An orientation theorem with parity conditions

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Abstract

Given a graph $G = (V, E)$ and a set $T \subseteq V$, an orientation of G is called T-odd if precisely the vertices of T get odd in-degree. We give a good characterization for the existence of a T-odd orientation for which there exist k edge-disjoint spanning arborescences rooted at a prespecified set of k roots. Our result implies Nash-Williams' theorem on covering the edges of a graph by k forests and a (generalization of a) theorem due to Nebeský on upper embeddable graphs. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Let $G = (V, E)$ be a graph. We denote the number of components of G by $c(G)$ and we use $\beta(G)$ to denote $|E(G)| - |V(G)| + c(G)$. Let G be a connected graph. An embedding of G into an orientable surface is called a 2-cell embedding if every region is homeomorphic to an open disk. The maximum integer g for which there exists a 2-cell embedding of G into an orientable surface of genus g is the *maximum genus* of G, denoted by $\gamma_M(G)$. It is known [13] that $\gamma_M(G) \leqslant \lfloor \beta(G)/2 \rfloor$. Graphs with $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$ are called *upper embeddable*.

The following theorem of Xuong gave the first characterization of maximum genus and was used later by Furst et al. [6] to construct a polynomial-time algorithm to compute $\gamma_M(G)$. For a spanning tree F of G let $x_G(F)$ denote the number of those components C of $G - E(F)$ for which $|E(C)|$ is odd. Let $\mathscr{F}(G)$ denote the set of spanning trees of G.

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Theorem 1.1 (Xuong [15]). *If G is a connected graph then*

$$\gamma_M(G) = (\beta(G) - \min\{x_G(F): F \in \mathscr{F}(G)\})/2. \tag{1}$$

Notice that $\beta(G) - x_G(F)$ is even for every spanning tree F. A spanning tree F of G is called an *even tree* of G if each connected component of $G - E(F)$ has an even number of edges. A special case of Theorem 1.1 (see also [9]) is as follows.

Corollary 1.2. *Let G be a connected graph with even $\beta(G)$. Then G is upper embeddable if and only if it has an even tree.*

Some years later Nebeský found the first NP∩co-NP characterization of $\gamma_M(G)$ by proving the following equality.

Theorem 1.3 (Nebeský [12]). *A connected graph $G = (V, E)$ satisfies*

$$\min\{x_G(F): F \in \mathscr{F}(G)\} = \max\{c(G - A) + b(G - A) - |A| - 1: A \subseteq E\}, \tag{2}$$

where $b(G - A)$ denotes the number of those components D of $G - A$ for which $\beta(D)$ is odd.

Specializing this result to upper embeddable graphs G of even $\beta(G)$ gives:

Theorem 1.4 (Nebeský [12]). *A connected graph $G = (V, E)$ has an even tree if and only if*

$$|A| \geqslant c(G - A) + b(G - A) - 1 \tag{3}$$

holds for every $A \subseteq E$.

For more results on maximum genus see the survey paper [14]. The starting point of our investigations is the purely graph theoretic Theorem 1.4 which involves parity as well as connectivity conditions. We shall prove (generalizations of) this result from a new viewpoint (namely, in terms of orientations of G) and hence give a new framework which links this result to other, probably more familiar results in graph theory.

Theorem 1.4 can be reformulated in terms of orientations of G. An *orientation* of an undirected graph G is a directed graph obtained from G by assigning an orientation to each edge of G. A subset F of edges of a directed graph is a *spanning arborescence rooted at* vertex r if F forms a spanning tree in the underlying undirected graph and each vertex has in-degree one in F, except the root r.

Let $G = (V, E)$ be a connected undirected graph and let $T \subseteq V$. An orientation of G is called *T-odd* if precisely the vertices of T get odd in-degree. It is easy to see that G has a T-odd orientation if and only if $|E| + |T|$ is even. (This can be seen by induction as follows. Take $v \in V$ for which $G - v$ is connected. If $v \in T$ and $d(v)$ is odd or $v \notin T$ and $d(v)$ is even then orient the edges incident to v towards v and apply induction on $G - v$ with respect to $T - v$ to obtain the orientation of the rest. If this is

not the case then orient all edges but one, say vx, towards v, and apply induction on $G - v$ with respect to $(T - v) \oplus \{x\}$. Here $\oplus$ denotes the symmetric difference.) From this fact we obtain the following observation (which was found earlier by Chevalier et al. [2], too).

Proposition 1.5. *Let $G = (V, E)$ be a connected graph for which $\beta(G)$ is even and let $r \in V$. Then G has an even tree if and only if there exists a $(V - r)$-odd orientation of G which contains a spanning arborescence rooted at r.*

Thus we can reformulate Theorem 1.4 as follows.

Corollary 1.6. *Let $G = (V, E)$ be a connected graph for which $\beta(G)$ is even and let $r \in V$. Then G has a $(V - r)$-odd orientation which contains a spanning arborescence rooted at r if and only if (3) holds for every $A \subseteq E$.*

Motivated by Theorem 1.4 and Corollary 1.6, we investigate more general problems concerning orientations of undirected graphs simultaneously satisfying connectivity and parity requirements. Namely, given an undirected graph $G = (V, E)$, $T \subseteq V$ and $k \geqslant 0$, our main result gives a necessary and sufficient condition for the existence of a T-odd orientation of G which contains k edge-disjoint spanning arborescences rooted at a given set of k roots. This good characterization generalizes Theorem 1.4 and at the same time slightly simplifies condition (3). Furthermore, it implies Nash-Williams' theorem on covering the edges of a graph by k forests as well. We also point out some connections to a related problem on k-edge-connected T-odd orientations which was investigated in [5]. These corollaries are discussed in Section 3.

The proof of our main result employs the proof method which was developed independently by Gallai and Anderson and which was first used to show an elegant proof for Tutte's theorem on perfect matchings of graphs, see [1]. In our case, the weaker result the proof hinges on (which is Hall's theorem in the previously mentioned proof for Tutte's result) is an orientation theorem of the first author (Theorem 1.9 below).

Let $R = \{r_1, \ldots, r_k\}$ be a *multiset* of vertices of G (that is, the elements of R are not necessarily pairwise distinct). By $T \oplus R$ we mean $((T \oplus r_1) \oplus \cdots) \oplus r_k$. For some $X \subseteq V$ the subgraph induced by X is denoted by $G[X]$. The number of edges in $G[X]$ is denoted by $i(X)$. For a partition $\mathscr{P} = \{V_1, \ldots, V_t\}$ of V with t *elements* the set of edges connecting different elements of $\mathscr{P}$ is denoted by $E(\mathscr{P})$. We set $e(\mathscr{P}) = |E(\mathscr{P})|$. The *in-degree* of a set $X \subseteq V$ in a directed graph $D = (V, E)$ is denoted by $\rho(X)$. The following well-known result is due to Edmonds.

Theorem 1.7 (Edmonds [3]). *Let R be a multiset of vertices of size k in a directed graph $D = (V, E)$. Then D contains k edge-disjoint spanning arborescences rooted at R if and only if*

$$\rho(X) \geqslant k - |X \cap R| \quad \text{for every } X \subseteq V. \tag{4}$$

The following result is due to Frank.

Theorem 1.8 (Frank [4, Theorem 2.1]). *Let $H = (V, E)$ be a graph, and let $g : V \rightarrow Z_+$ be a function. Then there exists an orientation of H whose in-degree function ρ satisfies $\rho(v) = g(v)$ for every $v \in V$ if and only if the following two conditions hold*:

$$g(V) = |E|, \tag{5}$$

$$g(X) \geqslant i(X) \quad for\ every\ \emptyset \neq X \subseteq V. \tag{6}$$

We shall rely on the following orientation theorem, which is easy to prove from Theorems 1.8 and 1.7.

Theorem 1.9. *Let $H = (V, E)$ be a graph, let $R = \{r_1, \ldots, r_k\}$ be a multiset of k vertices of H and let $g : V \rightarrow Z_+$ be a function. Then there exists an orientation of H whose in-degree function ρ satisfies $\rho(v) = g(v)$ for every $v \in V$ and for which there exist k edge-disjoint spanning arborescences with roots $\{r_1, \ldots, r_k\}$ if and only if (5) and the following condition hold*:

$$g(X) \geqslant i(X) + k - |X \cap R| \quad for\ every\ \emptyset \neq X \subseteq V. \tag{7}$$

Proof. It is easy to see that both (5) and (7) are necessary. To see the other direction let us take an orientation of H whose in-degree function ρ satisfies $\rho(v) = g(v)$ for every $v \in V$. By (5) and since (7) implies (6), such an orientation exists by Theorem 1.8. We claim that this directed graph satisfies (4) with respect to the multiset R of roots. Indeed, by observing that $\rho(X) = \sum_{x \in X} \rho(x) - i(X) = \sum_{x \in X} g(x) - i(X) = g(X) - i(X)$ for every $X \subseteq V$, by (7) we obtain $\rho(X) \geqslant i(X) + k - |X \cap R| - i(X) = k - |X \cap R|$, as required. Hence we are done by Theorem 1.7. □

Note that R is a multiset in Theorem 1.9, hence by $|X \cap R|$ in (7) we mean $|\{r_i \in R : r_i \in X,\ i = 1, \ldots, k\}|$. This convention will be used later on, whenever we take the intersection (or union) with a multiset.

Given $G = (V, E)$, $T \subseteq V$, $k \in Z_+$ and a partition $\mathscr{P} = \{V_1, \ldots, V_t\}$ of V, an element V_j $(1 \leqslant j \leqslant t)$ is called *odd* if $|V_j \cap T| - i(V_j) - k$ is odd, otherwise V_j is *even*. The number of odd elements of $\mathscr{P}$ is denoted by $s_G(\mathscr{P}, T, k)$ (where some parameters may be omitted if they are clear from the context). The following is our main result.

Theorem 1.10. *Let $G = (V, E)$ be a graph, $T \subseteq V$ and let $k \geqslant 0$ be an integer. For a multiset of k vertices $R = \{r_1, \ldots, r_k\}$ of V there exists a $T \oplus R$-odd orientation of G for which there exist k edge-disjoint spanning arborescences with roots $\{r_1, \ldots, r_k\}$ if and only if*

$$e(\mathscr{P}) \geqslant k(t - 1) + s(\mathscr{P}, T) \tag{8}$$

holds for every partition $\mathscr{P} = \{V_1, \ldots, V_t\}$ of V.

Note that taking $\mathcal{P} = \{V\}$ in (8) yields the condition $|T| - |E| - k$ is even. Also, note that the multiset R of roots plays no role in the characterization. This fact is easy to see directly by observing that if G has a proper $T \oplus R$-odd orientation for some R then G has a proper $T \oplus R'$-odd orientation for any other multiset R' of k roots. Indeed, by repeatedly taking two roots $r \in R - R'$ and $r' \in R' - R$ and a spanning arborescence F of root r in the proper $T \oplus R$-odd orientation and then reversing the edges of the directed $r - r'$ path in F, one obtains a proper $T \oplus R'$-odd orientation. In the proof, however, the most general formulation will be convenient.

2. The proof of Theorem 1.10

Proof of Theorem 1.10. To see the necessity of condition (8), consider an orientation of G with the required properties and some partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of V. The following fact is easy to observe.

Proposition 2.1. *For every $T \oplus R$-odd orientation of G and for every V_j $(1 \leqslant j \leqslant t)$ we have $|V_j \cap T| - i(V_j) - k \equiv \rho(V_j) - (k - |V_j \cap R|)$ (mod 2).*

Proof. Since the orientation is $T \oplus R$-odd, we obtain $\rho(V_j) + i(V_j) = \sum_{v \in V_j} \rho(v) \equiv |V_j \cap (T \oplus R)| \equiv |V_j \cap T| - |V_j \cap R|$ and the claim follows. □

Since there exist k edge-disjoint arborescences rooted at vertices of R, it follows from the easy part of Theorem 1.7 that $\rho(V_j) - (k - |V_j \cap R|) \geqslant 0$ for each V_j. If this number is odd (or, equivalently by Proposition 2.1, if V_j is odd) then it is at least one. This yields $e(\mathcal{P}) = \sum_{V_j \in \mathcal{P}} \rho(V_j) = \sum_{V_j \in \mathcal{P}} (\rho(V_j) - (k - |V_j \cap R|)) + kt - |V \cap R| \geqslant s(\mathcal{P}, T) + kt - k = k(t - 1) + s(\mathcal{P}, T)$, hence the necessity follows.

In what follows we prove that (8) is sufficient. An orientation is called *good* if the directed graph obtained contains k edge disjoint spanning arborescences rooted at R. Let us suppose that the statement of the theorem does not hold and let us take a counter-example (that is, a graph $G = (V, E)$ with T, R and k, for which (8) holds but no good $T \oplus R$-odd orientation exists) for which $|V| + |E|$ is as small as possible.

Proposition 2.2. $e(\mathcal{P}) \equiv k(t - 1) + s(\mathcal{P}, T)$ (mod 2) *for every partition $\mathcal{P}$.*

Proof. By choosing $\mathcal{P}_0 = \{V\}$ in (8) we obtain that $|T| - |E| - k$ is even. This implies $s(\mathcal{P}, T) \equiv \sum_1^t (|V_j \cap T| - i(V_j) - k) = |T| - (|E| - e(\mathcal{P})) - kt = |T| - |E| - k + e(\mathcal{P}) - k(t - 1) \equiv e(\mathcal{P}) - k(t - 1)$. □

We call a partition $\mathcal{P}$ of V consisting of t elements *tight* if $e(\mathcal{P}) = k(t-1) + s(\mathcal{P}, T)$ and $t \geqslant 2$.

Lemma 2.3. *There exists a tight partition of V.*

Proof. Let ab be an arbitrary edge of G. Focus on the graph $G' = G - ab$ and the modified set $T' = T \oplus b$. If there was a good $T' \oplus R$-odd orientation of G' then adding the arc ab would provide a good $T \oplus R$-odd orientation of G, which is impossible. Thus, by the minimality of G, there exists a partition $\mathscr{P}$ of V consisting of t elements violating (8) in G', that is, by Proposition 2.2, $e_{G'}(\mathscr{P}) + 2 \leqslant k(t - 1) + s_{G'}(\mathscr{P}, T')$. Clearly, $t \geqslant 2$ holds.

For the same partition in G we have $e_G(\mathscr{P}) \leqslant e_{G'}(\mathscr{P}) + 1$ and also $s_G(\mathscr{P}, T) \geqslant s_{G'}(\mathscr{P}, T') - 1$, since adding the edge ab and replacing $T \oplus b$ by T may change the parity of at most one element of the partition. Thus $k(t - 1) + s_G(\mathscr{P}, T) \geqslant k(t - 1) + s_{G'}(\mathscr{P}, T') - 1 \geqslant e_{G'}(\mathscr{P}) + 2 - 1 \geqslant e_G(\mathscr{P}) \geqslant k(t - 1) + s_G(\mathscr{P}, T)$, hence $\mathscr{P}$ is tight in G and the lemma follows. $\square$

Let us fix a tight partition $\mathscr{P} = \{V_1, \ldots, V_t\}$ in G for which t is maximal. Denote the number of odd components of $\mathscr{P}$ by s.

Lemma 2.4. *Every element V_j of $\mathscr{P}$ $(1 \leqslant j \leqslant t)$ has the following property*:
   (a) *if V_j is even, then (8) holds in $G[V_j]$, with respect to $T \cap V_j$,*
   (b) *if V_j is odd, then for each vertex $v \in V_j$, (8) holds in $G[V_j]$, with respect to* $(T \cap V_j) \oplus v$.

Proof. We handle the two cases simultaneously. Suppose that there exists a partition $\mathscr{P}'$ of t' elements in $G[V_j]$ violating (8) (with respect to $T \cap V_j$ in case (a) or with respect to $(T \cap V_j) \oplus v$, for some $v \in V_j$, in case (b)). By Proposition 2.2 this implies $k(t' - 1) + s' \geqslant e(\mathscr{P}') + 2$, where s' denotes the number of odd elements of $\mathscr{P}'$. Consider the partition $\mathscr{P}'' = (\mathscr{P} - V_j) \cup \mathscr{P}'$, consisting of t'' elements from which s'' are odd. Clearly, $e(\mathscr{P}'') = e(\mathscr{P}) + e(\mathscr{P}')$ and $t'' = t + t' - 1$. Furthermore, $s'' \geqslant s + s' - 2$, since the parity of at most two elements may be changed (these are V_j—only in case (b)—and the element in $\mathscr{P}'$ which contains the vertex v—only in case (b) again). Since (8) holds for $\mathscr{P}''$ by the assumption of the theorem, we have $k(t'' - 1) + s'' \leqslant e(\mathscr{P}'') = e(\mathscr{P}) + e(\mathscr{P}') \leqslant k(t - 1) + s + k(t' - 1) + s' - 2 = k((t + t' - 1) - 1) + s + s' - 2 \leqslant k(t'' - 1) + s''$. Thus $\mathscr{P}''$ is a tight partition with $t'' > t$, which contradicts the choice of $\mathscr{P}$. $\square$

Let us denote the graph obtained from G by contracting each element V_j of $\mathscr{P}$ into a single vertex v_j $(1 \leqslant j \leqslant t)$ by H. Let $R' = \{r'_1, \ldots, r'_k\}$ denote the multiset of vertices of H corresponding to the vertices of R in G (that is, every root in some V_j yields a new root v_j). Furthermore, let A denote those vertices of H which correspond to odd elements of $\mathscr{P}$ and let $B = V(H) - A$. Note that since $\mathscr{P}$ is tight, we have

$$|E(H)| = e(\mathscr{P}) = k(t - 1) + s(\mathscr{P}). \tag{9}$$

Now, define the following function g on the vertex set of H:

$$g(v_j) = \begin{cases} k + 1 - |V_j \cap R| & \text{if } v_j \in A, \\ k - |V_j \cap R| & \text{otherwise.} \end{cases}$$

Lemma 2.5. *There exists an orientation of H whose in-degree function is g and which contains k edge-disjoint spanning arborescences with roots $\{r'_1, \ldots, r'_k\}$.*

Proof. To prove the lemma we have to verify that the two conditions (5) and (7) of Theorem 1.9 are satisfied. First, we can see that $g(V(H)) = g(A) + g(B) = s(k+1) + (t-s)k - k = k(t-1) + s = |E(H)|$ by the definition of g and by (9). Thus (5) is satisfied.

To verify (7), let us choose an arbitrary non-empty subset X of $V(H)$. Let us define the partition $\mathscr{P}^*$ of $V(G)$ by $\mathscr{P}^* := \{V_j: v_j \in V(H) - X\} \cup \bigcup_{v_j \in X} V_j$. Then $\mathscr{P}^*$ has $t^* = t - |X| + 1$ elements and the number of its odd elements s^* is at least $s - |X \cap A|$. Applying (8) for $\mathscr{P}^*$, it follows that $k(t^* - 1) + s^* \leqslant e(\mathscr{P}^*)$. Hence $k((t - |X| + 1) - 1) + s - |X \cap A| \leqslant k(t^* - 1) + s^* \leqslant e(\mathscr{P}^*) = e(\mathscr{P}) - i(X) = k(t-1) + s - i(X)$. From this it follows that $i(X) + k \leqslant k|X| + |X \cap A|$. Therefore, $i(X) + k - |X \cap R'| \leqslant k|X| + |X \cap A| - |X \cap R'| = |X \cap A|(k+1) + |X \cap B|k - |X \cap R'| = g(X \cap A) + g(X \cap B) = g(X)$, proving that (7) is also satisfied. $\square$

Let us fix an orientation of H whose in-degree function $\rho_H = g$ and which contains a set $\mathscr{F}$ of k edge-disjoint spanning arborescences $\{F_1, \ldots, F_k\}$ with roots $\{r'_1, \ldots, r'_k\}$. Such an orientation exists by Lemma 2.5. Observe, that this orientation of H corresponds to a partial orientation of G (namely, an orientation of the edges of $E(\mathscr{P})$).

For any vertex v_j of H there are $g(v_j)$ arcs entering v_j. If V_j is even then each arc entering v_j belongs to some arborescence in $\mathscr{F}$. If V_j is odd then each arc entering v_j except exactly one belongs to some arborescence of $\mathscr{F}$, by the definition of g.

For an arbitrary $V_j \in \mathscr{P}$ let us denote by R_j^H the multiset of those vertices in V_j which are the heads of the arcs of this partial orientation entering V_j and belonging to some arborescence in $\mathscr{F}$. By the definition of g, we have $|R_j^H| = k - |V_j \cap R|$. Let $R_j = (V_j \cap R) \cup R_j^H$. Note that $|R_j| = |V_j \cap R| + |R_j^H| = k$. Furthermore, if V_j is odd then let us denote by a_j the vertex in V_j which is the head of the unique arc entering V_j and not belonging to any arborescence in $\mathscr{F}$. Let $T_j = T \cap V_j$ if V_j is even and let $T_j = (T \cap V_j) \oplus a_j$ if V_j is odd.

By the minimality of G and since $|V_j| < |V(G)|$ for each $1 \leqslant j \leqslant t$, Lemma 2.4 implies that for each j there exists a $T_j \oplus R_j$-odd orientation of $G[V_j]$ which contains k edge-disjoint spanning arborescences with roots in R_j. Combining these orientations of the subgraphs induced by the elements of $\mathscr{P}$ and the orientation of $E(\mathscr{P})$ obtained earlier, we get an orientation of G. This orientation is clearly a good $T \oplus R$-odd orientation of G, contradicting our assumption on G. This contradiction proves the theorem. $\square$

3. Corollaries

As we reformulated Theorem 1.4 in terms of odd orientations and spanning arborescences in Corollary 1.6, we can similarly reformulate Theorem 1.10 in terms of even components and spanning trees.

Theorem 3.1. *A graph $G = (V, E)$ has k edge-disjoint spanning trees $F_1, \ldots, F_k$ so that each connected component of $G - \bigcup_1^k E(F_i)$ has an even number of edges if and only if*

$$e(\mathscr{P}) \geqslant k(t-1) + s \tag{10}$$

holds for each partition $\mathscr{P} = \{V_1, \ldots, V_t\}$ of V, where s is the number of those elements of $\mathscr{P}$ for which $i(V_j) + k(|V_j| - 1)$ is odd.

Proof. As we observed, G has an orientation for which the in-degree of every vertex is even if and only if each connected component of G contains an even number of edges. Thus the desired spanning trees exist in G if and only if G has a $T \oplus R$-odd orientation which contains k edge-disjoint r-arborescences, where $T = V$, if k is odd, $T = \emptyset$, if k is even, and $R = \{r_1, \ldots, r_k\}$, $r_i = r$ $(i = 1, \ldots, k)$ for an arbitrary $r \in V$. Based on this fact, Theorem 1.10 proves the theorem (by observing that (8) specializes to (10) due to the special choice of T). $\quad\square$

The special case $k = 1$ of Theorem 3.1 corresponds to Theorem 1.4. Since (3) implies (10) if $k = 1$, Theorem 3.1 applies and we obtain a slightly simplified version of Nebeský 's result. Note also that our main result provides a proof of different nature for Theorem 1.4 by using Theorem 1.9.

Nash-Williams' classical theorem on forest covers can also be deduced from our main result.

Theorem 3.2 (Nash-Williams [11]). *The edges of a graph $G = (V, E)$ can be covered by k forests if and only if*

$$i(X) \leqslant k(|X| - 1) \tag{11}$$

holds for every $\emptyset \neq X \subseteq V$.

Proof. We consider the sufficiency of the condition. Let $G = (V, E)$ be a graph for which (11) holds. The first claim is that we can add new edges to G until the number of edges equals $k(|V| - 1)$ without destroying (11). To see this, observe that the addition of a new edge $e = xy$ (which may be parallel to some other edges already present in G) cannot be added if and only if $x, y \in Z$ for some $Z \subseteq V$ with $i(Z) = k(|Z| - 1)$. Such a set, satisfying (11) with equality, will be called *full*. It is well known that the function $i : 2^V \to Z_+$ is supermodular, that is, it satisfies $i(X) + i(Y) \leqslant i(X \cap Y) + i(X \cup Y)$ for every $X, Y \subseteq V$. Therefore, for two intersecting full sets Z and W, we have $k(|Z| - 1) + k(|W| - 1) = i(Z) + i(W) \leqslant i(Z \cap W) + i(Z \cup W) \leqslant k(|Z \cap W| - 1) + k(|Z \cup W| - 1) = k(|Z| - 1) + k(|W| - 1)$. Thus equality holds everywhere, and the sets $Z \cap W$ and $Z \cup W$ are also full. Now let F be a maximal full set (we may assume $F \neq V$) and $e = xy$ for some pair $x \in F$, $y \in V - F$. If we destroyed (11) by adding e, we would have a full set $x, y \in F'$ in G intersecting F, hence $F \cup F'$ would also be full by our previous observation. This contradicts the maximality of F.

Thus in the rest of the proof we may assume that $|E| = k(|V| - 1)$. We claim that there exist k edge-disjoint spanning trees in G. The existence of these trees immediately implies that G can be covered by k forests because $|E| = k(|V| - 1)$.

By Theorem 3.1, it is enough to prove that (10) holds in G. Let $\mathscr{P} = \{V_1, \ldots, V_t\}$ be a partition of V and let $V_1, \ldots, V_s$ denote the odd elements of $\mathscr{P}$ (with respect to k). Observe that for an odd element V_j the parity of $i(V_j)$ and $k(|V_j| - 1)$ must be different (this holds for even k and for odd k as well), hence these numbers cannot be equal. Thus we can count as follows: $e(\mathscr{P}) = |E| - \sum i(V_i) = k(|V| - 1) - \sum (i(V_i): V_i \text{ is even}) - \sum (i(V_j): V_j \text{ is odd}) \geqslant k(|V| - 1) - \sum (k(|V_i| - 1): V_i \text{ is even}) - \sum (k(|V_j| - 1) - 1: V_j \text{ is odd}) = k(|V| - 1) - k|V| + kt + s = k(t - 1) + s$, as required. $\square$

Finally, we point out some connections to a related problem which was solved recently by Frank and Király [5]. In an undirected graph $G = (V, E)$ a set $T \subseteq V$ is *G-even* if $|T| + |E|$ is even. As we remarked, G has a T-odd orientation if and only if T is G-even. Given a graph G and a G-even subset $T \subseteq V(G)$ one may ask for a necessary and sufficient condition for the existence of a T-odd orientation of G which is k-edge-connected. (A directed graph $G = (V, E)$ is *k-edge-connected* if $\rho(X) \geqslant k$ holds for every $\emptyset \neq X \subset V$, where $\rho(X)$ denotes the number of edges entering X.) This question is still open, even for $k = 1$. On the other hand, Frank and Király characterized those graphs G which have k-edge-connected T-odd orientations for *every* G-even T. One of their main result is the following.

Theorem 3.3 (Frank and Király [5]). *Let G be an undirected graph and k be a positive integer. Then G has a k-edge-connected T-odd orientation for every G-even subset $T \subseteq V$ if and only if*

$$e(\mathscr{P}) \geqslant (k + 1)t - 1 \tag{12}$$

holds for every partition $\mathscr{P} = \{V_1, \ldots, V_t\}$, $t \geqslant 2$, of V.

Our results imply a similar condition for those graphs G which have a $T \oplus R$-odd orientation with k edge-disjoint spanning arborescences rooted at R for *every* T and *every* set R of k roots.

Theorem 3.4. *Let $G = (V, E)$ be an undirected graph and k be a positive integer. Then G has a $T \oplus R$-odd orientation for which there exist k edge-disjoint spanning arborescences with roots $\{r_1, \ldots, r_k\}$ for every subset $T \subseteq V$ with $|T| + |E| - k$ even (and for an arbitrarily chosen multiset $R = \{r_1, \ldots, r_k\} \subseteq V$ of k roots) if and only if*

$$e(\mathscr{P}) \geqslant (k + 1)(t - 1) \tag{13}$$

holds for every partition $\mathscr{P} = \{V_1, \ldots, V_t\}$, $t \geqslant 2$, of V.

Proof. First suppose G satisfies (13) and take some $T \subseteq V$ with $|T| + |E| - k$ even. Clearly, $s(\mathscr{P}, T) \leqslant t$ for any partition $\mathscr{P}$ consisting of t classes. Thus $e(\mathscr{P}) \geqslant (k + 1)$

$(t-1) = k(t-1) + t - 1$ shows that G satisfies (8) for every partition $\mathscr{P}$ with $t \geqslant 2$ unless $e(\mathscr{P}) = (k+1)(t-1)$ and $s(\mathscr{P}, T) = t$. However, by Proposition 2.2 every partition $\mathscr{P}'$ satisfies $e(\mathscr{P}') \equiv k(t-1) + s(\mathscr{P}', T) \pmod 2$ and hence $(k+1)(t-1) \equiv k(t-1) + s(\mathscr{P}, T) \pmod 2$ follows. Thus the latter case cannot occur. This shows (8) holds for every T and every $\mathscr{P}$. Therefore, Theorem 1.10 implies that G has the desired $T \oplus R$-odd orientation for T and for every set R of k roots.

Conversely, suppose that for some partition $\mathscr{P} = \{V_1, \ldots, V_t\}$, $t \geqslant 2$, we have $e(\mathscr{P}) < (k+1)(t-1)$. Now by defining $T \subseteq V$ appropriately, we can make at least $t-1$ classes V_i "odd" and hence $s(\mathscr{P}, T) \geqslant t - 1$. (Since $|T| + |E| - k$ has to be even, the parity of $s(\mathscr{P}, T)$ is determined.) Thus for this T and $\mathscr{P}$ we have $e(\mathscr{P}) < (k+1)(t-1) = k(t-1) + t - 1 \leqslant k(t-1) + s(\mathscr{P}, T)$, showing that G violates (8). Thus, by the easy part of Theorem 1.10, the required orientation does not exist for every T in G. $\square$

Note that by a theorem of Tutte and Nash-Williams on edge-disjoint spanning trees, condition (13) is equivalent to the fact that G has $k+1$ edge-disjoint spanning trees. Using this, one can obtain a different proof for Theorem 3.4.

4. Remarks

The problem of finding a spanning tree of Theorem 1.4 with the required properties (or more generally, the problem of computing $\min\{x_G(F): F \in \mathscr{F}(G)\}$) can be formulated as a matroid parity problem in a certain co-graphic matroid, hence it can be solved in polynomial time. This was observed by Furst et al. [6] who gave a $O(en^2 \log^6 n)$ time algorithm based on results from [7]. With the help of this reduction one can obtain Theorem 1.3 from Lovász's characterization of the maximum size matroid-matching in a linear matroid [10].

A similar reduction, where the matroid is the dual of the sum of k graphic matroids, seems to work in the more general case of Theorem 3.1, too. However, from algorithmic point of view, such a reduction is not satisfactory, since it is not known how to represent the matroid in question. This difficulty can be handled by a useful observation of Iwata [8] which reduces the matroid parity problem on the sum of the k matroids to an instance of the matroid matching problem on the direct sum of the k matroids. It would be interesting to develop a more efficient algorithm for this problem as well as for the problem of Theorem 1.10.

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