optimal solution because we can not add diagonal edges to C_4. They also characterize graphs that fail the lower bound and show that one more edge is sufficient for them. The definition of the configurations are given in Section 3.

Theorem 4 (Bang-Jensen, Gabow, Jordán, Szigeti [3])**.** *Let $G = (V, E)$ be a graph, $\mathcal{P}$ a partition of V and $k \geq 2$ an integer. Then the minimum number of edges connecting distinct members of $\mathcal{P}$ to be added to G in order to make it k-edge-connected is equal to $\Phi(G, \mathcal{P}, k)$ unless G contains a C_4- or C_6-configuration, in which case it is equal to $\Phi(G, \mathcal{P}, k) + 1$.*

Another possible generalization is to study the problem for hypergraphs. Bang-Jensen and Jackson solved the problem of making a hypergraph $\mathcal{H} = (V, \mathcal{E})$ k-edge-connected by adding a minimum number of graph edges in [2]. Here again, besides $\alpha(\mathcal{H}, k)$, we have another lower bound : $\boldsymbol{\omega(\mathcal{H}, k)}= \max\{\#\text{component}(\mathcal{H} - \mathcal{F}) - 1 : \mathcal{F} \subseteq \mathcal{E}, |\mathcal{F}| = k - 1\}$. Indeed, let F be an edge set such that the new hypergraph $\mathcal{H} + F$ is k-edge-connected and let $\mathcal{F}$ be a set of $k - 1$ hyperedges of $\mathcal{H}$. Then, since F must connect the connected components of $\mathcal{H} - \mathcal{F}$, $|F| \geq \#\text{component}(\mathcal{H} - \mathcal{F}) - 1$, hence $\omega(\mathcal{H}, k)$ is a lower bound for the optimum value. They showed that the lower bound $\boldsymbol{\Phi(\mathcal{H}, k)}$=max$\{\alpha(\mathcal{H}, k), \omega(\mathcal{H}, k)\}$ can always be achieved.

Theorem 5 (Bang-Jensen, Jackson [2])**.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and $k \geq 1$ an integer. Then the minimum number of graph edges to be added to $\mathcal{H}$ in order to make it k-edge-connected is $\Phi(\mathcal{H}, k)$.*

Cosh considered the following problem in his thesis [7]. Given a hypergraph $\mathcal{H}$, a bipartition $\{P_1, P_2\}$ of its vertex set and an integer $k \geq 2$, find a minimum set of graph edges between P_1 and P_2 to be added to $\mathcal{H}$ in order to make it k-edge-connected. By the above arguments, the natural lower bound is $\boldsymbol{\Phi(\mathcal{H}, \{P_1, P_2\}, k)}= \max\{\alpha(\mathcal{H}, k), \beta(\mathcal{H}, \{P_1, P_2\}, k), \omega(\mathcal{H}, k)\}$. The definition of the configuration is given in Section 3.

Theorem 6 (Cosh [7])**.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $\{P_1, P_2\}$ a bipartition of V and $k \geq 2$ an integer. Then the minimum number of graph edges to be added between P_1 and P_2 in order to make $\mathcal{H}$ k-edge-connected is $\Phi(\mathcal{H}, \{P_1, P_2\}, k)$ if $\mathcal{H}$ contains no C_4-configuration, and $\Phi(\mathcal{H}, \{P_1, P_2\}, k) + 1$ otherwise.*

As a common generalization of all the above problems Cosh proposed the following: given a hypergraph $\mathcal{H}$, a partition $\mathcal{P}$ of its vertex set and an integer $k \geq 1$, find a minimum set of graph edges between different members of $\mathcal{P}$ to be added to $\mathcal{H}$ in order to make it k-edge-connected. For this problem the lower bound to be considered is $\boldsymbol{\Phi(\mathcal{H}, \mathcal{P}, k)}=$ $\max\{\alpha(\mathcal{H}, k), \beta(\mathcal{H}, \mathcal{P}, k), \omega(\mathcal{H}, k)\}$. He conjectured the following. The definition of the configurations are given in Section 3.

Conjecture 7 (Cosh [7])**.** *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{P}$ a partition of V and $k \geq 1$ an integer. Then the minimum number of graph edges to be added between different members of $\mathcal{P}$ in order to make $\mathcal{H}$ k-edge-connected is $\Phi(\mathcal{H}, \mathcal{P}, k)$ if $\mathcal{H}$ contains no configuration, and $\Phi(\mathcal{H}, \mathcal{P}, k) + 1$ otherwise.*

Our contribution is to provide a proof for this conjecture that is much shorther than the proof of Cosh [7] for the bipartite case. This result can be found in an extended abstract [12], in the Ph. D. Thesis [5] of Bernáth and in that [11] of Grappe. Many other generalizations of the edge-connectivity augmentation problem have also been studied in the literature. For a survey, we refer to [18].

Let us mention that Cosh [7] follows the method of Jordán [14] that uses the graph of the forbidden edges, while our approach is based on a new theorem on the number of splittable pairs to be proved in Section 4.1.

The outline of the paper is as follows. In Section 2 we recall basic definitions and state some useful facts. In Section 3 we give a complete description of the hypergraphs failing the lower bound defined above. A useful theorem about the number of splitting off one may choose is shown in Section 4.1, which helps us to prove the splitting off theorem of Section 4.2. We solve the main problem, and provide the augmentation theorem in Section 4.3. In Section 5, we provide the algorithmic details why the proof of our main theorem yields a strongly polynomial algorithm. Finally, we provide an application in Section 6 and two generalizations in Section 7.

2 Preliminaries

2.1 Definitions

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, where V is a finite set and $\mathcal{E}$ is a set of subsets of V, called **hyperedges**. A hyperedge of cardinality 2 is a **graph edge**. For a set $X \subseteq V$, let $\boldsymbol{\delta_{\mathcal{H}}(X)}$ be the set of hyperedges containing at least one vertex in X and at least one in $V - X$. The cardinality of this set is called the **degree** of X and is denoted by $\boldsymbol{d_{\mathcal{H}}(X)}$. When no confusion may arise we shall omit the subscript. Two sets X and Y are **crossing** when none of $X-Y, Y-X, X\cap Y, V-(X\cup Y)$ is empty. For a family $\mathcal{M}=\{M_1,\dots,M_l\}$ of subsets of V, let $\boldsymbol{M_0^\star}= \bigcap_{i=1}^l M_i$ and $\boldsymbol{M_i^\star}= M_i - \bigcup_{j\neq i} M_j$. For a partition $\{A_1,\dots,A_l\}$ of V and for $i=1,\dots,l$, sets A_i and A_{i+1} are called **consecutive**, where $A_{l+1}=A_1$. We denote by $\boldsymbol{X\subset V}$ that $X\subseteq V$ and $X\neq V$. A hypergraph $\mathcal{H}$ is ***k*-edge-connected** when $d_{\mathcal{H}}(X)\geq k$ for all nonempty $X\subset V$. The **local edge-connectivity** between two vertices x and y of $\mathcal{H}$ is defined by $\boldsymbol{\lambda_{\mathcal{H}}(x,y)}= \min\{d_{\mathcal{H}}(X): x\in X, y\notin X\}$. It is well known that the degree function satisfies the following equality for any subsets X and Y of V, where $\boldsymbol{d_0(X,Y)}$ (respectively $\boldsymbol{d_1(X,Y)}$) is the number of hyperedges intersecting $X-Y$ and $Y-X$ and none (resp. exactly one) of $X\cap Y$ and $V-(X\cup Y)$. $\boldsymbol{d(X,Y)}$ is the number of graph edges between $X-Y$ and $Y-X$.

$$d(X)+d(Y) \quad = \quad d(X\cap Y)+d(X\cup Y)+2d_0(X,Y)+d_1(X,Y). \tag{1}$$

Let $\mathcal{G}=(V+s,\mathcal{E})$ be a hypergraph where s is incident only to graph edges. Let $\boldsymbol{\Gamma(s)}$ denote the set of **neighbors** of s, that is the vertices of V that are adjacent to s. We say that $\mathcal{G}$ is ***k*-edge-connected in *V*** if, for any nonempty $X\subset V$, $d(X)\geq k$. In this section let $\mathcal{G}$ be such a hypergraph with $d(s)>0$ even. A set $X\subset V$ is called **tight** if $d(X)=k$ and **dangerous** if $d(X)\leq k+1$.

Let su, sv be edges of $\mathcal{G}$. We denote $\mathcal{G}-su-sv+uv$ by $\mathcal{G}_{uv}$. Replacing $\mathcal{G}$ by $\mathcal{G}_{uv}$ is called the **splitting off** su, sv and uv is a **split edge** of $\mathcal{G}_{uv}$. A pair or a splitting off su, sv is **admissible** if $\mathcal{G}_{uv}$ is still k-edge-connected in V. We will also say that su is admissible with sv. For a split edge uv of $\mathcal{G}$, $\boldsymbol{\mathcal{G}^{uv}}$ will denote the hypergraph where we **unsplit** uv, that is we undo the splitting off su, sv. For split edges e and f and edges su, sv of $\mathcal{G}$, we will use $\mathcal{G}^{e,f}$ and $\mathcal{G}^e_{uv}$ instead of $(\mathcal{G}^e)^f$ and $(\mathcal{G}^e)_{uv}$.

We say that a partition $\mathcal{P}$ of V is **$\mathcal{G}$-feasible** if $d_{\mathcal{G}}(s,P)\leq \frac{d_{\mathcal{G}}(s)}{2}$ for all $P\in\mathcal{P}$. We call $P\in\mathcal{P}$ **dominating** if $d_{\mathcal{G}}(s,P)=\frac{d_{\mathcal{G}}(s)}{2}$. The partition $\mathcal{P}$ can be considered as a coloring of the vertices. For a vertex v, $\boldsymbol{c(v)}$ denotes the color of v.

A **rainbow pair** is an admissible pair su, sv so that u and v are of different colors and any dominating color class contains one of u and v. A **complete rainbow splitting off** is a sequence of rainbow splittings that decreases the degree of s to zero.

2.2 Tight sets

Recall that $\mathcal{G}=(V+s,\mathcal{E})$ is a hypergraph that is k-edge-connected in V and s is incident only to graph edges with $d(s)>0$ is even.

The following claim can be proved by applying (1) for X and Y and for X and $V+s-Y$.

Claim 8. *Let X,Y be tight sets in $\mathcal{G}$. Then 1. if $X\cap Y\neq\emptyset$ and $X\cup Y\neq V$, then they are tight, 2. if $X-Y, Y-X\neq\emptyset$, then they are tight and $d(s,X\cap Y)=0$.*

We may define, by Claim 8, for a vertex $u\in\Gamma(s)$ that belongs to some tight sets, $\boldsymbol{X_u}$ as the minimal tight set containing u. The hypergraph $\mathcal{G}$ can be modified without destroying k-edge-connectivity in V as follows.

Claim 9. *Let $u\in\Gamma(s)$ and $u'\in X_u$. Then $\mathcal{G}-su+su'$ is k-edge-connected in V.*

Proof. Otherwise there exists a set Y of degree less than k in the new hypergraph. Then Y contains u (so $u\in X_u\cap Y$) but not u' (so $X_u-Y\neq\emptyset$) and Y is tight in $\mathcal{G}$. By the minimality of X_u, $Y-X_u\neq\emptyset$. Then, by Claim 8.2, $d(s,X_u\cap Y)=0$, a contradiction. $\square$

Claim 10. *Let $D\subseteq\delta_{\mathcal{G}}(s)$. Assume that each edge of D enters a tight set. Then there exists a partition $\mathcal{X}$ of $\bigcup_{su\in D} X_u$ such that $\sum_{X\in\mathcal{X}}(k-d_{\mathcal{G}-s}(X))\geq |D|$.*

Proof. By Claim 8, $\{X_u : su\in D\}$ form a laminar family. Let $\mathcal{X}$ be the set of maximal sets of this family. Then $\mathcal{X}$ is a partition of $\bigcup_{su\in D} X_u$ and $|D|\leq d_{\mathcal{G}}(s,\bigcup_{su\in D}X_u)=\sum_{X\in\mathcal{X}} d_{\mathcal{G}}(s,X)=\sum_{X\in\mathcal{X}}(k-d_{\mathcal{G}-s}(X))$. $\square$

2.3 Dangerous sets

We start this subsection by the characterization of admissible pairs, see [2]. In the light of Lemma 11 it is natural to study the properties of dangerous sets. The following technical lemmas will be applied throughout the paper.

Lemma 11. *[2] A pair of edges su, sv is admissible in $\mathcal{G}$ if and only if no dangerous set contains both u and v.*

Claim 12. *For a dangerous set Y, 1. $d(s,Y)\leq d(s,V-Y)$, and 2. if for a tight set X, $X\cap Y\neq\emptyset$ and $X\cup Y\neq V$, then $Y\cup X$ is dangerous.*

Proof. By Y dangerous, $d(V-Y)\geq k$, and (1) applied to Y and s, we have $1=k+1-k\geq d(Y)-d(V-Y)=2d(s,Y)-d(s)$, and then by $d(s)$ is even, 12.1 is satisfied. By (1) applied to X and Y and by $d(X\cap Y)\geq k$, 12.2 is satisfied. $\square$

Note that Claim 12.2 implies that if Y is a maximal dangerous set and X a tight set, then X and Y do not cross.

Claim 13. *Let $\mathcal{M} := \{M_1, M_2\}$ be a family of maximal dangerous sets. If $M_i^\star \cap \Gamma(s) \neq \emptyset$ for $i = 0, 1, 2$, then 1. $M_i^\star$ is tight for $i = 0, 1, 2$, 2. $d(s, M_0^\star) = 1$, 3. there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}| = 1 + 2d_0(M_0^\star, M_1^\star)$ and $\mathcal{F} = \delta(M_1^\star) \cap \delta(M_2^\star) = \delta(M_0^\star) \cap \delta(V - M_1 - M_2)$.*

Proof. Note that, by Claim 12.1, $M_1 \cup M_2 \neq V$. By maximality of M_1, $d(M_1 \cup M_2) \geq k + 2$, then apply (1) to M_1 and M_2 and then to M_1 and $\overline{M}_2 = V + s - M_2$ to get 13.1, 13.2, $d(M_1) = k + 1, d_0(M_1, M_2) = d_1(M_1, M_2) = d_1(M_1, \overline{M}_2) = 0$ and $d_0(M_1, \overline{M}_2) = 1$. Thus $\delta(M_1^\star) \cap \delta(M_2^\star) = \delta(M_0^\star) \cap \delta(V - M_1 - M_2)$. Let $\mathcal{F}$ be this set of hyperedges. By (1) applied to $M_0^\star$ and $M_1^\star$, $k + k = d(M_0^\star) + d(M_1^\star) = d(M_1) + 2d_0(M_0^\star, M_1^\star) + d_1(M_0^\star, M_1^\star) = k + 1 + 2d_0(M_0^\star, M_1^\star) + |\mathcal{F}|$, so 13.3 is proved. □

Claim 14. *Let $\mathcal{M} = \{M_1, \ldots, M_l\}$ be a family of maximal dangerous sets with $l \geq 3$. If $\Gamma(s) \bigcap M_i^\star \neq \emptyset$ for $i = 0, \ldots, l$, then for $i = 0, \ldots, l$, 1. $M_i^\star$ is tight, 2. $d(s, M_i^\star) = 1$, 3. $M_i = M_i^\star \cup M_0^\star$, 4. there exists a set $\mathcal{F}$ of $k - 1$ hyperedges of $\mathcal{E}$ intersecting every $M_i^\star$, 5. $d(M_j^\star \cup M_{j'}^\star) = k + 1$ for $1 \leq j < j' \leq l$.*

Proof. Let $t_i \in \Gamma(s) \cap M_i^\star$ for $i = 0, \ldots, l$. By applying Claim 13 to distinct pairs M_h and M_j of sets in $\mathcal{M}$, we get that t_0 belongs to the tight set $M_h \cap M_j$ and for $i = 1, \ldots, l$, t_i belongs to a tight set $M_h \setminus M_j$ for some h and j. By Claim 8.1, let Y_i be the maximal tight set containing t_i for $i = 0, \ldots, l$. Then, on the one hand, $M_0^\star \subseteq M_h \cap M_j \subseteq Y_0$ and $M_i^\star \subseteq M_h \setminus M_j \subseteq Y_i$ for $i = 1, \ldots, l$ and on the other hand, by the maximality of M_i and by the remark after Claim 12, we have $Y_i \subseteq M_i^\star$ for $i = 0, \ldots, l$. It follows that $Y_i = M_i^\star$, so 14.1 and 14.3 are satisfied for $i = 0, \ldots, l$. By Claim 13.3 applied to M_i and M_j for $i, j \in \{1, \ldots, l\}$, there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}| = 1 + 2d_0(M_0^\star, M_i^\star)$ and $\mathcal{F} = \delta(M_i^\star) \cap \delta(M_j^\star) = \delta(M_0^\star) \cap \delta(V - M_i - M_j)$. This last property implies that $d_0(M_0^\star, M_h^\star) = 0$ for all $h \neq 0, i, j$. Since it holds for all $i, j \in \{1, \ldots, l\}$, 14.4 is satisfied. Then, by M_i dangerous and $t_0, t_i \in \Gamma(s) \cap M_i^\star$ for $i = 1, \ldots, l$, we get 14.2 and then 14.5. □

Corollary 15. *If $\mathcal{G}$ has no admissible pair, then there exists a partition $V_1, \ldots, V_l$ of V and a hyperedge set $\mathcal{F} \subseteq \mathcal{E}$ of cardinality $k - 1$ such that V_i is tight, $d_{\mathcal{G}}(s, V_i) = 1$ and $\delta(V_i) = \mathcal{F} \cup sv_i$ with some $v_i \in V_i$ for every i.*

Proof. For each $t \in \Gamma(s)$, by Lemma 11, let $\mathcal{M}_t = \{M_1, \ldots, M_l\}$ be a minimal family of maximal dangerous sets containing t and covering $\Gamma(s)$. Note that, by the minimality of $\mathcal{M}_t$, $\Gamma(s) \bigcap M_i^\star \neq \emptyset$ for $i = 0, \ldots, l$. By Claim 12.1, $|\mathcal{M}_t| \geq 2$, and then by Claim 13 applied to M_1 and M_2 in $\mathcal{M}_t$, $d(s, t) = 1$ for every $t \in \Gamma(s)$. Since $|\mathcal{M}_t| = 2$ would imply $d(s) = 3$, a contradiction, we have $|\mathcal{M}_t| \geq 3$. By Claim 14 applied to $\mathcal{M}_t$, $\{V_i := M_i^\star : i = 0, \ldots, l\}$ is a subpartition of V satisfying the corollary. It is in fact a partition because if $Z := V - \bigcup_i V_i \neq \emptyset$, then every hyperedge $e \in \delta(Z)$ belongs to some $\delta(V_i)$ and hence to $\mathcal{F}$, so we have $k \leq d(Z) \leq |\mathcal{F}| = k - 1$, a contradiction. □

3 Ingredients

In this section, $\mathcal{H} = (V, \mathcal{E})$ will be a hypergraph, $\mathcal{P}$ a partition of V and k an integer. Let $\boldsymbol{OPT(\mathcal{H}, \mathcal{P}, k)}$ be the minimum number of desired edges in our augmentation problem. Following Frank's algorithm, we first describe in Subsection 3.1 how to do an optimal extension, that is how to add a new vertex s to $\mathcal{H}$ and a minimum number of graph edges between s and V in order to satisfy the partition and connectivity requirements. Afterwards, the aim is to split off rainbow pairs incident to s to get rid of s. In Subsection 3.2, we will characterize the hypergraphs for which this is impossible. To do so we will introduce obstacles. Finally, we characterize in Subsection 3.3 the hypergraphs for which the lower bound $\Phi = \Phi(\mathcal{H}, \mathcal{P}, k)$ defined in the Introduction may not be achieved. We will introduce configurations, the structures that force to have an obstacle in any optimal extension.

3.1 Optimal extension

Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a partition $\mathcal{P}$ of V and an integer k, we describe how to extend $\mathcal{H}$ that is how to add a new vertex s to $\mathcal{H}$ and a minimum number of graph edges between s and V in order to satisfy the partition and connectivity requirements.

Definition 16. *An **optimal extension** of $(\mathcal{H}, \mathcal{P})$ is a hypergraph $\hat{\mathcal{H}} = (V + s, \mathcal{E} + \delta_{\hat{\mathcal{H}}}(s))$ such that*

1. *$\hat{\mathcal{H}}$ is k-edge-connected in V,*
2. *$\delta_{\hat{\mathcal{H}}}(s)$ consists of 2Φ graph edges,*
3. *$\mathcal{P}$ is $\hat{\mathcal{H}}$-feasible.*

Theorem 17. *There exists an optimal extension of $(\mathcal{H}, \mathcal{P})$.*

Proof. We may find such an extension as follows. Recall that X_u is a minimal tight set containing u.
1. Introduce a new vertex s.
2. Add a minimum set of graph edges F between s and V such that $(V + s, \mathcal{E} + F)$ is k-edge-connected in V.

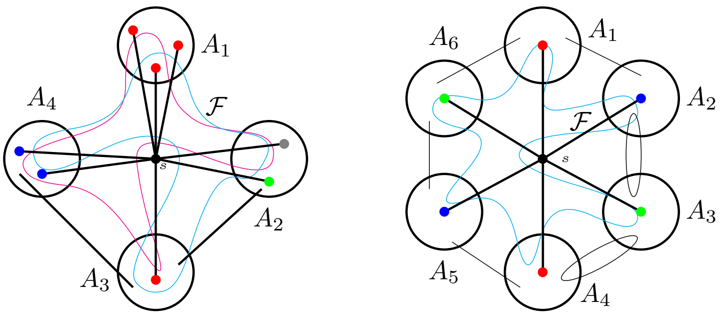

Figure 1: A $\mathcal{C}_4$-obstacle and a $\mathcal{C}_6$-obstacle

3. If $d(s)$ is odd, add an arbitrary edge incident to s to make $d(s)$ even. (Then $d(s) = 2\alpha(\mathcal{H}, k)$.)
4. Add some other edges incident to s if necessary so that $d(s) = \max\{2\alpha(\mathcal{H}, k), 2\omega(\mathcal{H}, k)\}$.
5. If some $P \in \mathcal{P}$ satisfies $d(s, P) > \frac{d(s)}{2}$, then proceed as follows.
   (a) If there exists an edge $su, u \in P$ such that $X_u \nsubseteq P$, then replace su by su', for some $u' \in X_u - P$. Note that the number of edges between s and P is decreased by 1. Repeat 5.
   (b) Otherwise for all $su, u \in P$, we have $X_u \subseteq P$. Add $2d(s, P) - d(s)$ edges between s and $V - P$.
6. Stop. Let $\hat{\mathcal{H}}$ be the resulting hypergraph.

By construction and by Claim 9, 16.1 and 16.3 are satisfied. Lemma 18 ensures that 16.2 is satisfied and hence we have obtained an optimal extension of $(\mathcal{H}, \mathcal{P})$.

Lemma 18. $d_{\hat{\mathcal{H}}}(s) = 2\Phi$.

Proof. Let $\mathcal{G}_i$ be the hypergraph obtained after Step i. Since $\alpha(\mathcal{H}, k)$ is a lower bound for the optimum, we have, by Claim 10 applied to $D = \delta_{\mathcal{G}_2}(s)$, $2\alpha(\mathcal{H}, k) \leq d_{\mathcal{G}_3}(s) \leq 2\alpha(\mathcal{H}, k)$. If we do not add edges in Step 5 then $2\Phi \leq d_{\mathcal{G}_5}(s) = \max\{2\alpha(\mathcal{H}, k), 2\omega(\mathcal{H}, k)\} \leq 2\Phi$. Otherwise we added $2d_{\mathcal{G}_{5a}}(s, P) - d_{\mathcal{G}_{5a}}(s)$ edges so $d_{\hat{\mathcal{H}}}(s) = 2d_{\mathcal{G}_{5a}}(s, P)$. Then, by Claim 10 applied to $D = \delta_{\mathcal{G}_{5a}}(s) \cap \delta_{\mathcal{G}_{5a}}(P)$, there exists a subpartition $\mathcal{Y}$ of P such that $2\Phi \geq 2\sum_{Y \in \mathcal{Y}}(k - d_{\mathcal{H}}(Y)) \geq 2|D| = 2d_{\mathcal{G}_{5a}}(s, P) = d_{\hat{\mathcal{H}}}(s)$. Note that in this case, for some subpartition $\mathcal{Y}'$ of some $P' \in \mathcal{P}$, by 16.1 and 16.3, $\Phi = \beta = \sum_{Y \in \mathcal{Y}'}(k - d_{\mathcal{H}}(Y)) \leq \sum_{Y \in \mathcal{Y}'} d_{\hat{\mathcal{H}}}(s, Y) \leq d_{\hat{\mathcal{H}}}(s, P) \leq \frac{1}{2}d_{\hat{\mathcal{H}}}(s)$, and we have equality. □

3.2 Obstacles

Let $\mathcal{G} = (V + s, \mathcal{E}')$ be a hypergraph that is k-edge-connected in V, where s is incident only to graph edges and $d_{\mathcal{G}}(s)$ is even, and $\mathcal{P}$ a $\mathcal{G}$-feasible partition of V. Below we describe two structures when no complete rainbow splitting off may be found.

Definition 19. *A partition $\mathcal{A} = \{A_1, \ldots, A_4\}$ of V is called a **$\mathcal{C}_4$-obstacle** of $\mathcal{G}$ if*

1. *$d_{\mathcal{G}}(A_i) = k$, for $i = 1, \ldots, 4$,*
2. *there exists $\mathcal{F} \subseteq \mathcal{E}'$ such that $k - |\mathcal{F}| \geq 3$ is odd and $\mathcal{F} = \delta_{\mathcal{G}}(A_1) \cap \delta_{\mathcal{G}}(A_3) = \delta_{\mathcal{G}}(A_2) \cap \delta_{\mathcal{G}}(A_4)$,*
3. *there exist $l \in \{1, 2\}$ and $P \in \mathcal{P}$ such that $(A_l \cup A_{l+2}) \cap \Gamma_{\mathcal{G}}(s) \subseteq P$.*

A $\mathcal{C}_4$-obstacle is called **simple** if $d(s, A_i) = 1$ for $i = 1, \ldots, 4$.

Definition 20. *A partition $\mathcal{A} = \{A_1, \ldots, A_6\}$ of V is called a **$\mathcal{C}_6$-obstacle** of $\mathcal{G}$ if*

1. *$d_{\mathcal{G}}(A_i) = k$, $d_{\mathcal{G}}(s, A_i) = 1$, $d_{\mathcal{G}}(A_i \cup A_{i+1}) = k + 1$ for $i = 1, \ldots, 6$,*
2. *there exists $\mathcal{F} \subseteq \mathcal{E}'$ so that $k - |\mathcal{F}| \geq 3$ is odd and $\mathcal{F} = \delta_{\mathcal{G}}(A_j) \cap \delta_{\mathcal{G}}(A_l)$ for all distinct non consecutive A_j and A_l,*
3. *there exist three distinct classes $P_1, P_2, P_3 \in \mathcal{P}$ such that $(A_j \cup A_{j+3}) \cap \Gamma_{\mathcal{G}}(s) \subseteq P_j$ for $j = 1, 2, 3$.*

An **obstacle** is either a $\mathcal{C}_4$- or a $\mathcal{C}_6$-obstacle. An **uncolored** $\mathcal{C}_4$- (respectively $\mathcal{C}_6$-) obstacle is a partition satisfying Definition 19.1-2 (resp. 20.1-2). Let $\mathcal{A}$ be an obstacle of $\mathcal{G}$. It is important to keep in mind that edges of the same color cannot enter consecutive sets of $\mathcal{A}$. We emphasize that 19.2 and 20.2 imply that the hyperedge set of $\mathcal{G}$ is composed of the following hyperedges: the edges incident to s, the set $\mathcal{F}$ of hyperedges intersecting every A_i, hyperedges intersecting only two consecutive sets and no others, and hyperedges lying inside the sets A_i.

Claim 21. *If $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle, then $d_{\mathcal{G}}(s, A_i \cup A_{i+2}) = \frac{d_{\mathcal{G}}(s)}{2}$ for $i = 1, 2$.*

Proof. $d_{\mathcal{G}-s}(A_1 \cup A_3) = d_{\mathcal{G}-s}(A_2 \cup A_4)$ because $V - (A_1 \cup A_3) = A_2 \cup A_4$. By (1) and 19.1-2, $d_{\mathcal{G}}(A_1 \cup A_3) = k + k - |\mathcal{F}| = d_{\mathcal{G}}(A_2 \cup A_4)$. It follows that $d_{\mathcal{G}}(s, A_1 \cup A_3) = d_{\mathcal{G}}(s, A_2 \cup A_4)$ and the claim is proved. □

By Claim 21, the color class $P \in \mathcal{P}$ in 19.3 is dominating. We also call the set $A_l \cup A_{l+2}$ in 19.3 **dominating**.

Claim 22. *If $\mathcal{A}$ is an uncolored obstacle, then $d(s, A_i) \geq 1$. Moreover, if $\mathcal{A}$ is a simple uncolored $\mathcal{C}_4$-obstacle or an uncolored $\mathcal{C}_6$-obstacle, then $d_0(A_i, A_{i+1}) = \frac{k-|\mathcal{F}|-1}{2} \geq 1$ and $d(A_i \cup A_{i+1}) = k+1$ for all i.*

Proof. For an uncolored $\mathcal{C}_6$-obstacle, (1) applied to A_i and A_{i+1}, and 20.1 imply the claim. Let $\mathcal{A}$ be an uncolored $\mathcal{C}_4$-obstacle. (1) applied to A_i and A_{i+1}, 19.1 and 19.2 imply that $d(A_i \cup A_{i+1}) - k$ is odd and then, by k-edge-connectivity in V, $d(A_i \cup A_{i+1}) \geq k+1$. It also follows that $d_0(A_i, A_{i+1}) \leq \frac{k-|\mathcal{F}|-1}{2}$. Then $0 = d(A_i) - k = |\mathcal{F}| + d(s, A_i) + d_0(A_i, A_{i+1}) + d_0(A_i, A_{i-1}) - k \leq |\mathcal{F}| + d(s, A_i) + 2\frac{k-|\mathcal{F}|-1}{2} - k = d(s, A_i) - 1$. Thus $d(s, A_i) \geq 1$ and if $d(s, A_i) = 1$ then we have equality everywhere and the claim follows by 19.2. □

Claim 23. *In an uncolored obstacle $\mathcal{A}$, no dangerous set may intersect distinct non consecutive sets A_i and A_j.*

Proof. Suppose that a maximal dangerous set Y intersects non consecutive A_i and A_j. We show that $A_i \cup A_j = Y$. By 19.1 or 20.1, and by Claim 12, $A_i \cup A_j \subseteq Y$. If Y intersected an other A_k, then by Claim 12, $A_k \subseteq Y$. Suppose that $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle. Then, by Claims 21, 22 and 12.1, $\frac{d(s)}{2} < d(s, Y) \leq \frac{d(s)}{2}$, is a contradiction. Now suppose that $\mathcal{A}$ is an uncolored $\mathcal{C}_6$-obstacle. Then, by Y is dangerous and by Claim 22, $k+1 \geq d(Y) \geq |\mathcal{F}| + 2\frac{k-|\mathcal{F}|-1}{2} + 3 = k+2$, is a contradiction. Thus $A_i \cup A_j = Y$. By Y is dangerous, (1) applied to A_i and A_j and by 19.1-2 or 20.1-2, $k+1 \geq d(Y) = d(A_i) + d(A_j) - 2d_0(A_i, A_j) - d_1(A_i, A_j) = 2k - 0 - |\mathcal{F}| \geq k+2$, a contradicton. □

Claim 24. *In an uncolored obstacle, A_i is a maximal tight set for all i.*

Proof. Suppose that a maximal tight set X intersects A_i and A_j for some $i < j$. By Claims 23 and 8, $j = i+1$ and $X = A_i \cup A_{i+1}$. Then by (1) applied to A_i and A_{i+1} and by 19.1 or 20.1, $k - |\mathcal{F}| = 2d_0(A_i, A_{i+1})$ is even, contradicting 19.2 or 20.2. □

Note that, by the remark after Claim 12, the above claim implies that if X is a maximal tight set and $\mathcal{A}$ an obstacle, then $X \in \mathcal{A}$. Moreover, if there is a partition of V into 4 (resp. 6) tight sets, then it must be the $\mathcal{C}_4$- (resp. $\mathcal{C}_6$-) obstacle in case one exists.

Claim 25. *If $\mathcal{A} = \{A_1, \ldots, A_6\}$ is a $\mathcal{C}_6$-obstacle, then splitting off any rainbow pair gives rise to a simple $\mathcal{C}_4$-obstacle.*

Proof. By Claim 23 and Lemma 11, 20.1 and 20.3, the only rainbow pairs are sa_{i-1}, sa_{i+1} for all i, where a_i is the unique neighbor of s in A_i. By (1) applied to $A_i \cup A_{i-1}$ and $A_i \cup A_{i+1}$, after splitting such a pair, $A_{i-1} \cup A_i \cup A_{i+1}$ is tight and $\{A_{i-1} \cup A_i \cup A_{i+1}, A_{i+2}, A_{i+3}, A_{i+4}\}$ is a simple $\mathcal{C}_4$-obstacle. □

Claim 26. *If $\mathcal{A} = \{A_1, \ldots, A_4\}$ is a $\mathcal{C}_4$-obstacle then there exists a rainbow splitting off, unless $\mathcal{A}$ is simple.*

Proof. Suppose that no rainbow splitting off exists. Let $a_i \in \Gamma(s) \cap A_i$ for all i. Since sa_i, sa_{i+1} and sa_i, sa_{i-1} are not admissible, there exist, by Lemma 11, maximal dangerous sets M_1 and M_2 containing a_i, a_{i+1} and a_i, a_{i-1} respectively. By Claim 23, $a_{i+1} \in M_1^\star, a_{i-1} \in M_2^\star$ and by 19.1 and Claim 12, $A_i \subseteq M_0^\star$. Then, by Claim 13.2, $1 = d(s, M_0^\star) \geq d(s, A_i) \geq 1$ and hence $\mathcal{A}$ is simple. □

Lemma 27. *If $\mathcal{G}$ contains an obstacle, then there is no complete rainbow splitting off, but there is a complete admissible splitting off.*

Proof. By Claims 25 and 26, it is enough to note that if $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle, then, by 19.3, splitting off any rainbow pair gives rise to a $\mathcal{C}_4$-obstacle. Hence, by Claim 22, there is no complete rainbow splitting off in $\mathcal{G}$. Since by Claim 23 there exists an admissible splitting off in a simple $\mathcal{C}_4$-obstacle, we get the last statement of the lemma. □

Corollary 28. *If $\mathcal{G}$ contains an obstacle then $d_{\mathcal{G}}(s) \geq 2\omega(\mathcal{G} - s)$.*

3.3 Configurations

Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$, a partition $\mathcal{P}$ of V and an integer k, we describe the structures of $\mathcal{H}$ for which the lower bound may not be achieved and then in the following lemma we make a link between configurations and obstacles.

Definition 29. *A partition $\{A_1, \ldots, A_4\}$ of V is a **$\mathcal{C}_4$-configuration** of $\mathcal{H}$ if*

1. *$k - d_{\mathcal{H}}(A_i) > 0$ for $i = 1, \ldots, 4$,*
2. *there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}|$ is odd and $\mathcal{F} = \delta_{\mathcal{H}}(A_1) \cap \delta_{\mathcal{H}}(A_3) = \delta_{\mathcal{H}}(A_2) \cap \delta_{\mathcal{H}}(A_4)$,*
3. *there exist $l \in \{1, 2\}$, $P \in \mathcal{P}$ and a subpartition $\mathcal{X}_j$ of $A_j \cap P$ such that $\sum_{X \in \mathcal{X}_j}(k - d_{\mathcal{H}}(X)) = k - d_{\mathcal{H}}(A_j)$ for $j = l, l+2$,*
4. *$\Phi = k - d_{\mathcal{H}}(A_1) + k - d_{\mathcal{H}}(A_3) = k - d_{\mathcal{H}}(A_2) + k - d_{\mathcal{H}}(A_4)$.*

Definition 30. *A partition $\{A_1,\dots,A_6\}$ of V is a* ***$\mathcal{C}_6$-configuration*** *of $\mathcal{H}$ if*

1. *$k-d_{\mathcal{H}}(A_i)=1$, $k-d_{\mathcal{H}}(A_i\cup A_{i+1})=1$ for $i=1,\dots,6$,*
2. *there exists $\mathcal{F}\subseteq\mathcal{E}$ such that $k-|\mathcal{F}|$ is odd and $\mathcal{F}=\delta_{\mathcal{H}}(A_j)\cap\delta_{\mathcal{H}}(A_l)$ for all distinct non consecutive A_j and A_l,*
3. *there exist $A'_i\subseteq A_i$ and three distinct classes $P_1,P_2,P_3\in\mathcal{P}$ such that $k-d_{\mathcal{H}}(A'_i)=1$ for $i=1,\dots,6$ and $A'_j\cup A'_{j+3}\subseteq P_j$ for $j=1,\dots,3$,*
4. *$\Phi=3$.*

A **configuration** is either a $\mathcal{C}_4$- or a $\mathcal{C}_6$-configuration. We mention that specializing these definitions to graphs we get the original definitions of C_4- and C_6-configurations given in [3].

Lemma 31. *Every optimal extension of $(\mathcal{H},\mathcal{P})$ contains an obstacle if and only if $\mathcal{H}$ contains a configuration.*

Proof. (of sufficiency) We show that if $\mathcal{A}$ is a $\mathcal{C}_4$-configuration (respectively $\mathcal{C}_6$-configuration) of $\mathcal{H}$ and $\hat{\mathcal{H}}$ is an optimal extension of $(\mathcal{H},\mathcal{P})$, then $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle (resp. $\mathcal{C}_6$-obstacle) of $\hat{\mathcal{H}}$. By 16.2, 29.4 (resp. 30.1 and 30.4) and 16.1, we have $\sum_i d_{\hat{\mathcal{H}}}(s,A_i)=d_{\hat{\mathcal{H}}}(s)=2\Phi=\sum_i(k-d_{\mathcal{H}}(A_i))\le\sum_i d_{\hat{\mathcal{H}}}(s,A_i)$. Hence $d_{\hat{\mathcal{H}}}(s,A_i)=k-d_{\mathcal{H}}(A_i)$ so $d_{\hat{\mathcal{H}}}(A_i)=k$ for all i, providing 19.1 (resp. the first part of 20.1. The second part of 20.1 comes from 30.1 which implies $d_{\hat{\mathcal{H}}}(A_i\cup A_{i+1})=d_{\mathcal{H}}(A_i\cup A_{i+1})+d_{\hat{\mathcal{H}}}(s,A_i)+d_{\hat{\mathcal{H}}}(s,A_{i+1})=(k-1)+1+1=k+1$). Note that $k-|\mathcal{F}|\ne 1$ otherwise $\frac{1}{2}|\mathcal{A}|=\Phi\ge\omega(\mathcal{H})\ge\#\text{component}(\mathcal{H}-\mathcal{F})-1=|\mathcal{A}|-1\ge\frac{1}{2}|\mathcal{A}|+1$, a contradiction. Hence 29.2 (resp. 30.2) implies 19.2 (resp. 20.2). By 29.3-4 (resp. 30.3-4), the edges between s and $A_l\cup A_{l+2}$ (resp. $A_j\cup A_{j+3}$) are between s and $\mathcal{X}_l\cup\mathcal{X}_{l+2}$ (resp. $A'_j\cup A'_{j+3}$), hence between s and P (resp. P_j), implying 19.3 (resp. 20.3) by 16.3.

Proof. (of necessity) Suppose that $(\mathcal{H},\mathcal{P})$ contains no configuration. By Theorem 17 there exists an optimal extension $\hat{\mathcal{H}}$ of $(\mathcal{H},\mathcal{P})$. Suppose that $\hat{\mathcal{H}}$ contains a $\mathcal{C}_4$-obstacle (respectively $\mathcal{C}_6$-obstacle) $\mathcal{A}$. 19.1-2 (resp. 20.1-2) imply 29.1-2 (resp. 30.1-2). 16.2 and Claim 21 (resp. 20.1) imply 29.4 (resp. 30.4). Therefore, since $(\mathcal{H},\mathcal{P})$ contains no configuration, 29.3 (resp. 30.3) does not hold. That is, by Claim 10, for any dominating $P\in\mathcal{P}$ (resp. there exists $P\in\mathcal{P}$) there exists $u\in P$ such that $X_u-P\ne\emptyset$ and we may replace su by su' with $u'\in X_u-P$ without violating feasibility of $\mathcal{P}$ and k-edge-connectivity in V by Claim 9. In the new hypergraph $\hat{\mathcal{H}}'$, 19.3 (resp. 20.3) is not satisfied so $\mathcal{A}$ is not a $\mathcal{C}_4$-obstacle (resp. $\mathcal{C}_6$-obstacle) anymore. Since X_u remains tight in $\hat{\mathcal{H}}'$ and by Claim 24, $\mathcal{A}$ is a partition of V into maximal tight sets in $\hat{\mathcal{H}}$, it is also in $\hat{\mathcal{H}}'$. Thus no obstacle can exist in $\hat{\mathcal{H}}'$ by Claim 24 and it is an optimal extension of $(\mathcal{H},\mathcal{P})$. ☐

4 Main results

4.1 A new theorem on admissible pairs

In this section we generalize and refine Theorem 2.12(b) of [3] on admissible pairs. It will help us to find a rainbow pair when no simple $\mathcal{C}_4$-obstacle exists but an admissible pair exists. Note that the partition constraints are not considered in the following theorem.

Theorem 32. *Let $\mathcal{G}=(V+s,\mathcal{E})$ be a hypergraph that is k-edge-connected in V, where s is incident only to graph edges and $d(s)$ is even. Suppose there is an admissible pair incident to s. Then either (i) there is an edge st that belongs to at least $\frac{d(s)}{2}$ distinct admissible pairs or (ii) $\mathcal{G}$ contains a simple uncolored $\mathcal{C}_4$-obstacle.*

Proof. For $t\in\Gamma(s)$, let $S_t\subseteq\delta(s)$ be the set of edges admissible with st, and, by Lemma 11, let $\mathcal{M}_t=\{M_1,\dots,M_l\}$ be a minimal family of maximal dangerous sets such that $t\in M_0^\star$ and $\delta(s)-S_t=\delta(s)\cap\delta(\bigcup_{i=1}^l M_i)$. Suppose that (i) is not satisfied that is $(*)$ $|S_t|\le\frac{d(s)}{2}-1$ for all $t\in\Gamma(s)$.

Claim 33. *For all $t\in\Gamma(s)$, $|\mathcal{M}_t|=2$, $d(M_0^\star)=k$, $d(s,M_0^\star)=1$ and $M_i=M_0^\star\cup M_i^\star$ for all $M_i\in\mathcal{M}_t$.*

Proof. If for some $t\in\Gamma(s)$, $|\mathcal{M}_t|=1$, then by Claim 12.1 and M_1 is dangerous, $d(s)-|S_t|=d(s,M_1)\le d(s,V-M_1)=|S_t|$ that contradicts $(*)$. Thus $|\mathcal{M}_t|\ge 2$ for all $t\in\Gamma(s)$. Claim 13, applied to pairs of sets of $\mathcal{M}_t$, and Claim 12.2, imply that $M_i=M_0^\star\cup M_i^\star$ for all i, $d(M_0^\star)=k=d(M_i^\star)$ and $d(s,M_0^\star)=1$. Suppose that for some $t_0\in\Gamma(s)$, $l=|\mathcal{M}_{t_0}|\ge 3$. Then, by Claim 14.4, there is a set $\mathcal{F}$ of $k-1$ hyperedges each intersecting $M_i^\star$ hence, by $d(M_i^\star)=k$, $\delta(M_i^\star)=\mathcal{F}\cup st_i$ for all $i=0,1\dots,l$. By Claim 14.5 and Lemma 11, $S_{t_i}=S_{t_0}$ for all $i=1,\dots,l$. The existence of an admissible pair implies that there exists $u\in S_{t_0}$. Since $st_0\in S_u$, $\{st_0,st_1,\dots,st_l\}\subseteq S_u$. Then $(*)$ applied to u and t_0 implies that $\frac{d(s)}{2}-1\ge|S_u|\ge l+1=d(s)-|S_{t_0}|\ge\frac{d(s)}{2}+1$, contradiction. ☐

By Claim 33 and $(*)$, for all $t\in\Gamma(s)$, $3=d(s,\bigcup\mathcal{M}_t)\ge\frac{d(s)}{2}+1$. Then, since $d(s)\ge 3$ is even, $d(s)=|\Gamma(s)|=4$. Let $\Gamma(s)=\{a_1,a_2,a_3,a_4\}$ so that sa_1,sa_3 is admissible. Let A_i be a maximal tight set containing a_i. By the claim below, (ii) is satisfied and the theorem is proved.

Claim 34. *$\mathcal{A}=\{A_1,A_2,A_3,A_4\}$ is a simple uncolored $\mathcal{C}_4$-obstacle.*

Proof. By Claims 33 and 12.2, $\mathcal{M}_{a_1} = \{A_1 \cup A_2, A_1 \cup A_4\}$ and $\mathcal{M}_{a_3} = \{A_2 \cup A_3, A_3 \cup A_4\}$. Since $A_1 \cup A_2$ is dangerous, so is $V - (A_1 \cup A_2)$ by $d(s) = 4$. By maximality, $V - (A_1 \cup A_2) = A_3 \cup A_4$ so $V = A_1 \cup A_2 \cup A_3 \cup A_4$. Claim 33 implies 19.1. Claim 13.3 applied to $\mathcal{M}_{a_1}$ provides 19.2, except $k - |\mathcal{F}| \geq 3$. It also holds by a parity argument because, by (1), $k + 2 \leq d(A_1 \cup A_3) = d(A_1) + d(A_3) - 2d_0(A_1, A_3) - d_1(A_1, A_3) = k + k - 0 - |\mathcal{F}|$. □

Corollary 35. *Let $\mathcal{G} = (V + s, \mathcal{E})$ be a hypergraph that is k-edge-connected in V, where s is incident only to graph edges and $d(s)$ is even, and $\hat{\mathcal{P}}$ a $\mathcal{G}$-feasible partition of V. Suppose there is an admissible pair but no rainbow pair incident to s. Then $\mathcal{G}$ contains a simple $\mathcal{C}_4$-obstacle.*

Proof. Theorem 32 applies to $\mathcal{G}$. Since no rainbow pair exists and $d_\mathcal{G}(s, P) \leq \frac{1}{2}d_\mathcal{G}(s)$ for all $P \in \mathcal{P}$, 32(i) does not hold. Then, by 32(ii), $\mathcal{G}$ contains a simple uncolored $\mathcal{C}_4$-obstacle. 19.3 also holds, otherwise, by Claim 23 and Lemma 11, there exists a rainbow pair, a contradiction. □

4.2 A new splitting off theorem

In this section, we prove the following splitting off result.

Theorem 36. *Let $\mathcal{G} = (V + s, \mathcal{E})$ be a hypergraph, where s is incident only to graph edges, $\mathcal{P}$ a partition of V and $k \geq 1$ an integer. There is a complete rainbow splitting off at s in $\mathcal{G}$ if and only if $\mathcal{G}$ is k-edge-connected in V, $d_\mathcal{G}(s) \geq 2\omega(\mathcal{G} - s, k)$ is even, $\mathcal{P}$ is $\mathcal{G}$-feasible and $\mathcal{G}$ contains no obstacle.*

Proof. (of necessity) Suppose there is a complete rainbow splitting off. Let F be the set of split edges. Then $\mathcal{G} + F$ is k-edge-connected. Since by splitting off we can not increase the edge-connectivity, $\mathcal{G}$ is k-edge-connected in V. Since $\omega(\mathcal{G} - s, k)$ is a lower bound for the augmentation problem for $\mathcal{G} - s$ and k, $d_\mathcal{G}(s) = 2|F| \geq 2\omega(\mathcal{G} - s, k)$ is even. The pairs that are split off are rainbow, so $d_\mathcal{G}(s, P) \leq \frac{1}{2}d_\mathcal{G}(s)$ for all $P \in \mathcal{P}$. By Lemma 27, no obstacle exists in $\mathcal{G}$.

Proof. (of sufficiency) Suppose now that all the conditions are satisfied. Let G and be the hypergraph obtained from $\mathcal{G}$ by performing any longest sequence of rainbow splittings. We must show that $d_G(s) = 0$. We suppose that this is not the case. Clearly, $d_G(s) = 2$ cannot be the case, so $d_G(s) \geq 4$.

Lemma 37. *G contains an admissible pair.*

Proof. Suppose not and let $\{V_1, \dots, V_l\}$ be the partition and $\mathcal{F}$ the set of hyperedges provided by Corollary 15.

Claim 38. *Each split edge uv of G is a cut edge in $G - s - \mathcal{F}$.*

Proof. We may assume $u, v \subseteq V_1$. Since we performed a longest sequence of rainbow splitting off, in G^{uv} we can not split consecutively admissible pairs su, sv_i and sv, sv_j for any sv_i and sv_j with suitable colors $(i, j \neq 1)$. Either both pairs are admissible in G^{uv} and splitting one of them destroys the other's admissibility, or one of these pairs is not admissible in G^{uv}. Therefore, by Lemma 11, there exists a dangerous set of G containing either u, v, v_i and v_j or exactly one of u and v, and at least one of v_i and v_j. Take a maximal such set Y. We may assume that $u, v_i \in Y$. By Claim 12.1, Y is disjoint from some V_r and by Claim 12.2, Y contains the tight set V_i, and if $u, v \in Y$, then $V_1 \subset Y$. Then $\mathcal{F} \cup sv_i \subseteq \delta(Y)$ and either $sv_1, sv_j \subseteq \delta(Y)$ or $uv \subseteq \delta(Y)$. Since $|\mathcal{F}| = k - 1$ and $d_G(Y) \leq k + 1$, we have $\mathcal{F} \cup sv_i \cup uv = \delta(Y)$ and uv is a cut edge in $G - s - \mathcal{F}$. □

Since $\delta(V_i) = \mathcal{F} \cup sv_i$, $G - s - \mathcal{F}$ has at least l connected components. Let F be the set of split edges in G. Then, by Claim 38, $G - s - \mathcal{F} - F = \mathcal{G} - s - \mathcal{F}$ has at least $l + |F|$ connected components. As $|\mathcal{F}| = k - 1$ and $l = d_G(s) \geq 4$, we get $1 + \omega(\mathcal{G} - s, k) \geq l + |F| = \frac{d_G(s)}{2} + \frac{d_G(s) + 2|F|}{2} \geq 2 + \frac{d_\mathcal{G}(s)}{2} \geq 2 + \omega(\mathcal{G} - s, k)$ a contradiction that proves Lemma 37. □

Since $\mathcal{G}$ contains no obstacle, but by Lemma 37 and Corollary 35, G contains a simple $\mathcal{C}_4$-obstacle $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, it follows that G contains a split edge. Let a_i be the unique neighbor of s in A_i.

Lemma 39. *For every split edge $e = xy$, G^e contains an obstacle. In particular, if $x \in A_i$ and $y \in A_{i+1}$ for some i, then $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle in G^e, and if $x, y \in A_i$ for some i, then G^e contains a $\mathcal{C}_6$-obstacle in which $A_{i+1}, A_{i+2}, A_{i+3}$ are consecutive sets.*

Proof. First suppose that $x \in A_1$ and $y \in A_2$. Then $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle in G^e. By Claim 23 and Lemma 11 applied in G^e, the pairs sx, sa_3 and sy, sa_4 are admissible. If 19.3 does not hold in G^e, then one of them, say sx, sa_3 is a rainbow pair. In $G^e_{xa_3}$, each A_i is tight and s has no neighbor in A_3 so by the remark after Claim 24 and by Claim 22, no simple $\mathcal{C}_4$-obstacle exists, a contradiction by Corollary 35. Hence 19.3 holds and then $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle in G^e.

Secondly, we may suppose, by 19.2, that $x, y \in A_1$. Let d denote the degree function in G^e. Since $c(x) \neq c(y)$, by possibly exchanging the role of x and y we can assume that $c(x) \neq c(a_3)$. In G, by Claim 23, no dangerous set contains both x and a_3. As it also holds in G^e, Lemma 11 implies that sx, sa_3 is a rainbow pair in G^e. Note that $G^e_{xa_3}$ is obtained by a longest sequence of rainbow splitting off, and therefore by Lemma 37 and Corollary 35, $G^e_{xa_3}$ contains a simple $\mathcal{C}_4$-obstacle $\mathcal{A}' = \{A'_1, A'_2, A'_3, A'_4\}$. Let a'_i be the unique neighbor of s in A'_i. We may assume $x \in A'_1$.

If $a_3 \notin A'_1$, then each A'_i is tight in G^e, and hence also in G. But then the remark after Claim 24 implies $A'_1 = A_1$, and we get the contradiction $k = d(A'_1) = d(A_1) = k+2$. Thus $a_3 \in A'_1$, which implies $d(A_1) = k+2 = d(A'_1)$. Moreover, since, in $G^e_{xa_3}$, A_3 is tight and A'_1 maximal tight by Claim 24, the remark after Claim 12 implies $A_3 \subseteq A'_1$

Let $X_1 = A_1 \cap A'_1$. Observe that $d(s, V - (A_1 \cup A'_1)) = d(s) - d(s, A_1) - d(s, A'_1) + d(s, A_1 \cap A'_1) = 6 - 3 - 3 + d(s, A_1 \cap A'_1) = d(s, X_1)$. Since $x \in X_1$, we have $d(s, X_1) > 0$, and therefore $A_1 \cup A'_1 \neq V$. In fact, $d(s, X_1) = 1$ holds. Indeed, otherwise we would have $a_2, a_4 \notin A'_1$, which would imply $A_2 \cup A_4 \subseteq V - A'_1$ because of the tightness of A_2, A_4 and A'_1 in $G^e_{xa_3}$, Claim 24 and the remark after Claim 12. In particular, $A'_1 - A_1 \subseteq A_3$, hence by 19.2 for $\mathcal{A}$ we would have $d_0(X_1, A'_1 - A_1) = 0$. Finally, (1) applied to X_1 and $A'_1 - A_1$ would provide the following contradiction, $k+2 = d(A'_1) = d(X_1) + d(A'_1 - A_1) - d_1(X_1, A'_1 - A_1) \geq k + k - |\mathcal{F}| \geq k+3$.

Then exactly one of a_2 and a_4 belongs to A'_1, as $d(s, A'_1) = 3$, $x, a_3 \in A'_1$ and $y, a_1 \notin A'_1$. We may assume that $a_4 \notin A'_1$. Now, since A'_1 is maximal tight in $G^e_{xa_3}$ and $A'_1 \cup A_4 \neq V$, then $A_4 \subseteq V - A'_1$. Hence A_4 is also a maximal tight set in $G^e_{xa_3}$ otherwise Claim 24 would be contradicted in G. So, by the remark after Claim 24, we have $A_4 \in \mathcal{A}' - A'_1$. By Claim 22 for $\mathcal{A}$, we have $d_0(A_3, A_4) \geq 1$ in G, and hence also in $G^e_{xa_3}$. Since $A_3 \subseteq A'_1$, it implies that $d_0(A'_1, A_4) \geq 1$ in $G^e_{xa_3}$. Thus, by 19.2 for $\mathcal{A}'$, and by $A_4 \in \mathcal{A}'$, we get that A'_1 and A_4 are consecutive in $\mathcal{A}'$. Hence we may assume that $A_4 = A'_4$. Recall that $d(s, V - (A_1 \cup A'_1)) = d(s, X_1) = 1$, which, since $a_4 \in V - (A_1 \cup A'_1)$, implies that $a_2 \in A'_1$ and $a'_2, a'_3 \in A_1$. Note that $A_1 \cup A'_1 \subseteq V - A_4$. Moreover, A'_2 and A'_3 being tight in G, the maximality of A_1 implies $A'_2 \cup A'_3 \subseteq A_1$. Now, $V - A_4 = A'_1 \cup A'_2 \cup A'_3 \subseteq A'_1 \cup A_1 \subseteq V - A_4$, hence there is equality everywhere. In particular, we get that $\{A'_2, A'_3\}$ is a partition of $A_1 - A'_1$. Similarly, $\{A_2, A_3\}$ is a partition of $A'_1 - A_1$.

Let $X_2 = A_2, X_3 = A_3, X_4 = A_4, X_5 = A'_3, X_6 = A'_2$. Then, $\{X_1, \dots X_6\}$ is a partition of V, and the following claim finishes the proof of Lemma 39.

Claim 40. *$\{X_1, X_2, X_3, X_4, X_5, X_6\}$ is a $\mathcal{C}_6$-obstacle in G^e.*

Proof. First we show 20.1-2: by (1) applied to A_1 and A'_1 and by $d(A_1 \cup A'_1) = d(V - (A_1 \cup A'_1)) + 4 = k+4$, $X_1 = A_1 \cap A'_1$ is tight and $d_0(X_6, X_2) = 0$. By 19.2 for $\mathcal{A}$ and $\mathcal{A}'$, there exists $\mathcal{F} \subseteq \mathcal{E}$ such that $k - |\mathcal{F}| \geq 3$ is odd and each hyperedge of $\mathcal{F}$ intersects $V - X_1$. By Claim 22 applied to $\mathcal{A}$ and $\mathcal{A}'$, $d_0(X_i, X_{i+1}) = \frac{k-|\mathcal{F}|-1}{2}$ for $i = 2, \dots, 5$. Since every X_i is tight and $d_0(X_6, X_2) = 0$, $d_0(X_i, X_{i+1}) = \frac{k-|\mathcal{F}|-1}{2}$ for $i = 6, 1$ and each hyperedge of $\mathcal{F}$ intersects X_1. Then $d(X_i \cup X_{i+1}) = k+1$ for $i = 1, \dots, 6$. Finally we show 20.3: let x_i be the unique neighbor of s in X_i. Since $A_1 \cup A'_2$ and $A'_2 \cup A'_3$ are dangerous and sx, sy is admissible, $y = a'_3$. Since a_2 and a_4 are in consecutive sets of $\mathcal{A}'$, $c(a_2) \neq c(a_4)$. Thus, by 19.3 applied for $\mathcal{A}$, $c(x_6) = c(a'_2) = c(a_3) = c(x_3)$. Similarly, $c(x_5) = c(x_2)$. By Corollary 35, $\{X_1, X_2, X_3 \cup X_4 \cup X_5, X_6\}$ is a simple $\mathcal{C}_4$-obstacle in $G^e_{x_3x_5}$ and $c(x_2) \neq c(x_6)$ so $c(x_1) = c(x_4)$. ☐

Lemma 41. *There exist two split edges e and f in G and a rainbow pair su, sv in $G^{e,f}$ such that $G' := G^{e,f}_{uv}$ contains no obstacle.*

Proof. By Lemma 39, we distinguish two cases.

Case 1: If every split edge in G connects consecutive members of $\mathcal{A}$ then $\mathcal{A}$ is an uncolored $\mathcal{C}_4$-obstacle in $\mathcal{G}$. In G, let a_i denote the neighbor of s in A_i for every i. By Lemma 39, $\mathcal{A}$ is a $\mathcal{C}_4$-obstacle in G^e for every split edge $e = xy$, hence one of x and y belongs to a dominating set in G^e, which is also a dominating set in G. If we had the same dominating set for all split edges, then $\mathcal{A}$ would be a $\mathcal{C}_4$-obstacle in $\mathcal{G}$ which is a contradiction. Thus there exist split edges $e = xy$ and $f = x'y'$ so that the different colors P of x and P' of x' are dominating in G and $y, y' \notin P \cup P'$. If $y \in A_j$, then $a_j \in P'$ and hence sx, sa_j is a rainbow pair. In $G^{e,f}_{xa_j}$, the sets of $\mathcal{A}$ are tight, so by Claim 24, no $\mathcal{C}_6$-obstacle exists. Moreover, there is no dominating color, so no $\mathcal{C}_4$-obstacle exists.

Case 2: If there exists a split edge e contained in A_i for some i then, by Lemma 39, G^e contains a $\mathcal{C}_6$-obstacle $\{B_1, \dots, B_6\}$. In G^e, let b_i be the neighbor of s in B_i for every i. Since $\mathcal{G}$ contains no obstacle, G^e has a split edge $f = xy$. We may assume that $x \in B_1$. By 20.3, Claim 23 and Lemma 11, $\{sb_4, sb_2\}$ is a rainbow pair in G^e. By Claim 25, $\{B_2 \cup B_3 \cup B_4, B_5, B_6, B_1\}$ is a simple $\mathcal{C}_4$-obstacle in $G^e_{b_4b_2}$. By Lemma 39, $G^{e,f}_{b_4b_2}$ contains an obstacle $\mathcal{A}'$. If $y \notin B_1$, then we may assume, by 20.2, that $y \in B_2$. By Lemma 39, $\mathcal{A}'$ is a $\mathcal{C}_4$-obstacle. Then, by 19.3, $c(b_3) = c(y)$. The same argument applied to $\{sb_4, sb_6\}$ shows that $c(b_5) = c(y)$. The edge sy should be of two different colors, a contradiction. If $y \in B_1$, then by Lemma 39, $\mathcal{A}'$ must be a $\mathcal{C}_6$-obstacle in which $B_2 \cup B_3 \cup B_4, B_5, B_6$ are consecutive sets, but since $c(b_3) = c(b_6)$, this gives a contradiction. ☐

Lemma 42. *G' contains a rainbow pair sw, sz.*

Proof. First we show that G' contains an admissible pair. Otherwise, by Corollary 15, $5 = d_{G'}(s) - 1 \leq \omega(G' - s, k) \leq \omega(G - s, k) + 2$. However, since G contains an admissible pair, $\omega(G - s, k) \leq \frac{d_G(s)}{2} = 2$, a contradiction. Secondly, if there was no rainbow pair in G', then by Corollary 35, G' would contain a simple $\mathcal{C}_4$-obstacle, hence $4 = d_{G'}(s) = 6$, a contradiction. ☐

Since $d_{G'_{wz}}(s) = 4 = d_G(s)$, the hypergraph G'_{wz} is obtained from $\mathcal{G}$ by performing a longest sequence of rainbow splittings, so by Lemma 39, G' contains an obtacle contradicting Lemma 41. The theorem is proved. ☐

4.3 The proof of the augmentation theorem

In this section we present and prove our main theorem. It states that the lower bound Φ can be achieved except when the starting hypergraph contains a configuration. In this case, we need one more edge. Recall that Φ was defined in Section 3.

Theorem 43. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $\mathcal{P}$ a partition of V and k an integer. Then the minimum number of graph edges to be added between different members of $\mathcal{P}$ in order to make $\mathcal{H}$ k-edge-connected is Φ if $\mathcal{H}$ contains no configuration, and $\Phi + 1$ otherwise.*

Proof. The following lemma proves the theorem.

Lemma 44. $\Phi \leq OPT(\mathcal{H}, \mathcal{P}, k) \leq \Phi + 1$*. Moreover,* $OPT(\mathcal{H}, \mathcal{P}, k) = \Phi$ *if and only if* $\mathcal{H}$ *contains no configuration.*

Proof. The first inequality was proved in the Introduction. By Theorem 17, there exists an optimal extension $\hat{\mathcal{H}}$ of $(\mathcal{H}, \mathcal{P})$. We can add two edges incident to s without violating feasibility of $\mathcal{P}$ to get $\mathcal{H}'$. By 16.2, $d_{\mathcal{H}'}(s) = 2\Phi + 2$. The two additional edges do not enter tight sets in $\mathcal{H}'$ so no obstacle exists by 19.1 and 20.1. Theorem 36 applied to $\mathcal{H}', \mathcal{P}$ and k, provides a complete rainbow splitting off and we have the second inequality.

If $\mathcal{H}$ contains no configuration, then, by Lemma 31, there exists an optimal extension $\hat{\mathcal{H}}$ of $(\mathcal{H}, \mathcal{P})$ that contains no obstacle. By 16.2, $d_{\hat{\mathcal{H}}}(s) = 2\Phi$. By Theorem 36 applied to $\hat{\mathcal{H}}, \mathcal{P}$ and k, there exists a complete rainbow splitting off in $\hat{\mathcal{H}}$, therefore $OPT(\mathcal{H}, \mathcal{P}, k) = \Phi$.

If $OPT(\mathcal{H}, \mathcal{P}, k) = \Phi$, then let F be an optimal solution and let $\mathcal{G}$ be obtained from $\mathcal{H} + F$ by adding a vertex s and by replacing every edge $uv \in F$ by the edges su and sv. Since $|F| = \Phi$, $\mathcal{G}$ is an optimal extension of $(\mathcal{H}, \mathcal{P})$. Since $\mathcal{H} + F$ is obtained from $\mathcal{G}$ by a complete rainbow splitting off, Lemma 27 implies that $\mathcal{G}$ contains no obstacle and hence, by Lemma 31, $\mathcal{H}$ contains no configuration. □

We emphasize that Theorem 43 specialized to graphs provides Theorem 4 and specialized to the partition composed of singletons provides Theorem 5.

5 Algorithmic aspects

In this section, we explain why the proof of our main theorem yields a strongly polynomial algorithm that finds a set of edges, of the desired cardinality, respecting the partition constraints and whose addition makes the hypergraph k-edge-connected.

Our algorithm starts as the algorithm of [3]: find an optimal extension of the starting hypergraph. What follows is quite different: we first decide if this optimal extension contains an obstacle. If it does, then we modify the extension to get another one that contains no obstacle. Eventually, the new extension will contain at most two more edges incident to s (it contains two more edges if and only if there existed a configuration). Then we find a complete rainbow splitting off that provides the set of edges of desired cardinality. We now explain why each of these steps is strongly polynomial.

Local edge-connectivity: Given a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and $x, y \in V$, we need a subroutine to compute the local edge-connectivity $\lambda(x, y)$ between x and y. By using a Max Flow-Min Cut algorithm in the capacitated bipartite incidence digraph of the hypergraph $\mathcal{H}$, this can be done in $O((n + m)^3)$, where $n = |V|$ and $m = |\mathcal{E}|$, for details see [2].

Deletion of edges incident to s: Given a hypergraph $\mathcal{G} = (V + s, \mathcal{E})$ that is k-edge-connected in V where $\delta(s)$ consists of graph edges, one can compute in $O(n^2(n + m)^3)$ the maximum number of copies of sx that can be removed from the hypergraph without destroying the k-edge-connectivity in V, see [2].

Splitting off: Given a hypergraph $\mathcal{G} = (V + s, \mathcal{E})$ that is k-edge-connected in V where $\delta(s)$ consists of graph edges and two neighbors x and y of s, one can compute the maximum number of admissible splittings sx, sy in $O(n(n + m)^3)$, see [2].

Tight sets: Given a hypergraph $\mathcal{G} = (V + s, \mathcal{E})$ that is k-edge-connected in V where $\delta(s)$ consists of graph edges, if a vertex $u \in V$ belongs to a tight set, then the above results allow us to compute the minimal one in $O(n(n+m)^3)$, namely $X_u = \{u\} \cup \{v \in V : \lambda(u, v) > k\}$. We can also compute the maximal one in $O(n^2(n + m)^3)$ as follows. For every vertex $v \in V$, contract s and v and find the minimal tight set $\bar{X}_v$ (if it exists) containing the resulting vertex. Then $V - \bar{X}_v$ is a maximal tight set in $\mathcal{G}$ that contains u but not v. Thus by n minimal tight set computations, we may find the maximal tight set containing u. We mention that if u is a neighbor of s, then the maximal tight set containing u is unique.

Optimal extension: The above facts imply that we may find an optimal extension of the starting hypergraph in $O(n^3(n+m)^3)$ using the algorithm given in the proof of Theorem 17: Steps 1. to 4. by results of [2], and Step 5. requires the computation of at most n minimal tight sets.

Obstacle: Deciding if V is partitioned into four or six maximal tight sets requires at most 6 computations of maximal tight sets. If it is the case, then checking if the partition is an obstacle is straightforward. Therefore, by Claim 24, deciding if the extension contains an obstacle is done in $O(n^2(n+m)^3)$.

Destroying obstacles: If the optimal extension contains an obstacle, then the proof of Lemma 31 gives the strongly polynomial algorithm that decides if the starting hypergraph contains a configuration. If there is no configuration, then it finds an optimal extension containing no obstacle by at most n computations of minimal tight sets. Otherwise, add two edges between s and V in order to ensure that no obstacle exists, as in the proof of Lemma 44. In both cases, by Theorem 36, there exists a complete rainbow splitting off in the resulting hypergraph.

Complete splitting off: To find a complete rainbow splitting off, we proceed in two steps. First, perform arbitrary rainbow splitting off as long as possible. The second step consists of unsplitting some split edges in order to find a longer sequence. This step considers two cases: either there are no admissible pairs, or there exists a simple $\mathcal{C}_4$-obstacle.

To start, perform an arbitrary sequence of rainbow splitting off until there are no rainbow pairs in the resulting hypergraph G. This can be done in $O(n^3(n+m)^3)$ because, for any two neigbors x and y of s, the maximum number of copies of sx, sy that can be split off can be computed in $O(n(n+m)^3)$. If the sequence is complete, then we are done. Otherwise, one of the following cases occurs.

(i) G contains no admissible pair. Then, by Corollary 15, there is a partition of V into tight set $\{V_1,\dots,V_\ell\}$ and a set $\mathcal{F}$ of $k-1$ hyperedges such that $\delta(V_i) = sv_i \cup \mathcal{F}$, where $v_i = V_i \cap \Gamma(s)$. In fact, each set V_i is a maximal tight set by Claim 12.2. As $d(s) \leq n$, we can compute $\{V_1,\dots,V_\ell\}$ and $\mathcal{F}$ in $O(n^3(n+m)^3)$. Repeating the following $(*)$ at most $\frac{d(s)}{2} \leq \frac{n}{2}$ times either completes the sequence of rainbow splitting off or puts us in Case (ii). By the proof of Lemma 37,

$(*)$ there exists a split edge $e = uv$, with $u, v \in V_i$ for some $i \in \{1,\dots,\ell\}$, and a rainbow pair sx, sv_j in G^e, with $x \in \{u,v\}$ and $j \neq i$, such that $G' := G^e_{xv_j}$ contains an admissible pair. If G' contains no rainbow pair, then we are in Case (ii). If G' contains a rainbow pair, then split it. Note that this pair is sy, sv_k for some $k \neq i, j$, where $y = \{u,v\} - x$. Thereafter, no admissible pair exists and $\{V_i \cup V_j \cup V_k\} \cup \{V_r : r \neq i,j,k\}$ is the partition of V into maximal tight sets.

Note that, by Claim 38, finding e means finding a split edge that is not a cut edge in $G - \mathcal{F} - s$. Hence $(*)$ was done in $O(n^2(n+m)^3)$ as it simply required to find admissible pairs containing su and sv.

(ii) G contains an admissible pair. Then, by Theorem 32, it contains a simple $\mathcal{C}_4$-obstacle. As seen previously, finding the simple $\mathcal{C}_4$-obstacle can be done in $O(n^2(n+m)^3)$. Now, the proofs of Lemmas 41 and 42 find one or two edges to unsplit in order to find a complete rainbow splitting off, and then directly give the edges to be split off.

We have sketched why our algorithm finds a set of edges of the desired cardinality in strongly polynomial time, and its overall complexity is in $O(n^3(n+m)^3)$. It relies mostly on the strongly polynomiality of the subroutine — finding a minimum cut in a hypergraph — due to the flow techniques of [2]. We emphasize that our algorithm is quite different from the algorithm to make a graph k-edge-connected with partition constraints of [3] because we may decide if an obstacle exists before the splitting off step.

6 Application

Given a graph G on n vertices and a permutation π of $\{1,\dots,n\}$, we define the **permutation graph G_π** as follows: we duplicate the graph G and we add a perfect matching defined by the permutation π between the two copies of the graph. In [10], the edge-connectivity of the permutation graph is investigated and the following theorem is proved.

Theorem 45 (Goddard, Raines, Slater [10])**.** *Let G be a simple graph of minimum degree $d \geq 1$. Then there exists a permutation π such that G_π is $(d+1)$-edge-connected if and only if G is not composed of two disjoint copies of the complete graph K_k on k vertices for some odd k.*

We define permutation hypergraphs as a natural generalization of permutation graphs. Given a hypergraph $\mathcal{G}$ on n vertices and a permutation π of $\{1,\dots,n\}$, we define the **permutation hypergraph $\mathcal{G}_\pi$** as follows: $\mathcal{G}_\pi = (V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2 \cup E_3)$ where $\mathcal{G}_1 = (V_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (V_2, \mathcal{E}_2)$ are 2 disjoint copies of $\mathcal{G}$, and $E_3 = \{v_i v_{\pi(i)} : v_i \in V_1, v_{\pi(i)} \in V_2\}$.

The following result from [13] characterizes hypergraphs that admit a k-edge-connected permutation hypergraph. It generalizes Theorem 45 and it can be proved by Theorem 43.

Theorem 46 (Jami, Szigeti [13])**.** *Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph and $k \geq 2$ an integer. Then there exists a permutation π such that $\mathcal{G}_\pi$ is k-edge-connected if and only if $d_{\mathcal{G}}(X) \geq k - |X|$ for all $\emptyset \neq X \subseteq V$, and $\mathcal{G}$ is not composed of two connected components, both of k vertices, k being odd.*

7 Generalizations

In this section we mention two generalizations of our problem. The first one, the partition constrained local edge-connectivity augmentation problem of a hypergraph is NP-complete, while the second one, the partition constrained covering of a symmetric crossing supermodular function by a graph is polynomial.

7.1 Partition constrained hypergraph local edge-connectivity augmentation by a graph

The problem of partition constrained hypergraph local edge-connectivity augmentation by a graph is defined as follows: given a hypergraph $\mathcal{H}$ on V, a symmetric requirement function $r(u,v) \in \mathbb{Z}\ \forall u,v \in V$ and a partition $\mathcal{P}$ of V, find a minimum set F of edges between different members of $\mathcal{P}$ so that the local-edge-connectivities in $\mathcal{H}+F$ satisfy the requirement function r, that is $\lambda_{\mathcal{H}+F}(u,v) \geq r(u,v)\ \forall u,v \in V$.

Let us consider the corresponding decision problem for the special case when the partition $\mathcal{P}$ is composed of singletons.

Hypergraph Local Edge-Connectivity Augmentation by a Graph
Instance: A hypergraph $\mathcal{H}$ on V, a symmetric requirement function $r(u,v) \in \mathbb{Z}\ \forall u,v \in V$, and $\gamma \in \mathbb{Z}_+$.
Question: Does there exist a graph $G=(V,E)$ with at most γ edges so that $\lambda_{\mathcal{H}+G}(u,v) \geq r(u,v)\ \forall u,v \in V$?

The following theorem shows that this problem is NP-complete.

Theorem 47 (Cosh, Jackson, Z. Király [15]). *The problem* Hypergraph Local Edge-Connectivity Augmentation by a Graph *is NP-complete.*

By Theorem 47, the even more general decision problem of partition constrained hypergraph local edge-connectivity augmentation by a graph is NP-complete.

7.2 Partition constrained covering of a symmetric crossing supermodular function by a graph

As a generalization of the global edge-connectivity augmentation in hypergraphs by adding graph edges, Benczúr and Frank [4] considered the following problem: given a symmetric, **positively crossing supermodular** ($p(X)+p(Y) \leq p(X\cap Y)+p(X\cup Y)$ for all $X,Y \subseteq V$ with $X-Y, Y-X, X\cap Y, V-(X\cup Y) \neq \emptyset$ and $p(X),p(Y)>0$) set function p, what is the minimum number of edges that cover p?

The subpartition and the component lower bounds can be extended for this problem: $\boldsymbol{\alpha(p)}=$ ⌈half of the maximum of the sum of the p-values of the sets in a subpartition of V⌉ and $\boldsymbol{dim(p)}-1 =$ one less than the maximum size of a p-full partition of V, where a partition is ***p*-full** if each union of some of its sets, has p-value at least one. Hence the lower bound for this problem is $\boldsymbol{\Phi(p)}=\max\{\alpha(p), dim(p)-1\}$.

For a hypergraph $\mathcal{H}$, the function $k-d_{\mathcal{G}}(X)$ is symmetric, positively crossing supermodular, thus the following theorem implies Theorem 5.

Theorem 48 (Benczúr, Frank [4]). *Let $p: 2^V \to \mathbb{Z}_+$ be a symmetric, positively crossing supermodular set function. Then the minimum number of edges that cover p is $\Phi(p)$.*

The following generalizes all the previously mentioned problems except the one of Subsection 7.1: given a symmetric, positively crossing supermodular set function p on V and a partition $\mathcal{P}$ of V, what is the minimum number of edges between different members of $\mathcal{P}$ that cover p? Now the lower bound to be considered is $\boldsymbol{\Phi(p,\mathcal{P})}=\max\{\alpha(p), \beta(p,\mathcal{P}), dim(p)-1\}$, where $\boldsymbol{\beta(p,\mathcal{P})}=\max\{\sum_{Y\in\mathcal{Y}}(p(Y)) : \mathcal{Y}$ subpartition of $P, P \in \mathcal{P}\}$. As a common generalization of Theorems 43 and 48 we proved in [6] the following theorem. For the definitions of C_4^*-, C_5^*-, and C_6^*-configurations see [6].

Theorem 49 (Bernáth, Grappe, Szigeti [6]). *Let $p: 2^V \to \mathbb{Z}$ be a symmetric, positively crossing supermodular set function and $\mathcal{P}$ a partition of V. Then the minimum number of edges between different members of $\mathcal{P}$ that covers p is $\Phi = \max\{\alpha(p), \beta(p,\mathcal{P}), dim(p)-1\}$ unless a C_4^*-, or a C_5^*-, or a C_6^*-configuration exists, in which case it is $\Phi+1$.*

Let us mention that the proof of Theorem 49 is much longer and significantly more complicated than the proof of the present paper.

8 Open problem

Finally, we propose the following open problem. Given a graph $G=(V,E)$, a partition $\mathcal{P}$ of V and a symmetric requirement function $r(u,v) \in \mathbb{Z}\ \forall u,v \in V$, find a minimum set F of edges between different members of $\mathcal{P}$ so that the local-edge-connectivities in $G+F$ satisfy the requirement function r, that is $\lambda_{G+F}(u,v) \geq r(u,v)\ \forall u,v \in V$.

For partial results, see the Ph. D. Thesis [19] of Végh.

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