Sandwich problems on orientations*

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Abstract

The graph sandwich problem for property Π is defined as follows: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ which satisfies property Π? We propose to study sandwich problems for properties Π concerning orientations, such as Eulerian orientation of a mixed graph and orientation with given in-degrees of a graph. We present a characterization and a polynomial-time algorithm for solving the m-orientation sandwich problem.

1 Introduction

Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with the same vertex set V and $E_1 \subseteq E_2$, a graph $G = (V, E)$ is called a *sandwich* graph for the pair G_1, G_2 if $E_1 \subseteq E \subseteq E_2$. The graph sandwich problem for property Π is defined as follows [13]:

GRAPH SANDWICH PROBLEM FOR PROPERTY Π
Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$.
Question: Is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ and G satisfies property Π?

We call E_1 the *mandatory* edge set, $E_0 = E_2 \setminus E_1$ the *optional* edge set and E_3 the *forbidden* edge set, where E_3 denotes the set of edges of the complementary graph $\overline{G}_2$ of G_2. Thus any sandwich graph $G = (V, E)$ for the pair G_1, G_2 must contain all mandatory edges, no forbidden edges and may contain a subset of the optional edges. Graph sandwich problems have attracted much attention lately arising from many applications and as a natural generalization of recognition problems [1, 2, 3, 7, 23, 25]. The recognition problem for a class of graphs $\mathcal{C}$ is equivalent to the graph sandwich problem in which $G_1 = G_2 = G$, where G is the graph we want to recognize and property Π is "to belong to class $\mathcal{C}$".

In this paper we propose to study sandwich problems for properties Π concerning orientations, such as Eulerian orientation of a mixed graph and orientation with given in-degrees of a graph, or more generally of a mixed graph.

The paper is organized as follows: Section 2 contains some basic definitions, notations and results. Section 3 contains some known results on degree constrained sandwich problems. We consider the undirected version and the directed version, the complexity, the characterization and the related optimization problems. We also define a simultaneous version and discuss its complexity. Section 4 focuses on Eulerian sandwich problems. We consider first undirected graphs and then directed graphs. These problems were already solved in [13], here we point out that the undirected case reduces to T-joins, while the directed case to circulations. We

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discuss the complexity of the problems and their characterizations and we also propose some mixed versions. In Section 5 we consider sandwich problems regarding an m-orientation, i.e., given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V, we show that it is polynomial to decide whether there exists a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has an orientation $\vec{G}$ whose in-degree vector is m that is $d^-_{\vec{G}}(v) = m(v)$ for all $v \in V$. This result stands in contrast to the strongly connected m-orientation sandwich problem which we show is NP-complete. Section 6 is devoted to a new kind of sandwich problem where we may contract (and not delete) optional edges and property Π is being bipartite.

2 Definitions

Undirected graphs. Let $G = (V, E)$ be an undirected graph. For vertex sets X and Y, the *cut* induced by X is defined to be the set of edges of G having exactly one end-vertex in X and is denoted by $\boldsymbol{\delta_G(X)}$. The *degree* $\boldsymbol{d_G(X)}$ (or $d_E(X)$) of X is the cardinality of the cut induced by X, that is $d_G(X) = |\delta_G(X)|$. The number of edges between $X \setminus Y$ and $Y \setminus X$ is denoted by $\boldsymbol{d_G(X, Y)}$. The number of edges of G having both (resp. at least one) end-vertices in X is denoted by $\boldsymbol{i_G(X)}$ or $i_E(X)$ or simply $i(X)$ (resp. $\boldsymbol{e_G(X)}$). It is well-known that (1) is satisfied for all $X, Y \subseteq V$,

$$d_G(X) + d_G(Y) \quad = \quad d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y). \tag{1}$$

We say that a vector m on V is the *degree vector* of G if $d_G(v) = m(v)$ for all $v \in V$. For a vector m on V, we consider m as a *modular* function, that is we use the notation: $\boldsymbol{m(X)}= \sum_{v \in X} m(v)$. Let us recall that $d_G(X)$ is the *degree function* of G. We define $\boldsymbol{\hat{d}_G}$ as the modular function defined by the degree vector $d_G(v)$ of G. Note that $\hat{d}_G(X) = d_G(X)+2i_G(X)\ \forall X \subseteq V$.

We denote by $\boldsymbol{T_G}$ the set of vertices of G of odd degree. For an edge set F of G, the subgraph induced by F, that is (V, F), is denoted by $\boldsymbol{G(F)}$. We say that G is *Eulerian* if the degree of each vertex is even, that is if $T_G = \emptyset$. Note that we do not suppose the graph to be connected.

Let T be a vertex set in G. An edge set F of G is called T*-join* if the set of odd degree vertices in the subgraph induced by F coincide with T, that is if $T_{G(F)} = T$. Given a cost vector on the edge set of G, a minimum cost T-join can be found in polynomial time by Edmonds and Johnson's algorithm [5].

Let f be a non-negative integer vector on V. An edge set F of G is called an f*-factor* of G if f is the degree vector of $G(F)$, that is $d_F(v) = f(v)$ for all $v \in V$. If $f(v) = 1$ for all $v \in V$, then we say that F is a 1-factor or a *perfect matching*. An f-factor - if it exists - can be found in polynomial time, see [21]. The graph G is called 3*-regular* if each vertex is of degree 3. Note that for a 3-regular graph, the existence of two edge-disjoint perfect matchings is equivalent to the existence of three edge-disjoint perfect matchings which is equivalent to the 3-edge-colorability of the graph.

Directed graphs. Let $D = (V, A)$ be a directed graph. For a vertex set X, the set of arcs of D entering (resp. leaving) X is denoted by $\boldsymbol{\varrho_D(X)}$ (resp. $\boldsymbol{\delta_D(X)}$). The *in-degree* $\boldsymbol{d^-_D(X)}$ (resp. *out-degree* $\boldsymbol{d^+_D(X)}$) of X is the number of arcs of D entering (resp. leaving) X, that is $d^-_D(X) = |\varrho_D(X)|$ (resp. $d^+_D(X) = |\delta_D(X)|$). The set of arcs of G having both end-vertices in X is denoted by $\boldsymbol{A(X)}$. The following equality will be used frequently without reference.

$$d^-_D(X) - d^+_D(X) = \sum_{v \in X} (d^-_D(v) - d^+_D(v)). \tag{2}$$

We say that a vector m on V is the *in-degree vector* of D if $d_D^-(v) = m(v)$ for all $v \in V$. Let us recall that $d_D^-(X)$ is the *in-degree function* of D. Let f be a non-negative integer vector on V. An arc set F of D is called a *directed f-factor* of D if f is the in-degree vector of $D(F)$, that is $d_F^-(v) = f(v)$ for all $v \in V$.

We say that D is *Eulerian* if the in-degree of v is equal to the out-degree of v for all $v \in V$, that is $d_D^-(v) = d_D^+(v)$ for all $v \in V$. Note that we do not suppose the graph to be connected.

Let f and g be two vectors on the arcs of D such that $f(e) \leq g(e)$ for all $e \in A$. A vector x on the arcs of D is a *circulation* if (3) and (4) are satisfied.

$$x(\delta_D(v)) = x(\varrho_D(v)) \quad \forall v \in V, \tag{3}$$
$$f(e) \leq x(e) \leq g(e) \quad \forall e \in A. \tag{4}$$

Note that if $f(e) = g(e) = 1$ for all $e \in A$, then D is Eulerian if and only if f is a circulation. We will use the following characterization when a circulation exists.

Theorem 1 (Hoffmann [16]) *Let $D = (V, A)$ be a directed graph and f and g two vectors on A such that $f(e) \leq g(e)$ $\forall e \in A$. There exists a circulation in D if and only if*

$$f(\varrho_D(X)) \leq g(\delta_D(X)) \quad \forall X \subseteq V. \tag{5}$$

We say that $H = (V, E \cup A)$ is a *mixed graph* if E is an edge set and A is an arc set on V. For an undirected graph $G = (V, E)$, if we replace each edge uv by the arc uv or vu, then we get the directed graph $\vec{\boldsymbol{G}}= (V, \vec{E})$. We say that $\vec{G}$ is an *orientation* of G.

Mixed graphs having Eulerian orientations are characterized as follows:

Theorem 2 (Ford, Fulkerson [8]) *A mixed graph $H = (V, E \cup A)$ has an Eulerian orientation if and only if*

$$d_A^-(v) + d_A^+(v) + d_E(v) \text{ is even} \quad \forall v \in V, \tag{6}$$
$$d_A^-(X) - d_A^+(X) \leq d_E(X) \qquad \forall X \subseteq V. \tag{7}$$

The following theorem characterizes graphs having an orientation with a given in-degree vector.

Theorem 3 (Hakimi [14]) *Given an undirected graph $G = (V, E)$ and a non-negative integer vector m on V, there exists an orientation $\vec{G}$ of G whose in-degree vector is m if and only if*

$$m(X) \geq i(X) \ \forall X \subseteq V, \tag{8}$$
$$m(V) = |E|. \tag{9}$$

Functions. Let b be a set function on the subsets of V. We say that b is *submodular* if for all $X, Y \subseteq V$,

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y). \tag{10}$$

The function b is called *supermodular* if $-b$ is submodular. A function is *modular* if it is supermodular and submodular. We will use frequently in this paper the following facts.

Claim 1 *The degree function $d_G(Z)$ of an undirected graph G and the in-degree function $d_D^-(Z)$ of a directed graph D are submodular and the function $i(Z)$ is supermodular.*

Theorem 4 *[18, 22] The minimum value of a submodular function can be found in polynomial time.*

Theorem 5 (Frank [10]) *Let b and p be a submodular and a supermodular set function on V such that $p(X) \leq b(X)$ for all $X \subseteq V$. Then there exists a modular function m on V such that $p(X) \leq m(X) \leq b(X)$ for all $X \subseteq V$. If b and p are integer valued then m can also be chosen integer valued.*

A pair (p, b) of set-functions on 2^V is a *strong pair* if p (resp. b) is supermodular (submodular) and they are *compliant*, that is, for every pairwise disjoint triple X, Y, Z,

$$b(X \cup Z) - p(Y \cup Z) \geq b(X) - p(Y).$$

Note that a pair (α, β) of modular functions is a strong pair if and only if $\alpha \leq \beta$. If (p, b) is a strong pair then the polyhedron

$$Q(p, b) = \{x \in \mathbb{R}^V : p(X) \leq x(X) \leq b(X), \text{ for every } X \subseteq V\}$$

is called a *generalized polymatroid* (or a *g-polymatroid*). When $\alpha \leq \beta$ are modular, we also call the g-polymatroid $Q(\alpha, \beta)$ a *box*.

Theorem 6 (Frank, Tardos [12]) *The intersection of an integral g-polymatroid $Q(p, b)$ and an integral box $Q(\alpha, \beta)$ is an integral g-polymatroid. It is nonempty if and only if $\alpha \leq b$ and $p \leq \beta$.*

Matroids. A set system $M = (V, \mathcal{F})$ is called a *matroid* if $\mathcal{F}$ satisfies the following three conditions:

$(I1)$ $\emptyset \in \mathcal{F}$,
$(I2)$ if $F \in \mathcal{F}$ and $F' \subseteq F$, then $F' \in \mathcal{F}$,
$(I3)$ if $F, F' \in \mathcal{F}$ and $|F| > |F'|$, then there exists $f \in F \setminus F'$ such that $F' \cup f \in \mathcal{F}$.

A subset X of V is called *independent* in M if $X \in \mathcal{F}$, otherwise it is called *dependent.* The maximal independent sets of V are the *basis* of M. Let $\mathcal{B}$ be the set of basis of M. Then $\mathcal{B}$ satisfies the following two conditions:

$(B1)$ $\mathcal{B} \neq \emptyset$,
$(B2)$ if $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$, then there exists $b' \in B' \setminus B$ such that $(B - b) \cup b' \in \mathcal{B}$.

Conversely, if a set system $(V, \mathcal{B})$ satisfies $(B1)$ and $(B2)$, then $M = (V, \mathcal{F})$ is a matroid, where $\mathcal{F} = \{F \subseteq V : \exists B \in \mathcal{B}, F \subseteq B\}$.

For $S \subset V$, the matroid $M \setminus S$ obtained from M by *deleting* S is defined as $M \setminus S = (V \setminus S, \mathcal{F}|_{V \setminus S})$, where $X \subseteq V \setminus S$ belongs to $\mathcal{F}|_{V \setminus S}$ if and only if $X \in \mathcal{F}$. For $S \in \mathcal{F}$, the matroid M/S obtained from M by *contracting* S is defined as $M/S = (V \setminus S, \mathcal{F}_S)$, where $X \subseteq V \setminus S$ belongs to $\mathcal{F}_S$ if and only if $X \cup S \in \mathcal{F}$. Let $\{V_1, \dots, V_l\}$ be a partition of V and $a_1, \dots, a_l$ a set of non-negative integers. Then $M = (V, \mathcal{F})$ is a matroid, where $\mathcal{F} = \{F \subseteq V : |F \cap V_i| \leq a_i\}$, we call it *partition matroid.* The *dual* matroid M^* of M is defined as follows : the basis of M^* are the complements of the basis of M.

Let $M = (V, \mathcal{F})$ be a matroid and c a cost vector on $V = \{v_1, \dots, v_n\}$. We can find a minimum cost basis F_n of M in polynomial time by the greedy algorithm: take a non-decreasing order of the elements of V : $c(v_1) \leq \dots \leq c(v_n)$. Let F_0 be empty and for $i = 1, \dots, n$, let $F_i = F_{i-1} + v_i$ if $F_{i-1} + v_i \in \mathcal{F}$, otherwise let $F_i = F_{i-1}$.

If M_1 and M_2 are two matroids on the same ground set V, then we can find a common basis of M_1 and M_2 in polynomial time (if there exists one) by the matroid intersection algorithm of Edmonds [4].

Theorem 7 (Edmonds, Rota [19]) *For an integer-valued, non-decreasing, submodular function b defined on a ground set S, the set $\{F \subseteq S; |F'| \leq b(F')$ for all $\emptyset \neq F' \subseteq F\}$ forms the set of independent sets of a matroid M_b whose rank function r_b is given by*

$$r_b(Z) = \min\{b(X) + |Z - X|, X \subseteq Z\}.$$

Given an undirected graph $G = (V, E)$ and a non-negative integer vector m on V, let $\bar{m}^G = \bar{m}$ be the set-function defined on E by $\bar{m}(F) = m(V(F))$ where $V(F)$ is the set of vertices covered by F. One can easily check that $\bar{m}$ is integer-valued, non-decreasing and submodular. Thus, by Theorem 7, $\bar{m}$ defines a matroid $M_{\bar{m}}$. The following claim is straightforward.

Claim 2 *The set $\{F \subseteq E : m(X) \geq i_F(X), \forall X \subseteq V\}$ is the set of independent sets of the matroid $M_{\bar{m}}$.*

3 Degree Constrained Sandwich Problems

Before studying sandwich problems on orientations of given in-degrees, let us start as a warming up by considering sandwich problems for undirected and directed graphs of given degrees. These problems reduce to the undirected and directed f-factor problems. We mention that the directed case is much easier than the undirected case because the addition of an arc in a directed graph contributes only to the in-degree of the head and not of the tail, while the addition of an edge in an undirected graph contributes to the degree of both end-vertices. This section contains no new results, we added it for the sake of completeness.

3.1 Undirected graphs

Undirected Degree Constrained Sandwich Problem
Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector f on V.
Question: Does there exist a sandwich graph $G = (V, E)$ $(E_1 \subseteq E \subseteq E_2)$ such that $d_G(v) = f(v)$ for all $v \in V$?

Complexity : It is in P because the answer is Yes if and only if there exists an $(f(v) - d_{G_1}(v))$-factor in the optional graph $G_0 = (V, E_0)$.
Characterization : The general f-factor theorem due to Tutte [26] can be applied to get a characterization.
Optimization : The minimum cost f-factor problem in undirected graphs can be solved in polynomial time, see Schrijver [21].

Simultaneous Undirected Degree Constrained Sandwich Problem
Instance: Given two edge-disjoint graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ in $G_3 = (V, E_3)$ and two non-negative integer vectors f_1 and f_2 on V.
Question: Do there exist simultaneously sandwich graphs $\hat{G}_1 = (V, \hat{E}_1)$ $(E_1 \subseteq \hat{E}_1 \subseteq E_3)$ and $\hat{G}_2 = (V, \hat{E}_2)$ $(E_2 \subseteq \hat{E}_2 \subseteq E_3)$ such that $\hat{E}_1 \cap \hat{E}_2 = \emptyset$ and $d_{\hat{G}_1}(v) = f_1(v)$ and $d_{\hat{G}_2}(v) = f_2(v)$ for all $v \in V$?

Complexity : It is NP-complete because it contains as a special case whether there exist two edge-disjoint perfect matchings so 3-edge-colorability of 3-regular graphs. Indeed, let $G = (V, E)$ be an arbitrary 3-regular graph. Let G_1 and G_2 be the edgeless graph on V, $G_3 = G$ and $f_1(v) = f_2(v) = 1$ for all $v \in V$. Then the sandwich graphs $\hat{G}_1$ and $\hat{G}_2$ exists if and only if $\hat{E}_1$ and $\hat{E}_2$ are edge-disjoint perfect matchings of G or equivalently, if there exists a 3-edge-coloring of G. Since the problem of 3-edge-colorability of 3-regular graphs is NP-complete [17], so is our problem.

3.2 Directed graphs

Directed Degree Constrained Sandwich Problem
Instance: Given directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ with $A_1 \subseteq A_2$ and a non-negative integer vector f on V.
Question: Does there exist a sandwich graph $D = (V, A)$ $(A_1 \subseteq A \subseteq A_2)$ such that $d^-_D(v) = f(v)$ for all $v \in V$?

Complexity + Characterization : It is in P because the answer is Yes if and only if there exists a directed $(f(v) - d^-_{D_1}(v))$-factor in the optional directed graph $D_0 = (V, A_0)$, hence we have the following.

Theorem 8 *The* Directed Degree Constrained Sandwich Problem *has a* Yes *answer if and only if $d^-_{D_2}(v) \geq f(v) \geq d^-_{D_1}(v)$ for all $v \in V$.*

Optimization : The feasible arc sets form the basis of a partition matroid, so the greedy algorithm provides a minimum cost solution.

Simultaneous Directed Degree Constrained Sandwich Problem 1
Instance: Given two arc-disjoint directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ in $D_3 = (V, A_3)$ and two non-negative integer vectors f_1 and f_2 on V.
Question: Do there exist simultaneously sandwich graphs $\hat{D}_1 = (V, \hat{A}_1)$ $(A_1 \subseteq \hat{A}_1 \subseteq A_3)$ and $\hat{D}_2 = (V, \hat{A}_2)$ $(A_2 \subseteq \hat{A}_2 \subseteq A_3)$ such that $\hat{A}_1 \cap \hat{A}_2 = \emptyset$ and $d^-_{\hat{D}_1}(v) = f_1(v)$ and $d^-_{\hat{D}_2}(v) = f_2(v)$ for all $v \in V$?

Complexity : It is in P because the answer is Yes if and only if $d^-_{D_3}(v) \geq f_1(v) + f_2(v)$, $f_1(v) \geq d^-_{D_1}(v)$ and $f_2(v) \geq d^-_{D_2}(v)$ for all $v \in V$.

Simultaneous Directed Degree Constrained Sandwich Problem 2
Instance: Given directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ with $A_1 \subseteq A_2$ and two non-negative integer vectors f and g on V.
Question: Does there exist a sandwich graph $D = (V, A)$ $(A_1 \subseteq A \subseteq A_2)$ such that $d^-_D(v) = f(v)$ and $d^+_D(v) = g(v)$ for all $v \in V$.
Complexity : The feasible arc sets for the in-degree constraint form the basis of a partition matroid and the feasible arc sets for the out-degree constraint form the basis of a partition matroid. The answer is Yes if and only if there exists a common basis in these two matroids. Thus it is in P by the matroid intersection algorithm of Edmonds [4].

4 Eulerian Sandwich Problems

In this section we consider first two problems that were already solved in [13]: Eulerian sandwich problems for undirected and directed graphs. We point out that the undirected case reduces to T-joins, while the directed case to circulations. We show that in both cases the simultaneous versions are NP-complete.

Then we propose to study the problem in mixed graphs. We show two cases that can be solved. The first case will be solved by the Discrete Separation Theorem 5 of Frank [10], while the second case reduces to the Directed Eulerian Sandwich Problem. The general case however remains open.

4.1 Undirected graphs

Undirected Eulerian Sandwich Problem
Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$.
Question: Does there exists a sandwich graph $G = (V, E)$ $(E_1 \subseteq E \subseteq E_2)$ that is Eulerian?

Complexity : It is in P because the answer is YES if and only if there exists a T_{G_1}-join in the optional graph G_0.

Characterization : The answer is YES if and only if each connected component of G_0 contains an even number of vertices of T_{G_1}.

Optimization : The minimum cost T-join problem can be solved in polynomial time [5].

SIMULTANEOUS UNDIRECTED EULERIAN SANDWICH PROBLEM
Instance: Given two edge-disjoint graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ in $G_3 = (V, E_3)$.
Question: Do there exist simultaneously Eulerian sandwich graphs $\hat{G}_1 = (V, \hat{E}_1)$ ($E_1 \subseteq \hat{E}_1 \subseteq E_3$) and $\hat{G}_2 = (V, \hat{E}_2)$ ($E_2 \subseteq \hat{E}_2 \subseteq E_3$) such that $\hat{E}_1 \cap \hat{E}_2 = \emptyset$?

Complexity : It is NP-complete because it contains as a special case whether there exist two edge-disjoint perfect matchings so 3-colorability of 3-regular graphs. Indeed, let $G = (V, E)$ be an arbitrary 3-regular graph. Let G_3 be obtained from G by adding 2 edge-disjoint perfect matchings M_1 and M_2 to G, let $G_1 = (V, M_1)$ and $G_2 = (V, M_2)$. Then the Eulerian sandwich graphs $\hat{G}_1$ and $\hat{G}_2$ exist if and only if $\hat{E}_1 \setminus M_1$ and $\hat{E}_2 \setminus M_2$ are edge-disjoint perfect matchings of G or equivalently, if there exists a 3-edge-coloring of G. Since the problem of 3-edge-colorability of 3-regular graphs is NP-complete [17], so is our problem.

4.2 Directed graphs

DIRECTED EULERIAN SANDWICH PROBLEM
Instance: Given directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ with $A_1 \subseteq A_2$.
Question: Does there exist a sandwich graph $D = (V, A)$ ($A_1 \subseteq A \subseteq A_2$) that is Eulerian?

Complexity : It is in P because it can be reformulated as a circulation problem: let $f(e) = 1, g(e) = 1$ if $e \in A_1$ and $f(e) = 0, g(e) = 1$ if $e \in A_0$. This way the arcs of A_1 are forced and the arcs of A_0 can be chosen if necessary.

Characterization : The answer is YES if and only if $d^-_{D_1}(X) \leq d^+_{D_2}(X)$ for all $X \subseteq V$ by Theorem 1.

Optimization : The minimum cost circulation problem can be solved in polynomial time, see Tardos [24].

SIMULTANEOUS DIRECTED EULERIAN SANDWICH PROBLEM
Instance: Given two arc-disjoint directed graphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ in $D_3 = (V, A_3)$.
Question: Do there exist simultaneously Eulerian sandwich graphs $\hat{D}_1 = (V, \hat{A}_1)$ ($A_1 \subseteq \hat{A}_1 \subseteq A_3$) and $\hat{D}_2 = (V, \hat{A}_2)$ ($A_2 \subseteq \hat{A}_2 \subseteq A_3$) such that $\hat{A}_1 \cap \hat{A}_2 = \emptyset$?

Complexity : It is NP-complete, it contains as a special case ($D_1 = (V, t_1s_1)$, $D_2 = (V, t_2s_2)$ and $D_3 = D$) the following directed 2-commodity integral flow problem that is NP-complete [6]: Given a directed graph D and two pairs of vertices, s_1, t_1 and s_2, t_2, decide whether there exist a path from s_1 to t_1 and a path from s_2 to t_2 that are arc-disjoint.

4.3 Mixed graphs

MIXED EULERIAN SANDWICH PROBLEM
Instance: Given mixed graphs $H_1 = (V, E_1 \cup A_1)$ and $H_2 = (V, E_2 \cup A_2)$ with $E_1 \subseteq E_2, A_1 \subseteq A_2$.
Question: Does there exist a sandwich mixed graph $H = (V, E \cup A)$ ($E_1 \subseteq E \subseteq E_2, A_1 \subseteq A \subseteq A_2$) that has an Eulerian orientation?

Complexity : We provide two special cases that can be treated, while the general problem remains open.

SPECIAL CASE 1: $E_1 = E_2 = E$ and $d^+_{A_2}(X) - d^-_{A_1}(X) + \hat{d}_E(X)$ is even for all $X \subseteq V$.
Characterization + Complexity : We show that the problem is in P and we provide a characterization.

Theorem 9 *The* MIXED EULERIAN SANDWICH PROBLEM *with $E_1 = E_2 = E$ and $d^+_{A_2}(X) - d^-_{A_1}(X) + \hat{d}_E(X)$ is even for all $X \subseteq V$ has a* YES *answer if and only if*

$$d^-_{A_1}(X) - d^+_{A_2}(X) \leq d_E(X) \;\forall X \subseteq V. \tag{11}$$

In particular, this problem is in P.

Proof. By the result of Section 4.2, the answer is YES if and only if there exists an orientation $\vec{E}$ of E such that $d^-_{A_1 \cup \vec{E}}(X) \leq d^+_{A_2 \cup \vec{E}}(X) \;\forall X \subseteq V$, or equivalently

$$d^-_{\vec{E}}(X) - d^+_{\vec{E}}(X) \leq d^+_{A_2}(X) - d^-_{A_1}(X) \;\forall X \subseteq V. \tag{12}$$

Let m be the in-degree vector of $\vec{E}$. Then $d^-_{\vec{E}}(X) - d^+_{\vec{E}}(X) = \sum_{v \in X}(d^-_{\vec{E}}(v) - d^+_{\vec{E}}(v)) = \sum_{v \in X}(2d^-_{\vec{E}}(v) - d_E(v)) = 2m(X) - \hat{d}_E(X)$, and (12) becomes

$$2m(X) \leq d^+_{A_2}(X) - d^-_{A_1}(X) + \hat{d}_E(X). \tag{13}$$

Let $b(X) = \frac{1}{2}(d^+_{A_2}(X) - d^-_{A_1}(X) + \hat{d}_E(X))$. Then b, being the sum of a modular function and a submodular function ($b(X) = \frac{1}{2}\sum_{v \in X}(d^+_{A_1}(v) - d^-_{A_1}(v) + d_E(v)) + d^+_{A_0}(X)$), is a submodular function and, by the assumption, it is integer valued. By Theorem 3, an orientation $\vec{E}$ satisfying (12) exists if and only if there exists a vector m such that $i_E(X) \leq m(X) \leq b(X)$, that is by Claim 1 and Theorem 5, if and only if $i_E(X) \leq b(X)$. This is equivalent to (11) and can be decided in polynomial time by Theorem 4, namely the submodular function $b'(X) = b(X) - i_E(X)$ must have minimum value 0. □

SPECIAL CASE 2: $E_1 = \emptyset$.
Characterization + Complexity : It is in P because it can be reformulated as the following problem : We create two copies of each edge in E_2 and orient them in opposite directions. Denote this arc set by $\overrightarrow{E_2^2}$. It is not difficult to see that the graph $(V, E_2 \cup A_2)$ has a subgraph containing (V, A_1) with an Eulerian orientation if and only if the graph $(V, \overrightarrow{E_2^2} \cup A_2)$ has a directed Eulerian subgraph containing (V, A_1). Indeed, in such a graph, if every edge of E_2 is used at most once, we are done. If some edge of E_2 is used twice, as two arcs in opposite directions, we can just remove these two arcs, the obtained graph remaining Eulerian and containing (V, A_1). Now applying the result for DIRECTED EULERIAN SANDWICH PROBLEM we have :

Theorem 10 *The* MIXED EULERIAN SANDWICH PROBLEM *with $E_1 = \emptyset$ has a* YES *answer if and only if*

$$d^-_{A_1}(X) - d^+_{A_2}(X) \leq d_{E_2}(X) \;\forall X \subseteq V. \tag{14}$$

In particular, this problem is in P.

Proof. Let $D_1 = (V, A_1)$ and $D_2 = (V, A_2 \cup \overrightarrow{E_2^2})$. By the arguments above, the MIXED EULERIAN SANDWICH PROBLEM with $E_1 = \emptyset$ has a solution if and only if there is an Eulerian sandwich graph for D_1 and D_2 or equivalently, $d^-_{D_1}(X) \leq d^+_{D_2}(X)$ for all $X \subseteq V$. By $d^+_{D_2}(X) = d^+_{A_2}(X) + d_{E_2}(X)$, we have $d^-_{A_1}(X) - d^+_{A_2}(X) \leq d_{E_2}(X)$ for all $X \subseteq V$. Note that $d_{E_2}(X) + d^+_{A_2}(X) - d^-_{A_1}(X)$ is a submodular function, and hence by Theorem 4, (14) can be verified in polynomial time. □

5 m-orientation Sandwich Problems

In this section we consider the sandwich problem where the property Π is to have an orientation of given in-degrees.

5.1 m-Orientation

m-orientation Sandwich Problem
Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V.
Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has an orientation $\vec{G}$ whose in-degree vector is m that is $d^-_{\vec{G}}(v) = m(v)$ for all $v \in V$?

Characterization : We prove the following theorem.

Theorem 11 *The following assertions are equivalent.*

(a) The m-orientation Sandwich Problem has a Yes answer.

(b) E_1 is independent in $M_{\bar{m}}$ and $M_{\bar{m}}$ has an independent set of size $m(V)$.

(c) $r_{\bar{m}}(E_1) = |E_1|$ and $r_{\bar{m}}(E_2) \geq m(V)$.

(d) $i_{E_1}(X) \leq m(X) \leq e_{E_2}(X)$ for all $X \subseteq V$.

Proof. (a) implies (d) Let $X \subseteq V$. Since each edge of G_1 in X contributes 1 to $m(X)$, we have $i_{E_1}(X) \leq m(X)$. On the other hand, the edges of G_2 that have no end-vertex in X can not contribute 1 to $m(X)$, so we have $m(X) \leq e_{E_2}(X)$.

(d) implies (c). Let F be a subset of E_1 and $X = V(F)$. The condition $i_{E_1}(X) \leq m(X)$ implies $|F| \leq m(V(F)) = \bar{m}(F)$, that is $|E_1| \leq \bar{m}(F) + |E_1 \setminus F|$. By Theorem 7, $r_{\bar{m}}(E_1) \geq |E_1|$, or equivalently $r_{\bar{m}}(E_1) = |E_1|$. Let now F be a subset of E_2 and $X = V \setminus V(F)$. The condition $m(X) \leq e_{E_2}(X)$ implies that $m(V) \leq m(V(F)) + e_{E_2}(V - V(F)) \leq \bar{m}(F) + |E_2 \setminus F|$. By Theorem 7, $r_{\bar{m}}(E_2) \geq m(V)$.

(c) implies (b). By definition.

(b) implies (a). By (b), E_1 is independent in $M_{\bar{m}}$ and there exists an independent in $M_{\bar{m}}$ of size $m(V)$. Therefore, by $(I3)$, there exists an independent set E of size $m(V)$ that contains E_1. By Theorem 3 and Claim 2, E is a solution of the m-orientation Sandwich Problem. □

We say that a subset F of E_0 is *feasible* if $(V, F \cup E_1)$ has an m-orientation. The next corollary of Theorem 11 characterizes the feasible sets.

Corollary 1 *If the m-orientation Sandwich Problem has a Yes answer then a subset F of E_0 is feasible if and only if F is a base of the matroid $M_{\bar{m}}/E_1$.*

Complexity : The condition (d) of Theorem 11 can be verified in polynomial time by Theorem 4, so the m-orientation Sandwich Problem is in P.

Optimization : The minimum cost version of the problem can be solved in polynomial time. First, we find an optimal feasible subset F by greedy algorithm. Then we can orient the edges of $F \cup E_1$ using a known algorithm. (See [11] for example).

Corollary 1 and the matroid intersection algorithm of Edmonds [4] imply that the two following simultaneous versions of the m-orientation Sandwich Problem are also in P.

Simultaneous m-orientation Sandwich Problem 1
Instance: Given two edge-disjoint undirected subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of an undirected graph $G_3 = (V, E_3)$ and two non-negative integer vectors m_1 and m_2 on V.
Question: Do there exist simultaneously edge-disjoint sandwich graphs $\hat{G}_1 = (V, \hat{E}_1)$ ($E_1 \subseteq \hat{E}_1 \subseteq E_3$) and $\hat{G}_2 = (V, \hat{E}_2)$ ($E_2 \subseteq \hat{E}_2 \subseteq E_3$) such that $\hat{G}_i$ has an orientation whose in-degree vector is m_i for $i \in \{1, 2\}$?

Note that, the two input matroids for the matroid intersection algorithm must be taken as $(M^{G_1}_{\bar{m}_1}/E_1) \setminus E_2$ and the dual matroid of $(M^{G_2}_{\bar{m}_2}/E_2) \setminus E_1$.

Simultaneous m-orientation Sandwich Problem 2
Instance: Given two undirected subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ of an undirected graph $G_3 = (V, E_3)$ and two non-negative integer vectors m_1 and m_2 on V.
Question: Does there exist an edge set F in $E_3 \setminus (E_1 \cup E_2)$ such that the graph $G_i = (V, E_i \cup F)$ admits an orientation whose in-degree vector is m_i for $i \in \{1, 2\}$?

5.2 Strongly Connected m-Orientation

Strongly Connected m-Orientation Sandwich Problem
Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and a non-negative integer vector m on V.
Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has a strongly connected orientation $\vec{G}$ whose in-degree function is m?

Complexity : It is NP-complete because the special case $E_1 = \emptyset, m(v) = 1\ \forall v \in V$ is equivalent to decide if G_2 has a Hamiltonian cycle.

5.3 (m_1, m_2)-Orientation

(m_1, m_2)-orientation Sandwich Problem
Instance: Given undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subseteq E_2$ and non-negative integer vectors m_1 and m_2 on V.
Question: Does there exist a sandwich graph $G = (V, E)$ ($E_1 \subseteq E \subseteq E_2$) that has an orientation $\vec{G}$ whose in-degree vector is m_1 and whose out-degree vector is m_2 ?

Complexity : The problem is NP-complete since it contains as a special case ($E_1 = \emptyset$) the NP-complete problem of [20].

5.4 Mixed m-Orientation

Mixed m-orientation Sandwich Problem
Instance: Given mixed graphs $G_1 = (V, E_1 \cup A_1)$ and $G_2 = (V, E_2 \cup A_2)$ with $E_1 \subseteq E_2$, $A_1 \subseteq A_2$ and an non-negative integer vector m on V.
Question: Does there exist a sandwich mixed graph $G = (V, E \cup A)$ with $E_1 \subseteq E \subseteq E_2$ and $A_1 \subseteq A \subseteq A_2$, that has an orientation $\overrightarrow{G} = (V, \overrightarrow{E} \cup A)$ whose in-degree vector is m?

Characterization : Suppose that $E_1 \subseteq E \subseteq E_2$ has been chosen and oriented, then the problem is reduced to the Directed Degree Constrained Sandwich Problem with $m_1(v) = m(v) - d^-_{\overrightarrow{E}}(v)$ which, by Theorem 8, has a solution if and only if $d^-_{A_2}(v) \geq m(v) - d^-_{\overrightarrow{E}}(v) \geq d^-_{A_1}(v)$ for all $v \in V$. Hence the Mixed m-orientation Sandwich Problem has a solution if and only if there exists $E_1 \subseteq E \subseteq E_2$ which admits an orientation $\overrightarrow{E}$ with $m(v) - d^-_{A_1}(v) \geq d^-_{\overrightarrow{E}}(v) \geq m(v) - d^-_{A_2}(v)$ for all $v \in V$. Let $m_2 : V \to \mathbb{Z}$ satisfy

$m(v) - d^-_{A_2}(v) \le m_2(v) \le m(v) - d^-_{A_1}(v)$. By Theorem 11, there exists $E_1 \subseteq E \subseteq E_2$ which admits an orientation $\overrightarrow{E}$ with $d^-_{\overrightarrow{E}}(v) = m_2(v)$ if and only if $i_{E_1}(X) \le m_2(X) \le e_{E_2}(X)$ for all $X \subseteq V$. Therefore we have

Claim 3 *The* Mixed m-orientation Sandwich Problem *has a* Yes *answer if and only if there exists an integer valued function $m_2 : V \to \mathbb{Z}$ such that*

$$\begin{array}{rcll} m(v) - d^-_{A_2}(v) \le & m_2(v) & \le m(v) - d^-_{A_1}(v) & \forall v \in V, \\ i_{E_1}(X) \le & m_2(X) & \le e_{E_2}(X) & \forall X \subseteq V. \end{array}$$

Claim 4 *The pair (i_{E_1}, e_{E_2}) is a strong pair.*

Proof. Let X, Y, Z be three pairwise disjoint subset of V. We show that $e_{E_2}(X \cup Z) - i_{E_1}(Y \cup Z) \ge e_{E_2}(X) - i_{E_1}(Y)$. In fact, we have $i_{E_1}(Y \cup Z) - i_{E_1}(Y) = i_{E_1}(Z) + d_{E_1}(Y, Z) \le i_{E_2}(Z) + d_{E_2}(Y, Z)$, and $e_{E_2}(X \cup Z) - e_{E_2}(X) = i_{E_2}(Z) + d_{E_2}(Z) - d_{E_2}(X, Z)$. As X, Y, Z are pairwise disjoint, $d_{E_2}(Y, Z) + d_{E_2}(X, Z) \le d_{E_2}(Z)$. The claim follows by Claim 1. □

By Claim 3, 4 and Theorem 6 applied for $\alpha(v) = m(v) - d^-_{A_2}(v), \beta(v) = m(v) - d^-_{A_1}(v), p = i_{E_1}, b = e_{E_2}$, we have

Theorem 12 *The* Mixed m-orientation Sandwich Problem *has a* Yes *answer if and only if*

$$i_{E_1}(X) + \hat{d}^-_{A_1}(X) \le m(X) \le e_{E_2}(X) + \hat{d}^-_{A_2}(X) \tag{15}$$

for every subset X of V.

Note that Theorem 12 implies Theorems 8 and 11.

Complexity : The condition (15) can be verified in polynomial time by Theorem 4. If it is satisfied, then a vector m_2 satisfying the conditions in Claim 3 can be found using greedy algorithm for g-polymatroids. Then we find and orient an edge set E ($E_1 \subseteq E \subseteq E_2$) with in-degree m_2 (m-orientation Sandwich Problem). Last, we choose an arc set A ($A_1 \subseteq A \subseteq A_2$) such that $d^-_A(v) = m_1(v) = m(v) - m_2(v)$, for all $v \in V$ (Directed Degree Constrained Sandwich Problem).

6 Contracting Sandwich Problems

In this section, we propose to consider a new type of sandwich problems. Instead of deleting edges from the optional graph, we are interested in contracting edges. We solve the problem for the property Π being a bipartite graph.

Contracting Sandwich Problem
Instance: Given an undirected graph $G = (V, E)$ and $E_0 \subseteq E$.
Question: Does there exist $F \subseteq E_0$ such that contracting F results in a bipartite graph?

Complexity: Since a graph is bipartite if and only if all its cycles have an even length, the problem is equivalent to finding $F \subseteq E_0$ such that, for all cycles C, $|C \cap F| \equiv |C| \bmod 2$.

Fixe a spanning forest T of G. For $e \in E \setminus T$, denote $C(T, e)$ the unique cycle contained in $T \cup e$. By [19, Theorem 9.1.2], if C is a cycle of G then $C = \Delta_{e \in C} C(T, e)$, where Δ denotes the symmetric difference of sets. Therefore, $|C \cap F| \equiv \sum_{e \in C} |C(T, e) \cap F| \bmod 2$. Let $\mathcal{C}_T$ denote the collection of cycles $C(T, e)$ of G. The problem is reduced to finding $F \subseteq E_0$ such that, for all $C \in \mathcal{C}_T$, $|C \cap F| \equiv |C| \bmod 2$, or equivalently, finding an $F' (= E \setminus F) \supseteq E_1 = E \setminus E_0$ such that $|F' \cap C| \equiv 0 \bmod 2$, for all $C \in \mathcal{C}_T$.

Consider now the matrix M defined as the following. The rows of M correspond to $C \in \mathcal{C}_T$ and the columns correspond to the edges of G; the entry M_{Ce} is 1 if $e \in C$ and is 0 otherwise. For $X \subseteq E$, let χ_X denote the characteristic vector of X. For a vector $x \in \{0,1\}^E$, let $x_{|X}$ denote the projection of x on X. Let $\mathbf{1}$ be the all-one vector in $\{0,1\}^E$. A subset $F' \subseteq E$ satisfies $|F' \cap C| \equiv 0 \bmod 2$, for all $C \in \mathcal{C}_T$, if and only if $\chi_{F'} \in \operatorname{Ker} M$ in $\mathbb{F}_2$. Such an F' is the solution of the Contracting Sandwich Problem if and only if $\chi_{F'|E_1} = \mathbf{1}_{|E_1}$.

Let B be a basis of the kernel of M in $\mathbb{F}_2$. (This can be computed in polynomial time using the Gauss elimination.) Consider the projections B' of B on E_1. Then the Contracting Sandwich Problem has a solution if and only if $\mathbf{1}_{|E_1}$ is in the subspace of $\{0,1\}^{E_1}$ spanned by B', that is $\operatorname{rank} B' = \operatorname{rank} B' \cup \mathbf{1}_{|E_1}$. This can be decided in polynomial time using the Gauss elimination. We conclude that the Contracting Sandwich Problem is in P.

We finish with a related problem. For a fixed integer k, solving the Contracting Sandwich Problem when $E_0 = E$ with extra requirement $|F| \leq k$ is known to be tractable in polynomial time [15]. However the authors mention that finding a solution of minimum cardinality is NP-complete.

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