On packing time-respecting arborescences

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Abstract

We present a slight generalization of the result of Kamiyama and Kawase [7] on packing time-respecting arborescences in acyclic pre-flow temporal networks. Our main contribution is to provide the first results on packing time-respecting arborescences in non-acyclic temporal networks. As negative results, we prove the NP-completeness of the decision problem of the existence of 2 arc-disjoint spanning time-respecting arborescences and of a related problem proposed in this paper.

1 Introduction

Temporal networks were introduced to model the exchange of information in a network or the spread of a disease in a population. We are given a directed graph D and a time label function τ on the arcs of D, the pair (D, τ) is called a temporal network. Intuitively, for an arc a of D, $\tau(a)$ is the time when the end-vertices of a communicate, that is when the tail of a can transmit a piece of information to the head of a. Then the information can propagate through a path P if it is time-respecting, meaning that the time labels of the arcs of P in the order they are passed are non-decreasing. For a nice introduction to temporal networks, see [8].

Problems about packing arborescences in temporal networks were investigated in [7]. An arborescence is called time-respecting if all the directed paths it contains are time-respecting. The main result of [7] provides a packing of time-respecting arborescences, each vertex belonging to many of them, if the network is pre-flow and acyclic. Here pre-flow means intuitively that each vertex different from the root has at least as many arcs entering as leaving, while acyclic means that no directed cycle exists. Kamiyama and Kawase [7] presented examples to show that these conditions can not be dropped.

Two questions naturally arise from these results: Must all kinds of directed cycles be forbidden? Does high time-respecting root-connectivity imply the existence of 2 arc-disjoint spanning time-respecting arborescences in a non-pre-flow temporal network?

Let us now present our contributions that give an answer to those questions.

We first propose a generalized version of the result of [7] with a simplified proof in Theorem 2.

Our main result, Theorem 4, is about packing time-respecting arborescences in pre-flow temporal networks that may contain directed cycles. The condition in Theorem 4 is that the arcs in the same strongly connected component must have the same τ-value. If this condition holds then our intuition would be to use regular arborescences in the strongly connected components and then to try to extend them to obtain a packing of time-respecting arborescences in the temporal network. This idea is a step in the right direction, however the exact process used in the proof is a bit more complex, see Section 4.

By the famous result of Edmonds [3], we know that k-root-connectivity implies the existence of a packing of k spanning s-arborescences. The authors of [8] show that for any positive integer k, time-respecting k-root-connectivity does not imply the existence of 2 arc-disjoint spanning time-respecting arborescences in a temporal network. To explain this construction (or more precisely, a slightly modified version of it), we point out and recall in Section 5 the close relation between packings of spanning time-respecting arborescences, packings of Steiner arborescences and proper

2-colorings of hypergraphs. We remark in Theorem 12 that the decision problem, whether there exist 2 arc-disjoint spanning time-respecting arborescences, is NP-complete.

We show in Theorem 11 that time-respecting $(n-1)$-root-connectivity implies the existence of a packing of 2 spanning time-respecting s-arborescences in an arbitrary temporal network on n vertices. This result becomes more interesting if we note that the examples of Figure 1 show that time-respecting $(n-3)$-root-connectivity is not enough.

Finally, in Theorem 13, we show that in an acyclic temporal network (D, τ), it is NP-complete to decide whether there exists a spanning arborescence whose directed paths consist of arcs of the same τ-value.

2 Definitions

Let $\boldsymbol{D} = (V \cup s, A)$ be a directed graph with a special vertex $\boldsymbol{s}$, called *root*, such that no arc enters s. The set of arcs entering, leaving a vertex set X of D is denoted by $\boldsymbol{\rho_D(X)}$, $\boldsymbol{\delta_D(X)}$, respectively. Sometimes we use $\rho_A(X)$ for $\rho_D(X)$ and similarly $\delta_A(X)$ for $\delta_D(X)$. We denote $|\rho_D(X)|$ and $|\delta_D(X)|$ by $\boldsymbol{d^-_D(X)}$ and $\boldsymbol{d^+_D(X)}$, respectively. We call the directed graph D *acyclic* if D contains no directed cycle. If $d^-_D(v) = d^+_D(v)$ for all $v \in V$, then D is called *Eulerian*. We say that D is *pre-flow* if $d^-_D(v) \geq d^+_D(v)$ for all $v \in V$. A subgraph $F = (V' \cup s, A')$ of D is called an *s-arborescence* if F is acyclic and $d^-_F(v) = 1$ for all $v \in V'$. We say that F is *spanning* if $V' = V$. For $U \subseteq V$, F is called a *Steiner s-arborescence* or an *(s, U)-arborescence* if F is an s-arborescence and it contains all the vertices in U. A *packing* of arborescences means a set of arc-disjoint arborescences. For $v \in V$, a path from s to v is called an *(s, v)-path* and $\boldsymbol{\lambda_D(s, v)}$ denotes the maximum number of arc-disjoint (s, v)-paths in D. For some $k \in \mathbb{N}$, we say that D is *k-root-connected* if $\lambda_D(s, v) \geq k$ for all $v \in V$. For some $U \subseteq V$ and $k \in \mathbb{N}$, we say that D is *Steiner k-root-connected* if $\lambda_D(s, v) \geq k$ for all $v \in U$. We call a directed graph $D' = (V \cup \{s, t\}, A')$ *almost Eulerian* if $d^-_{D'}(v) = d^+_{D'}(v)$ for all $v \in V$ and $d^-_{D'}(s) = 0 = d^+_{D'}(t)$.

For a function $\boldsymbol{\tau} : A \to \mathbb{N}$, $\boldsymbol{N} = (D, \tau)$ is called a *temporal network*. For $i \in \mathbb{N}$, let $\boldsymbol{\rho^i_N(v)}:= \{a \in \rho_D(v) : \tau(a) \leq i\}$ and $\boldsymbol{\delta^i_N(v)}:= \{a \in \delta_D(v) : \tau(a) \leq i\}$. We call the temporal network N *acyclic* if D is acyclic. We say that N is *pre-flow* if $|\rho^i_N(v)| \geq |\delta^i_N(v)|$ for all $i \in \mathbb{N}$ and for all $v \in V$. Note that if a temporal network (D, τ) is pre-flow, then the directed graph D is pre-flow. We say that (D, τ) is *consistent* if arcs of different τ-values cannot belong to the same strongly connected component of D. In this case in each strongly connected component Q of D that contains at least one arc, each arc has the same τ-value, that we denote by $\boldsymbol{\tau(Q)}$. A directed path P of D, consisting of the arcs $a_1, \ldots, a_\ell$ in this order, is called *time-respecting* or *τ-respecting* if $\tau(a_i) \leq \tau(a_{i+1})$ for $1 \leq i \leq \ell - 1$. An s-arborescence F of D is called *time-respecting* or *τ-respecting* if for every vertex v of F, the unique (s, v)-path in F is τ-respecting. For $v \in V$, $\boldsymbol{\lambda_N(s, v)}$ denotes the maximum number of arc-disjoint τ-respecting (s, v)-paths in D. We say that N is *time-respecting k-root-connected* if $\lambda_N(s, v) \geq k$ for all $v \in V$. If $N' = (D', \tau')$ is a temporal network where $D' = (V \cup \{s, t\}, A')$ is almost Eulerian, then for a vertex $v \in V$, we call a bijection μ'_v from $\delta_{D'}(v)$ to $\rho_{D'}(v)$ *τ'-respecting* if $\tau'(\mu'_v(f)) \leq \tau'(f)$ for all $f \in \delta_{D'}(v)$.

A hypergraph $\boldsymbol{\mathcal{H}} = (V, \mathcal{E})$ is defined by its vertex set V and its hyperedge set $\mathcal{E}$ where a hyperedge is a subset of V. For some $r \in \mathbb{N}$, the hypergraph $\mathcal{H}$ is called *r-uniform* if each hyperedge in $\mathcal{E}$ is of size r and *r-regular* if each vertex in V belongs to exactly r hyperedges. A 2-coloring of the vertex set V is called *proper* if each hyperedge in $\mathcal{E}$ contains vertices of both colors, in other words no monochromatic hyperedge exists in $\mathcal{E}$. We call $\mathcal{E}' \subseteq \mathcal{E}$ an *exact cover* of $\mathcal{H}$ if each vertex in V belongs to exactly one hyperedge in $\mathcal{E}'$.

3 Packing time-respecting arborescences in acyclic pre-flow temporal networks

The aim of this section is to generalize the following result of Kamiyama and Kawase [7] on packing time-respecting arborescences in acyclic pre-flow temporal networks.

Theorem 1 ([7]) *Let $N = ((V \cup s, A), \tau)$ be an acyclic pre-flow temporal network and $k \in \mathbb{N}$. There exists a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to* $\min\{k, \lambda_N(s, v)\}$ *of them.*

Note that Theorem 1 implies that in a time-respecting k-root-connected acyclic pre-flow temporal network there exists a packing of k spanning time-respecting s-arborescences.

We now present our first result, a slight extension of Theorem 1.

Theorem 2 *Let $N = ((V \cup s, A), \tau)$ be an acyclic temporal network and $k \in \mathbb{N}$ such that*

$$\min\{k, |\rho^i_N(v)|\} \geq \min\{k, |\delta^i_N(v)|\} \quad \textit{for all } i \in \mathbb{N}, \textit{ for all } v \in V. \tag{1}$$

There exists a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to $\min\{k, d^-_A(v)\}$ *of them.*

We will partially follow the proof of [7] but we will point out that Lemmas 3 and 4 in [7] are not needed to prove Theorem 2. Hence the proof of Theorem 2 is simpler than that of Theorem 1. The following algorithm is a slightly modified version of the algorithm of Kamiyama and Kawase [7]. Its input is an acyclic temporal network $N = ((V \cup s, A), \tau)$ and $k \in \mathbb{N}$ such that (1) is satisfied. Its output is a packing of τ-respecting s-arborescences $T_1, \dots, T_k$ such that each vertex v in V belongs to $\min\{k, d^-_A(v)\}$ of them. For every $v \in V$, let $\boldsymbol{I(v)}$ be a set of arcs of smallest τ-values entering v of size $\min\{k, d^-_A(v)\}$. The algorithm will use arcs only in $\bigcup_{v \in V} I(v)$. The algorithm heavily relies on the fact that the network is acyclic. It is well-known that a directed graph D is acyclic if and only if a *topological ordering* $v_1, \dots, v_n$ of its vertex set exists, that is if $v_i v_j$ is an arc of D then $i > j$. Since no arc enters s, we may suppose that in a topological ordering $v_n = s$.

Algorithm "PACKING TIME-RESPECTING ARBORESCENCES"

Let $\boldsymbol{v_n} = s, \dots, \boldsymbol{v_1}$ be a topological ordering of $V \cup s$.
Let $\boldsymbol{A_i} = \emptyset$ for all $1 \leq i \leq k$.
For $j = 1$ to $n - 1$, let
  $\boldsymbol{I} = \{1 \leq i \leq k : \delta_{A_i}(v_j) \neq \emptyset\}$,
  $\boldsymbol{a_i}$ be an arc in $\delta_{A_i}(v_j)$ of minimum τ-value for all $i \in I$,
  $\{\boldsymbol{\bar{a}_1}, \dots, \boldsymbol{\bar{a}_{|I|}}\}$ be an ordering of $\{a_i : i \in I\}$ such that $\tau(\bar{a}_1) \leq \dots \leq \tau(\bar{a}_{|I|})$,
  $\boldsymbol{\pi} : I \to \{1, \dots, |I|\}$ be the bijection such that $a_i = \bar{a}_{\pi(i)}$ for all $i \in I$,
  $\boldsymbol{J}$ be a subset of $\{1, \dots, k\} \setminus I$ of size $|I(v_j)| - |I|$,
  $\boldsymbol{\sigma} : J \to \{1, \dots, |J|\}$ be a bijection,
  $\{\boldsymbol{e_1, \dots, e_{|I|}, f_1, \dots, f_{|J|}}\}$ be an ordering of $I(v_j)$ such that
    $\tau(e_1) \leq \dots \leq \tau(e_{|I|}) \leq \tau(f_1) \leq \dots \leq \tau(f_{|J|})$,
  $\boldsymbol{A_i} = A_i \cup e_{\pi(i)}$ for all $i \in I$,
  $\boldsymbol{A_i} = A_i \cup f_{\sigma(i)}$ for all $i \in J$.
Let $\boldsymbol{T_i} = (V_i, A_i)$ where $\boldsymbol{V_i}$ is the vertex set of the arc set A_i for all $1 \leq i \leq k$.
Stop.

Theorem 3 *Given an acyclic temporal network $N = ((V \cup s, A), \tau)$ and $k \in \mathbb{N}$ such that* (1) *is satisfied, Algorithm* PACKING TIME-RESPECTING ARBORESCENCES *outputs a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to* $\min\{k, d^-_A(v)\}$ *of them.*

Proof For all $1 \leq j \leq n - 1$, in the j^{th} iteration of the algorithm, by the definition of I, (1) and the definition of $I(v_j)$, we have $|I| \leq \min\{k, d^+_A(v_j)\} \leq \min\{k, d^-_A(v_j)\} = |I(v_j)|$. This implies that J exists. By construction, the digraphs $T_1, \dots, T_k$ are pairwise arc-disjoint and the in-degree of each vertex $v_j \in V_i - s$ is 1 in T_i. Then, since N is acyclic, T_i is an s-arborescence for all $1 \leq i \leq k$. Moreover, $|\{1 \leq i \leq k : v_j \in V_i\}| = |I| + |J| = |I(v_j)| = \min\{k, d^-_A(v_j)\}$ for all $1 \leq j \leq n - 1$. To see that T_i is time-respecting for all $1 \leq i \leq k$, let v_j be a vertex in $V_i - s$ and $a \in \delta_{A_i}(v_j)$. Then $e_{\pi(i)} \in \rho_{A_i}(v_j)$. Suppose on the contrary that $\tau(e_{\pi(i)}) > \tau(a)$. Since $\tau(g) \geq \tau(e_{\pi(i)}) > \tau(a)$ for all $g \in \rho_A(v_j) \setminus \{e_1, \dots, e_{\pi(i)-1}\}$, we have $|\rho^{\tau(a)}_N(v_j)| \leq |\{e_1, \dots, e_{\pi(i)-1}\}| = \pi(i) - 1$. Since $\tau(a) \geq \tau(a_i) = \tau(\bar{a}_{\pi(i)}) \geq \tau(\bar{a}_\ell)$ for all $1 \leq \ell \leq \pi(i)$ and $\pi(i) \leq |I| \leq k$, we have $\pi(i) =$

$|\{\bar{a}_1,\ldots,\bar{a}_{\pi(i)}\}| \leq \min\{|\delta_N^{\tau(a)}(v_j)|,k\}$. Thus $|\rho_N^{\tau(a)}(v_j)| < \min\{|\delta_N^{\tau(a)}(v_j)|,k\}$ that contradicts (1). This contradiction completes the proof. ■

Note that Theorem 3 implies Theorem 2. Note also that Theorem 2 implies Theorem 1. Indeed, if N is pre-flow, then (1) is satisfied, so, by Theorem 2, there exists a packing of k τ-respecting s-arborescences such that each vertex v in V belongs to exactly $\min\{k, d_A^-(v)\}$ of them. This implies that $\min\{k, \lambda_N(s,v)\} = \min\{k, d_A^-(v)\}$ and hence Theorem 1 follows.

4 Packing time-respecting arborescences in non-acyclic pre-flow temporal networks

In [7], Kamiyama and Kawase provide an example of 7 vertices and 12 arcs that shows that in Theorem 1 one can not delete the condition that D is acyclic. Here we provide a smaller example with 5 vertices and 7 arcs, see the first temporal network in Figure 1. Note that this temporal network contains a directed cycle whose arcs are not of the same τ-values and hence the temporal network is not consistent.

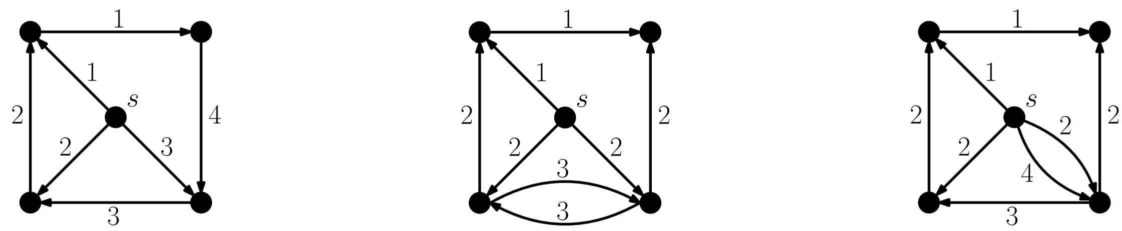

Figure 1: Three temporal networks N where the τ-value of an arc is presented on the arc. The first two are non-acyclic pre-flow, the second one is consistent. The third one is acyclic but not pre-flow. They contain no 2 arc-disjoint τ-respecting s-arborescences such that each vertex v belongs to $\min\{2, \lambda_N(s,v)\}$ of them.

The second temporal network in Figure 1 is another example that shows that in Theorem 1 one can not delete the condition that D is acyclic. Here the temporal network contains one directed cycle C and all the arcs of C are of the same τ-values and hence the temporal network is consistent. Note that in this example there exists a packing of three τ-respecting s-arborescences such that each vertex v belongs to exactly $\lambda_N(s,v)$ of them.

Kamiyama and Kawase [7] also provide an example of 7 vertices and 12 arcs that shows that in Theorem 1 one can not delete the condition that D is pre-flow. Here we provide a smaller example with 5 vertices and 8 arcs, see the third temporal network in Figure 1.

We now present the main result of this paper on packing of time-respecting arborescences in consistent pre-flow temporal networks where only the natural upper bound is given on the number of arborescences.

Theorem 4 *Let $N = (D = (V \cup s, A), \tau)$ be a consistent pre-flow temporal network. There exists a packing of $d_D^+(s)$ τ-respecting s-arborescences, each vertex v in V belonging to $\lambda_N(s,v)$ of them.*

To prove Theorem 4, we need an easy observation on almost Eulerian acyclic pre-flow temporal networks. A similar result has already been presented in [7].

Proposition 1 *If $N = (D = (V \cup \{s,t\}, A), \tau)$ is an almost Eulerian acyclic temporal network and μ_v is a τ-respecting bijection from $\delta_D(v)$ to $\rho_D(v)$ for all $v \in V$, then D decomposes into $d_D^+(s)$ τ-respecting (s,t)-paths such that each vertex $v \in V$ belongs to $d_D^-(v)$ of them.*

Proof We prove the claim by induction on $d_D^+(s)$. If $d_D^+(s) = 0$, then, since D is almost Eulerian and acyclic, we have $d_D^-(v) = 0$ for all $v \in V$ and we are done. Otherwise, there exists an arc leaving s. Then, using the bijections μ_v^{-1} and the facts that D is acyclic and μ_v is a τ-respecting, we find a τ-respecting directed (s,t)-path P. By deleting the arcs of P and applying the induction, the claim follows. ■

We also need the following result of Bang-Jensen, Frank, Jackson [2].

Theorem 5 ([2]) *Let $D = (V \cup s, A)$ be a pre-flow directed graph. There exists a packing of s-arborescences, each vertex $v \in V$ belonging to $\lambda_D(s,v)$ of them.*

We are ready to prove our main result.

Proof (of Theorem 4) First we transform the instance into another one $\boldsymbol{N'} = (D', \tau')$ as follows. The directed graph $\boldsymbol{D'} = (V \cup \{s,t\}, A \cup A')$ is obtained from D by adding a new vertex $\boldsymbol{t}$ and $d^-_D(v) - d^+_D(v)$ parallel arcs from v to t for all $v \in V$ and we define $\boldsymbol{\tau'}(a)$ to be equal to $\tau(a)$ if $a \in A$ and to M if $a \in A'$, where $\boldsymbol{M} = \max\{\tau(a) : a \in A\}$. Since N is pre-flow, so is D, that is $d^-_D(v) - d^+_D(v) \geq 0$ for all $v \in V$ and hence the construction is correct. This way we get an instance which remains consistent ($\{t\}$ is a new strongly connected component) and pre-flow (by the definition of M) and D' is almost Eulerian.

For each vertex $v \in V$, let us fix orderings of $\rho_{D'}(v)$ and $\delta_{D'}(v)$ such that $\tau'(e_1) \leq \cdots \leq \tau'(e_{d^-_{D'}(v)})$ and $\tau'(f_1) \leq \cdots \leq \tau'(f_{d^+_{D'}(v)})$, respectively. Then $\mu'_v(f_j) = e_j$ for all $1 \leq j \leq d^+_{D'}(v)$ is a τ'-respecting bijection for all $v \in V$. Indeed, if there exists j such that $\tau'(e_j) = \tau'(\mu'_v(f_j)) > \tau'(f_j) =: i$, then $|\rho^i_{N'}(v)| \leq j-1 < j \leq |\delta^i_{N'}(v)|$ that contradicts the fact that N' is pre-flow.

To reduce the problem to an easy acyclic problem that can be treated by Proposition 1 and some problems that can be treated by Theorem 5, let us denote the strongly connected components of D' by $\boldsymbol{Q'_1}, \dots, \boldsymbol{Q'_\ell}$. Let $\boldsymbol{U_j}$ denote the vertex set of Q'_j for all $1 \leq j \leq \ell$. Then the directed graph $\boldsymbol{D''}$ obtained from D' by contracting each Q'_j into a vertex $\boldsymbol{q''_j}$ is acyclic. By changing the indices if it is necessary, we may suppose that $q''_\ell = s, \dots, q''_1 = t$ is a topological ordering of the vertices of D''. Let $\boldsymbol{N''} = (D'', \tau'')$ be the temporal network where $\boldsymbol{\tau''}(a) = \tau'(a)$ for all $a \in A(D'')$. Note that since D' is almost Eulerian, so is D''. Indeed, we have $d^-_{D''}(q''_j) - d^+_{D''}(q''_j) = d^-_{D'}(U_j) - d^+_{D'}(U_j) = \sum_{v \in U_j}(d^-_{D'}(v) - d^+_{D'}(v)) = 0$ for all $2 \leq j \leq \ell - 1$. Note also that $d^+_D(s) = d^+_{D'}(s) = d^+_{D''}(s)$.

To define a convenient τ''-respecting bijection μ''_j from $\delta_{D''}(q''_j) = \delta_{D'}(U_j)$ to $\rho_{D''}(q''_j) = \rho_{D'}(U_j)$ for all $2 \leq j \leq \ell-1$, let us fix such a $\boldsymbol{j}$ and let us define the following sets:

$\boldsymbol{R^1_j} = \{vw \in \delta_{D'}(U_j) : \tau'(\mu'_v(vw)) > \tau'(Q'_j)\}$,
$\boldsymbol{R^2_j} = \{vw \in \delta_{D'}(U_j) : \tau'(vw) < \tau'(Q'_j)\}$,
$\boldsymbol{R^3_j} = \delta_{D'}(U_j) \setminus (R^1_j \cup R^2_j)$,
$\boldsymbol{S^1_j} = \{\mu'_v(vw) : vw \in R^1_j\}$,
$\boldsymbol{S^2_j} = \{\mu'_v(vw) : vw \in R^2_j\}$ and
$\boldsymbol{S^3_j} = \rho_{D'}(U_j) \setminus (S^1_j \cup S^2_j)$.

Claim 1 *$\{R^1_j, R^2_j, R^3_j\}$ is a partition of $\delta_{D'}(U_j)$ and $\{S^1_j, S^2_j, S^3_j\}$ is a partition of $\rho_{D'}(U_j)$.*

Proof If $vw \in R^1_j$, $v'w' \in R^2_j$, $uv = \mu'_v(vw) \in S^1_j$ and $u'v' = \mu'_{v'}(v'w') \in S^2_j$, then, since μ'_v and $\mu'_{v'}$ are τ'-respecting bijections, we have $\tau'(vw) \geq \tau'(\mu'_v(vw)) = \tau'(uv) > \tau'(Q'_j) > \tau'(v'w') \geq \tau'(\mu'_v(v'w')) = \tau'(u'v')$. Thus $vw \neq v'w'$ and $uv \neq u'v'$, so $R^1_j \cap R^2_j = \emptyset$ and $S^1_j \cap S^2_j = \emptyset$. By the definition of R^1_j and R^2_j, we have $R^1_j \cup R^2_j \subseteq \delta_{D'}(U_j)$. If $vw \in R^1_j$, then $\tau'(\mu'_v(vw)) > \tau'(Q'_j)$. If $vw \in R^2_j$, then, since μ'_v is a τ'-respecting bijection, we get $\tau'(\mu'_v(vw)) \leq \tau'(vw) < \tau'(Q'_j)$. Then, using that each arc in Q'_j has τ'-value $\tau'(Q'_j)$, we have $S^1_j \cup S^2_j \subseteq \rho_{D'}(U_j)$. By the definition of R^3_j and S^3_j, Claim 1 follows. ■

We now start to define μ''_j. For $vw \in R^1_j \cup R^2_j$, let $\boldsymbol{\mu''_j}(vw) = \mu'_v(vw)$. Since each μ'_v is τ'-respecting, we have $\tau''(vw) = \tau'(vw) \geq \tau'(\mu'_v(vw)) = \tau''(\mu''_v(vw))$. Note that for all $xy \in R^3_j$ and for all $uv \in S^3_j$, $\tau'(xy) \geq \tau'(Q'_j) \geq \tau'(uv)$. However, we cannot take an arbitrary bijection from R^3_j to S^3_j because we have to guarantee that the vertices in Q'_j also belong to the required number of arborescences. In order to do this, let us define the temporal network $\boldsymbol{N'_j} = (D'_j, \tau'_j)$ where the directed graph $\boldsymbol{D'_j}$ is obtained from D' by contracting $\bigcup_{i>j} U_i$ into a vertex $\boldsymbol{s_j}$, contracting $\bigcup_{i<j} U_i$ into a vertex $\boldsymbol{t_j}$ and deleting the arcs from s_j to t_j and $\boldsymbol{\tau'_j}(a) = \tau'(a)$ for all $a \in A(D'_j)$.

Claim 2 *N'_j satisfies the following.*

(a) D'_j is almost Eulerian,

(b) $\lambda_{D'_j}(s_j, t_j) = d^-_{D'_j}(t_j)$,

(c) $\lambda_{N'_j}(s_j, v) \geq \lambda_{N'}(s, v)$ for all $v \in U_j$.

Proof (a) Since D' is almost Eulerian, so is D'_j. Indeed, we have $d^-_{D'_j}(v) = d^-_{D'}(v) = d^+_{D'}(v) = d^+_{D'_j}(v)$ for all $v \in U_j$.

(b) By (a) and $d^-_{D'_j}(s_j) = 0 = d^+_{D'_j}(t_j)$, (b) easily follows. Indeed, let $\boldsymbol{r_j} = d^-_{D'_j}(t_j)$ and let us define $\boldsymbol{D^*_j}$ by adding r_j arcs $\{\boldsymbol{h_1}, \ldots, \boldsymbol{h_{r_j}}\}$ from t_j to s_j in D'_j. Then, by (a), D^*_j is Eulerian. Thus it decomposes into directed cycles. Let $\boldsymbol{C_1, \ldots, C_{r_j}}$ be the arc-disjoint directed cycles that contain the arcs $h_1, \ldots, h_{r_j}$. Then $\boldsymbol{P_1} = C_1 - h_1, \ldots, \boldsymbol{P_{r_j}} = C_{r_j} - h_{r_j}$ are arc-disjoint directed (s_j, t_j)-paths. Hence $r_j \leq \lambda_{D'_j}(s_j, t_j) \leq r_j$, and we have (b).

(c) For all $v \in U_j$, any τ'-respecting (s, v)-path in N' provides a τ'_j-respecting (s_j, v)-path in N'_j, and (c) follows. ■

To be able to use normal arborescences (not time-respecting ones), we have to modify D'_j. No τ-respecting directed path in D may contain an arc in S^1_j and an arc in Q'_j, hence the corresponding arcs in R^1_j and S^1_j will be deleted from D'_j. A τ-respecting s-arborescence in D may contain an arc $\mu'_v(vw)$ in S^2_j (where $vw \in R^2_j$) and an arc in Q'_j, but this arborescence must contain vw. To guarantee this property we use a trick: we replace the corresponding two arcs in R^2_j and S^2_j in D'_j by two convenient arcs. More precisely, let $\boldsymbol{H_j}$ be obtained from D'_j by deleting s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^1_j$ and replacing s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^2_j$ by $\boldsymbol{e_{vw}} = s_jt_j$ and $\boldsymbol{f_{vw}} = t_jv$. Let $\boldsymbol{E_j} = \{e_{vw} : vw \in R^2_j\}$ and $\boldsymbol{F_j} = \{f_{vw} : vw \in R^2_j\}$.

Claim 3 *H_j satisfies the following.*

(a) H_j is pre-flow,

(b) $\lambda_{H_j}(s_j, t_j) = d^-_{H_j}(t_j)$,

(c) $\lambda_{H_j}(s_j, v) \geq \lambda_{N'_j}(s_j, v) - d^-_{S^1_j}(v)$ for all $v \in U_j$.

Proof (a) By Claim 2(a), D'_j is almost Eulerian. Then, by $\delta_{D'_j}(t_j) = \emptyset$, D'_j is pre-flow. By deleting from D'_j the arcs s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^1_j$, we decreased the in-degree and the out-degree of each vertex by the same value so the directed graph obtained this way remained pre-flow. By replacing s_jv and vt_j that correspond to $\mu'_v(vw)$ and vw for all $vw \in R^2_j$ by s_jt_j and t_jv, we may decrease the out-degrees of the vertices in Q'_j but the in-degrees remained unchanged. Further, $d^+_{H_j}(t_j) = d^+_{D'_j}(t_j) + |F_j| = |E_j| \leq d^-_{H_j}(t_j)$. It follows that H_j is pre-flow.

(b) Note that for all $t_j \in X \subseteq U_j \cup t_j$, $d^-_{H_j}(X) = d^-_{D'_j}(X) - |R^1_j|$. Then, by Claim 2(b), we have $d^-_{H_j}(t_j) \geq \lambda_{H_j}(s_j, t_j) \geq \lambda_{D'_j}(s_j, t_j) - |R^1_j| = d^-_{D'_j}(t_j) - |R^1_j| = d^-_{H_j}(t_j)$ and (b) follows.

(c) On the one hand, by deleting the arcs corresponding to $\rho_{S^1_j}(v)$, we destroyed at most $d^-_{S^1_j}(v)$ τ'_j-respecting (s_j, v)-paths in N'_j and we did not destroy a τ'_j-respecting (s_j, u)-path in N'_j for $u \in U_j \setminus v$ because each arc in Q'_j has τ'_j-value $\tau'_j(Q'_j)$ and each arc in $\rho_{S^1_j}(v)$ has τ'_j-value strictly larger than $\tau'_j(Q'_j)$. On the other hand, if a τ'_j-respecting (s_j, u)-path P contains s_jv (corresponding to $\mu'_v(vw)$ for some $vw \in R^2_j$) in N'_j then $P - s_jv + e_{vw} + f_{vw}$ is a directed (s_j, u)-path in H_j. These arguments imply (c). ■

By Claim 3(a) and Theorem 5, there exists a packing $\mathcal{B}_j$ of s_j-arborescences $\boldsymbol{T^i_j}$ in H_j, each vertex $v \in U_j \cup t_j$ belonging to $\lambda_{H_j}(s_j, v)$ of them. Let us choose such a packing $\boldsymbol{\mathcal{B}_j}$ that minimizes the size of the set $\boldsymbol{F_{\mathcal{B}_j}}$ of the arcs $f_{vw} \in F_j$ such that an arborescence $\boldsymbol{T^{f_{vw}}_j}$ in $\mathcal{B}_j$ contains f_{vw} but not e_{vw}.

Claim 4 *$\mathcal{B}_j$ satisfies the following.*

(a) $d^+_{H_j}(s_j) = |\mathcal{B}_j| = d^-_{H_j}(t_j)$,

(b) $F_{\mathcal{B}_j} = \emptyset$,

(c) $\{T^i_j - s_j - t_j : T^i_j \in \mathcal{B}_j\}$ *is a packing of arborescences in* Q'_j, *each vertex* $v \in U_j$ *belonging to* $\lambda_{H_j}(s_j, v)$ *of them.*

Proof (a) By Claim 3(b), t_j belongs to $\lambda_{H_j}(s_j,t_j) = d^-_{H_j}(t_j)$ of the s_j-arborescences in $\mathcal{B}_j$. Thus each arc entering t_j belongs to some s_j-arborescence in $\mathcal{B}_j$ and $d^-_{H_j}(t_j) \leq |\mathcal{B}_j|$. Moreover, by construction and since D'_j is almost Eulerian, we have $d^-_{H_j}(t_j) = d^-_{D'_j}(t_j) - |R^1_j| = d^+_{D'_j}(s_j) - |S^1_j| = d^+_{H_j}(s_j) \geq |\mathcal{B}_j|$, and (a) follows.

(b) Suppose that $F_{\mathcal{B}_j} \neq \emptyset$. Let $\boldsymbol{E_{\mathcal{B}_j}} = \{e_{vw} : f_{vw} \in F_{\mathcal{B}_j}\}$. By (a), every $e_{vw} \in E_{\mathcal{B}_j}$ is contained in an s_j-arborescence $\boldsymbol{T^{e_{vw}}_j}$ in $\mathcal{B}_j$.

First suppose that for some $\boldsymbol{e_{vw}} \in E_{\mathcal{B}_j}$, $T^{e_{vw}}_j$ contains only the arc e_{vw}. Note that $T^{f_{vw}}_j - f_{vw}$ consists of an s_j-arborescence T'_j and a v-arborescence T''_j. Let $\boldsymbol{\mathcal{B}'_j}$ be obtained from $\mathcal{B}_j$ by replacing $T^{f_{vw}}_j$ by T'_j and $T^{e_{vw}}_j$ by $e_{vw} + f_{vw} + T''_j$. Then $\mathcal{B}'_j$ is a packing of s_j-arborescences in H_j such that each vertex $v \in U_j \cup t_j$ belongs to $\lambda_{H_j}(s_j, v)$ of them. Moreover, f_{vw} and e_{vw} belong to the same s_j-arborescence in $\mathcal{B}'_j$, that is $|F_{\mathcal{B}'_j}| < |F_{\mathcal{B}_j}|$ and we have a contradiction.

We may hence suppose that for every $e_{vw} \in E_{\mathcal{B}_j}$, $T^{e_{vw}}_j$ contains another arc, so by (a), contains an arc in $F_{\mathcal{B}_j}$. Let $\mathcal{B}'_j$ be the set of those s_j-arborescences in $\mathcal{B}_j$ that contain an arc of $F_{\mathcal{B}_j}$. Then $|F_{\mathcal{B}_j}| = |E_{\mathcal{B}_j}| \leq |\mathcal{B}'_j| \leq |F_{\mathcal{B}_j}|$. Hence we have equality everywhere. It follows that every s_j-arborescences in $\mathcal{B}'_j$ contains exactly one arc from both $F_{\mathcal{B}_j}$ and $E_{\mathcal{B}_j}$. Then for every $f_{vw} \in F_{\mathcal{B}_j}$, $T^{f_{vw}}_j$ contains an arc $e_{v'w'} \in E_{\mathcal{B}_j}$. Let $\boldsymbol{\mathcal{B}''_j}$ be obtained from $\mathcal{B}_j$ by replacing $e_{v'w'}$ by $e_{vw} \in E_{\mathcal{B}_j}$ in $T^{f_{vw}}_j$ for every $f_{vw} \in F_{\mathcal{B}_j}$. Then $\mathcal{B}''_j$ is a packing of s_j-arborescences in H_j such that each vertex $v \in U_j \cup t_j$ belongs to $\lambda_{H_j}(s_j, v)$ of them. Moreover, $F_{\mathcal{B}''_j} = \emptyset$ and we have a contradiction.

(c) follows from the definition of $\mathcal{B}_j$, (a) and (b). ■

We now finish the definition of μ''_j. Let $vw \in R^3_j$. Then vw corresponds in H_j to an arc $\boldsymbol{g_{vw}} = vt_j$ entering t_j. By Claim 4(a), g_{vw} belongs to an s_j-arborescence $\boldsymbol{T^{g_{vw}}_j}$ in $\mathcal{B}_j$. Let us define $\boldsymbol{\mu''_j}(vw) \in S^3_j$ to be the arc xq''_j of D'' that corresponds to the arc $s_j u$ in H_j of the unique (s_j, t_j)-path of $T^{g_{vw}}_j$. Then $\tau''_j(vw) = \tau'_j(vw) \geq \tau'_j(Q'_j) \geq \tau'_j(xq''_j) = \tau''_j(\mu''_j(vw))$ for all $vw \in R^3_j$.

By the definition of μ''_j and Claim 1, we have a τ''-respecting bijection μ''_j from $\delta_{D''}(q''_j)$ to $\rho_{D''}(q''_j)$ for all $2 \leq j \leq \ell - 1$. Recall that D'' is acyclic and almost Eulerian. Then, by Proposition 1 and $d^+_D(s) = d^+_{D''}(s)$, D'' decomposes into τ''-respecting (s,t)-paths $\boldsymbol{P_1, \dots, P_{d^+_D(s)}}$ such that each vertex $q''_j \neq s$ belongs to $d^-_{D''}(q''_j)$ of them. These paths can be extended, using from Claim 4(c) the arborescences $T^i_j - s_j - t_j$ in Q'_j for $1 \leq i \leq d^+_{H_j}(s_j)$ and $2 \leq j \leq \ell - 1$, to get s-arborescences in D' such that each vertex $v \in V$ belongs to $\lambda_{H_j}(s_j, v) + d^-_{S^1_j}(v) \geq \lambda_{N'_j}(s_j, v) \geq \lambda_{N'}(s, v)$ of them, by Claims 3(b) and 2(c). Since the directed paths $P_1, \dots, P_{d^+_D(s)}$ are τ''-respecting, that is τ'-respecting and D' is consistent, the arborescences constructed are τ'-respecting. Hence N' has a packing of τ'-respecting s-arborescences $\boldsymbol{T'_1, \dots, T'_{d^+_D(s)}}$ such that each vertex v of D' distinct from s and t belongs to $\lambda_{N'}(s,v) = \lambda_N(s,v)$ of them, and hence $\{\boldsymbol{T_1} = T'_1 - t, \dots, \boldsymbol{T_{d^+_D(s)}} = T'_{d^+_D(s)} - t\}$ is a packing of τ-respecting s-arborescences such that each vertex v of D distinct from s belongs to $\lambda_N(s,v)$ of them. ■

5 Arc-disjoint spanning time-respecting arborescences

Edmonds' arborescence packing theorem [3] states that k-root-connectivity from s implies the existence of a packing of k spanning s-arborescences. The following observation of [8] shows that the natural extension of Edmonds theorem for $k = 1$ is true for temporal networks.

Theorem 6 ([8]) *Any τ-respecting root-connected temporal network $N = ((V \cup s, A), \tau)$ contains a spanning τ-respecting s-arborescence.*

The authors of [8] show that high time-respecting root-connectivity of a temporal network does not imply the existence of 2 arc-disjoint spanning time-respecting arborescences.

Theorem 7 ([8]) *For all $k \in \mathbb{N}^+$, there exist temporal networks $N = ((V \cup s, A), \tau)$ such that $\lambda_N(s, v) \geq k$ for all $v \in V$ and no packing of 2 spanning τ-respecting s-arborescences exists in N.*

Their construction contains directed cycles but it can be easily modified to get an acyclic example. This acyclic example for $k = 2$ is presented in Figure 2 in [7].

We now relate the spanning time-respecting arborescence packing problem to known problems, namely the Steiner arborescence packing problem and the hypergraph proper 2-coloring problem. To do that we explain how the above mentioned modified construction can be obtained in 3 steps. First, take a k-uniform hypergraph without proper 2-coloring. Then construct a directed graph that is Steiner k-root-connected without 2 arc-disjoint Steiner arborescences. Finally, construct an acyclic temporal network that is time-respecting k-root-connected without 2 arc-disjoint spanning time-respecting arborescences.

There exist many constructions for k-uniform hypergraphs without proper 2-coloring, see [1], [8] and Exercice 13.45(b) of [9]. We mention that, by a result of Erdős [4], all examples contain exponentially many hyperedges in k.

Theorem 8 *[4] Any k-uniform hypergraph without a proper 2-coloring contains at least 2^{k-1} hyperedges.*

We now show that starting from an arbitrary k-uniform hypergraph $\mathcal{H}_k = (V_k, \mathcal{E}_k)$ without proper 2-coloring how to construct an acyclic directed graph D_k and a vertex set U_k such that $\lambda_{D_k}(s, u) = k$ for all $u \in U_k$ and there exists no packing of two (s, U_k)-arborescences in D_k. Let $\boldsymbol{G_k}$:= $(V_k, U_k; E_k)$ be the bipartite incidence graph of the hypergraph $\mathcal{H}_k$, where the elements of $\boldsymbol{U_k}$ correspond to the hyperedges in $\mathcal{E}_k$. Let $\boldsymbol{D_k} = (V_k \cup U_k \cup s, A_k)$ be obtained from G_k by adding a vertex s and an arc sv for all $v \in V_k$ and directing each edge of E_k from V_k to U_k. By construction D_k is acyclic. Since $\mathcal{H}_k$ is k-uniform, we have $\lambda_{D_k}(s, u) = k$ for all $u \in U_k$.

Theorem 9 *D_k has no packing of two (s, U_k)-arborescences.*

Proof Suppose that there exists a packing of 2 (s, U_k)-arborescences F_1 and F_2 in D_k. Using this packing, we can define a 2-coloring of V_k: let $v \in V_k$ be colored by 1 if $sv \in A(F_1)$ and by 2 otherwise. Since each vertex in U_k belongs to both F_1 and F_2, no hyperedge of $\mathcal{E}_k$ is monochromatic, that is the above defined 2-coloring of $\mathcal{H}_k$ is proper. This contradicts the fact that $\mathcal{H}_k$ has no proper 2-coloring. ■

As a next step, we show that starting from the acyclic directed graph D_k and the vertex set U_k, how to construct a temporal network N_k such that $\lambda_{N_k}(s, v) = k$ for all vertices v and no packing of 2 spanning time-respecting s-arborescences exists in N. Let us define $\boldsymbol{N_k} := (D_k^*, \tau_k^*)$ as follows: $\boldsymbol{D_k^*}$ is obtained from D_k by adding the set of arcs $\boldsymbol{A_k^*}$ consisting of $k-1$ parallel arcs from s to all $v \in V_k$ and we define $\boldsymbol{\tau_k^*}(a) = 1$ if $a \in A_k$ and 2 if $a \in A_k^*$. Note that since D_k is acyclic, so is D_k^*. Then a spanning s-arborescence F^* of D_k^* is τ_k^*-respecting if and only if $F^* - A_k^*$ is an (s, U_k)-arborescence in D_k. Thus a packing of 2 spanning τ_k^*-respecting s-arborescences in D_k^* would provide a packing of 2 (s, U_k)-arborescences in D_k. Hence, the following result is an immediate consequence of Theorem 9.

Theorem 10 *For all $k \in \mathbb{N}^+$, there exist acyclic temporal networks $N = ((V \cup s, A), \tau)$ such that $\lambda_N(s, v) \geq k$ for all $v \in V$ and no packing of 2 spanning τ-respecting s-arborescences exists in N.*

These examples of acyclic temporal networks that are time-respecting k-root-connected without 2 arc-disjoint spanning time-respecting arborescences contain, by Theorem 8, exponentially many vertices in k. In other words, $k \leq log(n)$ where n is the number of vertices. In the light of this fact, it is natural to ask whether there exist 2 arc-disjoint spanning time-respecting arborescences in a temporal network if k is linear in n. The examples of Figure 1 show that time-respecting $(n-3)$-root-connectivity does not imply the existence of 2 arc-disjoint spanning time-respecting arborescences. We propose the first steps in this direction. We first remark that n-root-connectivity is enough.

Claim 5 *Let $N = ((V \cup s, A), \tau)$ be a temporal network on $n \geq 1$ vertices such that $\lambda_N(s, v) \geq n$ for all $v \in V$. Then there exists a packing of 2 spanning τ-respecting s-arborescences in N.*

Proof Since $\lambda_N(s, v) \geq n \geq 1$ for all $v \in V$, there exists, by Theorem 6, a spanning τ-respecting s-arborescence $\boldsymbol{F}$ in N. Further, there exist n arc-disjoint τ-respecting (s, v)-paths $\boldsymbol{P_1^v, \ldots, P_n^v}$ for all $v \in V$. By deleting the arcs of F, we can destroy at most $|A(F)|$ of the (s, v)-paths $P_1^v, \ldots, P_n^v$ for all $v \in V$. Since $|A(F)| = n - 1$, this implies that $\lambda_{N-A(F)}(s, v) \geq n - (n - 1) = 1$ for all $v \in V$. Then, there exists, by Theorem 6, a spanning τ-respecting s-arborescence $\boldsymbol{F'}$ in $N - A(F)$, and we are done. ■

With some effort we can improve the previous result by 1.

Theorem 11 *Let $N = ((V \cup s, A), \tau)$ be a temporal network on $n \geq 2$ vertices such that $\lambda_N(s, v) \geq n - 1$ for all $v \in V$. Then there exists a packing of 2 spanning τ-respecting s-arborescences in N.*

Proof Since $\lambda_N(s, v) \geq n - 1 \geq 1$ for all $v \in V$, there exists, by Theorem 6, a spanning τ-respecting s-arborescence $\boldsymbol{F}$ in N. Let $\boldsymbol{F(v)}$ be the unique arc of F entering v for all $v \in V$. Note that $A(F) = \{F(v) : v \in V\}$. If $\lambda_{N-A(F)}(s, v) \geq 1$ for all $v \in V$ then there exists, by Theorem 6, a spanning τ-respecting s-arborescence in $N - A(F)$, and we are done.

Otherwise, $\lambda_{N-A(F)}(s, u) = 0$ for some $\boldsymbol{u} \in V$. By assumption, there exist $n - 1$ arc-disjoint τ-respecting (s, u)-paths $\boldsymbol{P_1, \ldots, P_{n-1}}$. Then, since $|V| = n - 1$, there exists a bijection $\boldsymbol{\pi}$ from V to $\{1, \ldots, n - 1\}$ such that $F(v)$ is contained in $P_{\pi(v)}$ for all $v \in V$. It follows that no arc leaves u in F. Let $\boldsymbol{w} \in V - u$ be a vertex for which $\tau(F(w))$ is maximum. Let the last arc of $P_{\pi(w)}$ be denoted by $\boldsymbol{xu}$. Then, since $F(u)$ is the last arc of the path $P_{\pi(u)}$ and the paths are arc-disjoint, $F(u) \neq xu$. By the choice of w and since $P_{\pi(w)}$ is τ-respecting, we have $\tau(F(x)) \leq \tau(F(w)) \leq \tau(xu)$. We obtain that $\boldsymbol{F'} := F - F(u) + xu \neq F$ is also a spanning τ-respecting s-arborescence in N.

By assumption and $|A(F) - F(u)| = n - 2$, we have $\lambda_{N-(A(F)-F(u))}(s, v) \geq (n-1) - (n-2) = 1$ for all $v \in V$. Then, by Theorem 6, there exists a spanning τ-respecting s-arborescence $\boldsymbol{F''}$ in $N - (A(F) - F(u))$. Since F'' contains a unique arc entering u, it does not contain either $F(u)$ or xu. Thus, F'' is arc-disjoint from either F or F', and we are done. ■

We conjecture that the following is true.

Conjecture 1 *Let $N = ((V \cup s, A), \tau)$ be an acyclic temporal network on $n \geq 4$ vertices such that $\lambda_N(s, v) \geq \frac{n}{2}$ for all $v \in V$. Then a packing of 2 spanning τ-respecting s-arborescences exists in N.*

The third example presented in Figure 1 is of 5 vertices, acyclic, time-respecting 2-root-connected and has no packing of 2 spanning τ-respecting s-arborescences. It follows that time-respecting $\frac{2n}{5}$-root-connectivity is not enough to have a packing of 2 spanning time-respecting s-arborescences in acyclic temporal networks.

6 Complexity results

Lovász [10] proved that the problem of 2-colorings of k-uniform hypergraphs is NP-complete. This implies that the problem of packing 2 Steiner arborescences is also NP-complete. An easier way to see this is to use the NP-complete problem of two arc-disjoint directed paths in a directed graph D, one from r to t and the other from t to r. (See [6].) Construct D' from D by adding a new vertex s and the two arcs sr and st. Then D has an (r, t)-path and a (t, r)-path that are arc-disjoint if and only if D' has a packing of 2 $(s, \{r, t\})$-arborescences. This with the construction presented in the previous section finally imply the following.

Theorem 12 *The problem of packing k spanning time-respecting arborescences is NP-complete even for $k = 2$.*

Let us check what happens if we replace the inequality with equality in the definition of time-respecting directed paths and we consider the values of τ as colors. Then we get monochromatic directed paths. We may hence study the following problem MoChPaSpAr:

Problem 1 *Given a directed graph $D = (Z \cup s, A)$ and a coloring c of the arcs, decide whether there exists a spanning s-arborescence containing only monochromatic directed paths.*

We show that this decision problem is difficult. We will reduce the exact cover in 3-regular 3-uniform hypergraphs problem (RXC3) to our problem. In RXC3, we are given a 3-regular 3-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$, and the problem consists of determining whether there exists a subset $\mathcal{E}'$ of $\mathcal{E}$ such that each vertex in V occurs in exactly one hyperedge in $\mathcal{E}'$. Gonzalez proved in [5] that RXC3 is NP-complete.

Theorem 13 *The problem* MoChPaSpAr *is NP-complete even for acyclic directed graphs and for two colors.*

Proof It is clear that MoChPaSpAr is in NP. Let us take an instance of RXC3, that is let $\mathcal{H}$ be a 3-regular 3-uniform hypergraph. We construct a polynomial size instance (D, c) of MoChPaSpAr such that $\mathcal{H}$ has an exact cover if and only if (D, c) has a spanning s-arborescence containing only monochromatic directed paths. Since $\mathcal{H}$ is a 3-regular 3-uniform hypergraph, the number of vertices of $\mathcal{H}$ and the number of hyperedges of $\mathcal{H}$ coincide. Let us denote the vertices of $\mathcal{H}$ by $V = \{v_1, \ldots, v_h\}$ and the hyperedges of $\mathcal{H}$ by $\mathcal{E} = \{H_1, \ldots, H_h\}$.

Let $\boldsymbol{D} = (Z \cup s, A)$ be the directed graph where $\boldsymbol{Z} = U \cup V \cup W$ and $\boldsymbol{A} = A_1 \cup A_2 \cup A_3 \cup A_4$ with $\boldsymbol{U} = \{u_1, \ldots, u_h\}$, $\boldsymbol{W} = \{w_{i,j} : H_i \cap H_j \neq \emptyset\}$, $\boldsymbol{A_1} = \{e_i^1 = su_i : 1 \leq i \leq h\}$, $\boldsymbol{A_2} = \{e_i^2 = su_i : 1 \leq i \leq h\}$, $\boldsymbol{A_3} = \{u_i v_j : u_i \in U, v_j \in V, v_j \in H_i\}$ and $\boldsymbol{A_4} = \{u_i w_{i,j}, u_j w_{i,j} : u_i, u_j \in U, w_{i,j} \in W\}$. Let $\boldsymbol{c}(a)$ be equal to black if $a \in A_1 \cup A_3$ and grey if $a \in A_2 \cup A_4$. Note that D is acyclic and c uses only two colors. For an example see Figure 2.

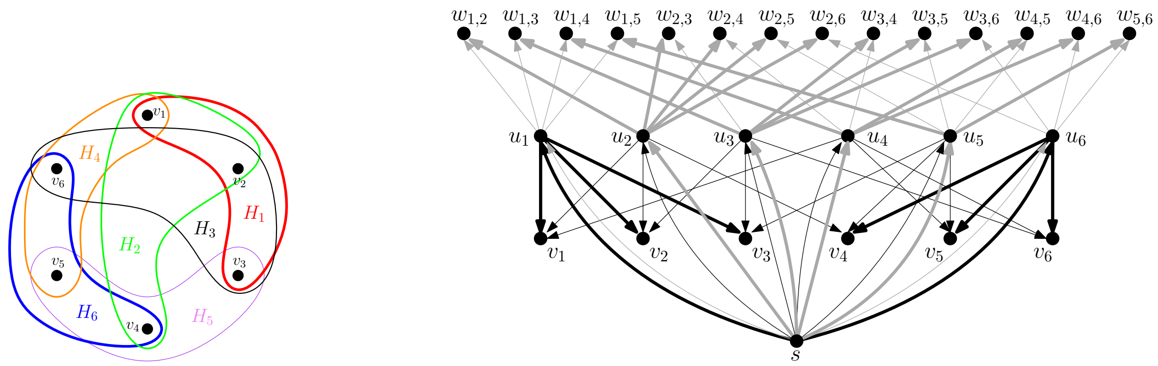

Figure 2: A 3-regular 3-uniform hypergraph and the constructed colored directed graph for it.

The size of D is polynomial in h. Indeed, since $\mathcal{H}$ is a 3-regular 3-uniform hypergraph, $|W| \leq \frac{1}{2} \cdot 3 \cdot 2 \cdot h$, so $|Z \cup s| = |U| + |V| + |W| + 1 \leq h + h + 3h + 1 = 5h + 1$ and $|A| = |A_1| + |A_2| + |A_3| + |A_4| \leq h + h + 3h + 2 \cdot 3h = 11h$.

Suppose first that $\mathcal{H}$ has an exact cover $\boldsymbol{\mathcal{H}'}$. Let $\boldsymbol{Z'}$ be the set of vertices of D that can be reached from s by a black directed path starting with an arc su_i with $H_i \in \mathcal{H}'$ and $\boldsymbol{Z''}$ by a grey directed path starting with an arc su_i with $H_i \notin \mathcal{H}'$. Since $\mathcal{H}'$ is a cover, we have $Z' = V \cup \{u_i : H_i \in \mathcal{H}'\}$. Since the hyperedges in $\mathcal{H}'$ are disjoint, we have $Z'' = \{u_i : H_i \notin \mathcal{H}'\} \cup W$. Since $Z' \cap Z'' = s$, the desired spanning s-arborescence containing only monochromatic directed paths exists. In the example of Figure 2, $\mathcal{H}' = \{H_1, H_6\}$, $Z' = V \cup \{u_1, u_6\}$, $Z'' = \{u_2, u_3, u_4, u_5\} \cup W$ and the arborescence is represented by bold arcs.

Now suppose that (D, c) has a spanning s-arborescence $\boldsymbol{F}$ containing only monochromatic directed paths. Let $\boldsymbol{\mathcal{H}'} = \{H_j : u_j \in U, v_i \in V, u_j v_i \in F\}$. Since F is a spanning s-arborescence, each vertex v_i has exactly one black arc $u_j v_i$ in F entering. This implies that $\mathcal{H}'$ covers V. Let H_j, H_k $(j < k)$ be hyperedges in $\mathcal{H}'$. If $w_{j,k} \in W$, then, since the directed paths are monochromatic in F, su_j and su_k are black and hence $u_j w_{j,k}$ and $u_k w_{j,k}$ are not contained in F that contradicts the fact that F is a spanning s-arborescence. Thus H_j and H_k are disjoint. It follows that $\mathcal{H}'$ is an exact cover. ■

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