Old and new results on packing arborescences

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March 7, 2016

Abstract

We propose a further development in the theory of packing arborescences. First we review some of the existing results on packing arborescences and then we provide common generalizations of them. We introduce and solve the problem of reachability-based packing of matroid-rooted hyperarborescences and the one of matroid-based packing of matroid-rooted mixed hyperarborescences.

Keywords Arborescence, Packing, Directed Hypergraph, Matroid

1 Introduction

The director of a secret agency, an intelligent woman, runs her network that has been carefully constructed by herself: from each agent to any other agent some (zero, one or more) secret channels are available.

Yesterday she created, from a given set of information, some messages that she assigned to agents. To avoid the interception of all the information, each message was assigned to one agent and an agent could have been assigned to zero, one or more messages. The messages can then be propagated through the network: any agent may send any message they know to any of their contacts.

Today the security rules changed: from now on, the transmission of at most one message is allowed via the same channel. The new rules pose serious questions that the director must consider. Is it possible for each agent to receive all the messages? She then realizes that it is not guaranteed that all the agents could have received all the messages with the old rules. So is it possible today that each agent receives the messages that he or she could have received with the old rules?

She knew that all the messages she gave to her agents were not independent: it is possible that given a subset of messages, one would get no extra information by adding another message to the set. The director currently wonders whether it is now possible that each agent receives only independent messages that contain all the information, or all the information they could have received before the new rules.

The director could have constructed her network in a more efficient way: a channel would be from a whole group of agents to one agent and any agent from that group may send a message to that contact. After having this idea, she reconsiders all her previous questions in this more general framework.

For each channel (from a (group of) agent(s) to a contact) it must be decided which message will be sent (if any) and from which agent. The director then knows that the minimal set of channels through which the same message is sent forms an arborescence.

We study packings of arborescences in this paper. The famous result of Edmonds [4] on packing spanning arborescences, Theorem 2 in this paper, has extensions in many directions. For our purposes let us mention four of them: Theorem 4 on packing *reachability* arborescences (Kamiyama, Katoh, Takizawa [9]), Theorem 5 on packing matroid-rooted arborescences with *matroid* constraint (Durand de

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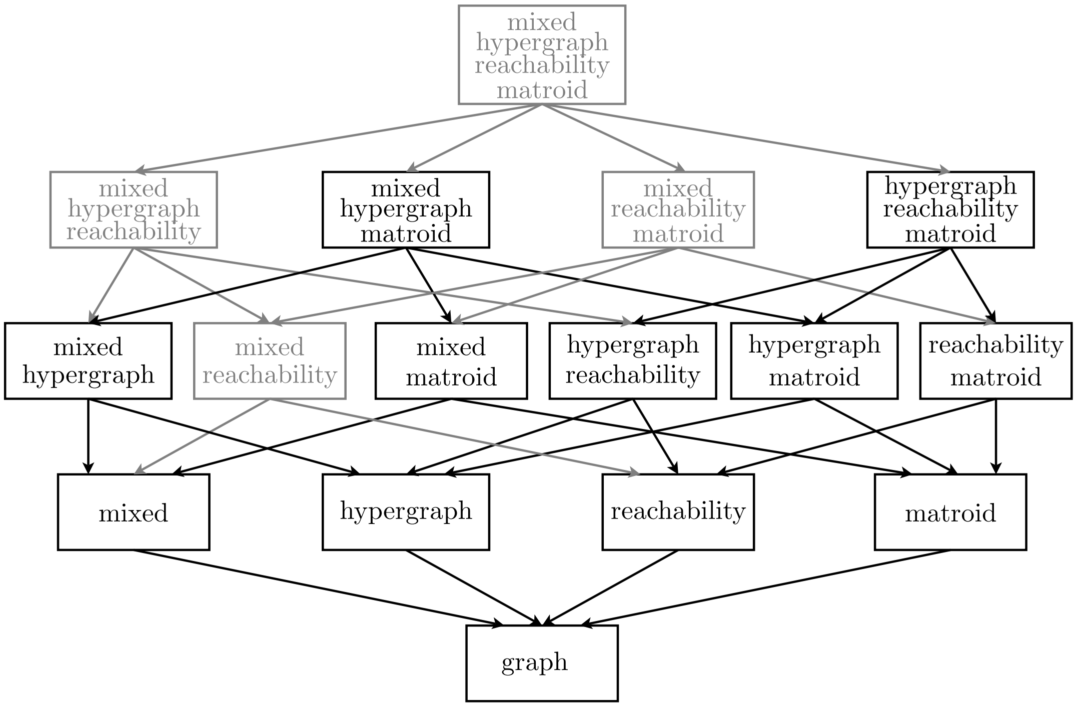

Figure 1: All possible common generalizations of the 4 problems mentioned in the introduction.

Gevigney, Nguyen, Szigeti [3]), Theorem 7 on packing spanning *hyper*arborescences (Frank, T. Király, Kriesell [7]) and Theorem 12 on packing spanning *mixed* arborescences (Frank [5]). Figure 1 shows all the possible common generalizations of these extensions. The results corresponding to black boxes of the diagram are presented in this paper, the ones in gray are yet to be proved.

Here we propose a common generalization of three of the above four extensions, not the mixed one, namely the problem of reachability-based packing of matroid-rooted hyperarborescences in directed hypergraphs. As the main contribution of the this work we present the solution to this problem, by reducing it, using the trimming operation, to its special case in digraphs. The latter result, Theorem 6, on reachability-based packing of matroid-rooted arborescences in digraphs was proved by Cs. Király [11] and is a common generalization of the above mentioned results on packing reachability arborescences [9] and on packing matroid-rooted arborescences with matroid constraint [3]. By Theorem 1, the reduction by trimming can be applied if the directed hypergraph covers an intersecting supermodular function. In our case the reduction is not so straightforward since the function, which appears in the necessary condition of the problem, is not intersecting supermodular and hence we cannot apply Theorem 1.

We also consider a generalization of other three of the above four extensions, not the reachability one this time, namely the problem of matroid-based packing of matroid-rooted mixed hyperarborescences. Using a new orientation result (Theorem 20) on hypergraphs covering intersecting supermodular functions, we reduce this problem to its directed version, the problem of matroid-based packing of matroid-rooted hyperarborescences, which in turn is a special case of the problem of reachability-based packing of matroid-rooted hyperarborescences. The proof of Theorem 20 imitates the proof (see in [5]) of its special case in graphs.

The techniques of this paper and many of its results are presented and explained in the recent book of Frank [5].

The rest of this paper is organized as follows. In Section 2 we provide all the basic definitions needed in the other sections. In Sections 3 and 4 we present some results about rooted digraphs and matroid-rooted digraphs. Section 5 contains results about rooted directed hypergraphs. We introduce and prove our main result on matroid-rooted directed hypergraphs in Section 6. Sections 7 and 8 consider undirected graphs, mixed graphs and matroid-rooted mixed graphs. In Sections 9 and 10 we present results about rooted mixed hypergraphs and matroid-rooted mixed hypergraphs. To prove the results about mixed hypergraphs we need an orientation theorem which we present and demonstrate in Section 11. Finally, Section 12 contains some remarks on further generalizations.

2 Definitions

Let $\boldsymbol{\mathcal{H}} = (V, \mathcal{E})$ be a hypergraph where $\boldsymbol{V}$ denotes the set of vertices and $\boldsymbol{\mathcal{E}}$ denotes the set of hyperedges of $\mathcal{H}$. We suppose that all the hyperedges in $\mathcal{E}$ are of size at least 2. For a vertex set X, $\boldsymbol{i_{\mathcal{E}}(X)}$ denotes the number of hyperedges in $\mathcal{E}$ that are contained in X, while $\boldsymbol{j_{\mathcal{E}}(X)}$ denotes the number of hyperedges in $\mathcal{E}$ that intersect X (that is contains at least one element of X). Note that the functions $i_{\mathcal{E}}$ and $j_{\mathcal{E}}$ are closely related: every hyperedge is either completely contained in X or intersects $V \setminus X$ therefore the following formula holds.

$$i_{\mathcal{E}}(X) + j_{\mathcal{E}}(V \setminus X) = |\mathcal{E}|. \tag{1}$$

Let $\boldsymbol{\mathcal{P}} = \{V_0, V_1, \ldots, V_\ell\}$ be a partition of V where $\boldsymbol{V_0}$ can be empty but the other V_i's cannot. We denote by $\boldsymbol{e_{\mathcal{E}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ intersecting at least two members of $\mathcal{P}$. Since every hyperedge is either completely contained in some V_i or intersects at least two V_i's, the following formula holds.

$$e_{\mathcal{E}}(\mathcal{P}) + \sum_0^\ell i_{\mathcal{E}}(V_i) \quad = \quad |\mathcal{E}|. \tag{2}$$

For a set function h on V, we say that the hypergraph $\mathcal{H}$ is *h-subpartition-connected* if

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_1^\ell h(V_i) \text{ for every partition } \mathcal{P} = \{V_0, V_1, \ldots, V_\ell\} \text{ of } V. \tag{3}$$

Please note that in (3) the index i starts at 1 (and not at 0). This means that we consider here all the subpartitions of V.

Let $\boldsymbol{\vec{\mathcal{H}}} = (V, \mathcal{A})$ be a directed hypergraph (shortly *dypergraph*) where $\boldsymbol{V}$ denotes the set of vertices and $\boldsymbol{\mathcal{A}}$ denotes the set of hyperarcs of $\vec{\mathcal{H}}$. By a *hyperarc* we mean a pair (Z, z) such that $z \in Z \subseteq V$, where z is the *head* of the hyperarc (Z, z) and the elements of $Z \setminus z$ are the *tail*s of the hyperarc (Z, z). We suppose that each hyperarc has one head and at least one tail. Let $X, Y \subseteq V$. We say that the hyperarc (Z, z) *enters* X if the head of (Z, z) is in X and at least one tail of (Z, z) is not in X, that is $z \in X$ and $(Z \setminus z) \cap (V \setminus X) \neq \emptyset$. We define the *in-degree* $\boldsymbol{\rho_{\mathcal{A}}(X)}$ of X as the number of hyperarcs in $\mathcal{A}$ entering X. We denote by $\boldsymbol{d_{\mathcal{A}}(X, Y)}$ the number of hyperarcs in $\mathcal{A}$ that are contained in $X \cup Y$ and intersect both $X \setminus Y$ and $Y \setminus X$ and by $\boldsymbol{i_{\mathcal{A}}(X)}$ the number of hyperarcs in $\mathcal{A}$ that are contained in X. The following equalities are well known for dypergraphs:

$$\rho_{\mathcal{A}}(X) + i_{\mathcal{A}}(X) \quad = \quad \sum_{v \in X} \rho_{\mathcal{A}}(v), \tag{4}$$

$$\rho_{\mathcal{A}}(X) + \rho_{\mathcal{A}}(Y) - d_{\mathcal{A}}(X, Y) \quad = \quad \rho_{\mathcal{A}}(X \cap Y) + \rho_{\mathcal{A}}(X \cup Y). \tag{5}$$

For a set function h on V, we say that the dypergraph $\vec{\mathcal{H}}$ *covers* h if

$$\rho_{\mathcal{A}}(X) \geq h(X) \text{ for all } X \subseteq V. \tag{6}$$

By *trimming* the dypergraph $\vec{\mathcal{H}}$ we mean replacing each hyperarc (K, v) of $\vec{\mathcal{H}}$ by an arc uv where u is one of the tails of the hyperarc (K, v). If a dypergraph $\vec{\mathcal{H}}$ covers a function h then it is a natural question to wonder when $\vec{\mathcal{H}}$ can be trimmed to a digraph $\vec{H}$ that also covers h. The proof of Theorem 7.4.9 given in [5] with the necessary straightforward modifications can be extended for intersecting supermodular functions and hence we have the following theorem.

Theorem 1. *Let $\vec{\mathcal{H}} = (V, \mathcal{A})$ be a dypergraph and h an integer-valued, intersecting supermodular function on V such that $h(\emptyset) = 0 = h(V)$. If $\vec{\mathcal{H}}$ covers h, then $\vec{\mathcal{H}}$ can be trimmed to a digraph $\vec{H}$ that covers h.*

By an *orientation* of $\mathcal{H}$, we mean a dypergraph $\vec{\mathcal{H}}$ obtained from $\mathcal{H}$ by choosing, for every $Z \in \mathcal{E}$, an orientation of Z, that is by choosing a head z for Z.

For a vector $m : V \to \mathbb{Z}$ and a set $X \subseteq V$, we define $\boldsymbol{m(X)}$ as usual, that is

$$m(X) = \sum_{v \in X} m(v). \tag{7}$$

Let p be a set function on V. We call p *supermodular* if for every $X, Y \subseteq V$,

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{8}$$

We say that p is *intersecting supermodular* if (8) is satisfied for every $X, Y \subseteq V$ when $X \cap Y \neq \emptyset$. A set function b is called *submodular* if $-b$ is supermodular. It is well known that $i_{\mathcal{E}}$ is supermodular and that $\rho_{\mathcal{A}}$ and $j_{\mathcal{E}}$ are submodular.

We also need some basic definitions from matroid theory. Let $\boldsymbol{\mathcal{M}}$ be a matroid on S with rank function $\boldsymbol{r_{\mathcal{M}}}$. It is well known that $r_{\mathcal{M}}$ is monotone non-decreasing and submodular. Let $\mathsf{Q} \subseteq \mathsf{S}$. We say that Q is *independent* if $r_{\mathcal{M}}(\mathsf{Q}) = |\mathsf{Q}|$. Recall that every subset of an independent set is independent. A maximal independent set in Q is a *base* of Q. $\mathcal{M}$ is called a *free matroid* if each subset of S is independent, that is the only base of $\mathcal{M}$ is S. We define $\textbf{Span}_{\boldsymbol{\mathcal{M}}}\textbf{(Q)} = \{\mathsf{s} \in \mathsf{S} : r_{\mathcal{M}}(\mathsf{Q} \cup \{\mathsf{s}\}) = r_{\mathcal{M}}(\mathsf{Q})\}$. Recall that $\mathrm{Span}_{\mathcal{M}}$ is monotone and if B is a base of Q, then $\mathsf{Q} \subseteq \mathrm{Span}_{\mathcal{M}}(\mathsf{B})$.

A *matroid-rooted vertex-set* is a quadruple $(V, \mathcal{M}, \mathsf{S}, \pi)$ where V is a vertex-set, $\mathcal{M}$ is a matroid on the set $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$ with rank function $r_{\mathcal{M}}$ and π is a map from S to V. In general, π is not injective; different elements of S may be mapped to the same vertex of V. The elements $\{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$ mapped to the vertices of V are called the *roots*. For $X \subseteq V$, we denote by $\boldsymbol{\mathsf{S}_X}$ the set of roots mapped to X by π. We say that π is $\mathcal{M}$-*independent* if S_v is independent in $\mathcal{M}$ for all $v \in V$.

3 Rooted digraphs

Let $\vec{G} = (V, A)$ be a digraph. For a set $X \subseteq V$, we denote by $\boldsymbol{P_A(X)}$ the set of vertices from which X can be reached in $\vec{G}$ and by $\boldsymbol{Q_A(X)}$ the set of vertices that can be reached from X in $\vec{G}$. Note that $X \subseteq P_A(X), X \subseteq Q_A(X)$ and if $u \in P_A(v)$ then $v \in Q_A(u)$. Let $\boldsymbol{R} = \{r_1, \ldots, r_t\}$ be a list of t not necessarily distinct vertices of $\vec{G}$. We call the pair $(\vec{G}, R)$ a *rooted digraph*. For $X \subseteq V$, we define $\boldsymbol{q_A^R(X)}$ as the number of r_i's, which do not belong to X but from which X is reachable in $\vec{G}$, in other words: $q_A^R(X) = |\{i : r_i \notin X, Q_A(r_i) \cap X \neq \emptyset\}|$.

3.1 Packing spanning arborescences

Let $\vec{G} = (V, A)$ be a digraph and r a vertex of $\vec{G}$. A subgraph $\vec{T} = (U, B)$ of $\vec{G}$ is called an r-*arborescence* if $r \in U$, $\rho_B(r) = 0$, $\rho_B(u) = 1$ for all $u \in U \setminus r$ and $\rho_B(X) \geq 1$ for all $X \subseteq V \setminus r, X \cap (U \setminus r) \neq \emptyset$. We mention that this definition is equivalent to the usual definitions of arborescences. This version is convenient for us for later generalizations. Please note that the single vertex r is an r-arborescence. We use later without any reference that any vertex of an r-arborescence can be reached from r. We call $\vec{T}$ a *reachability* r-*arborescence* in $\vec{G}$ if U contains the set $Q_A(r)$ of vertices that can be reached from r in $\vec{G}$ (that is, $U = Q_A(r)$) in other words, if $Q_B(r) = Q_A(r)$. Please note that a reachability r-arborescence exists always for any vertex r in $\vec{G}$. If all the vertices can be reached from r in $\vec{G}$ (that is $Q_A(r) = V$) then a reachability r-arborescence is called *spanning*.

Our starting point is the result of Edmonds [4] on packing of spanning r-arborescences.

Theorem 2 (Edmonds [4]). *There exist k arc-disjoint spanning r-arborescences in a digraph $\vec{G} = (V, A)$ if and only if*

$$\rho_A(X) \geq k \qquad \text{for all non-empty } X \subseteq V \setminus r. \tag{9}$$

Note that (9) is satisfied if and only if each vertex in $\vec{G}$ is reachable from r and the following holds for $R = \{r, \ldots, r\}$ (k times).

$$\rho_A(X) \geq q_A^R(X) \qquad \text{for all } X \subseteq V. \tag{10}$$

In the following we provide lots of generalizations of Theorem 2. The first one considers the case when the roots of the arborescences can be different. Let $(\vec{G}, R)$ be a rooted digraph with $R = \{r_1, \ldots, r_t\}$. A *packing of spanning R-arborescences* is a set $\{\vec{T}_1, \ldots, \vec{T}_t\}$ of pairwise arc-disjoint spanning r_i-arborescences $\vec{T}_i$ in $\vec{G}$.

Theorem 3 (Edmonds [4]). *There exists a packing of spanning R-arborescences in a rooted digraph $(\vec{G}, R)$ if and only if each vertex in $\vec{G}$ is reachable from each element of R and* (10) *holds.*

Note that Theorem 3 reduces to Theorem 2 when $\{r_1, \ldots, r_t\}$ is equal to $\{r, \ldots, r\}$ (k times).

3.2 Packing reachability arborescences

Let $(\vec{G}, R)$ be a rooted digraph with $R = \{r_1, \ldots, r_t\}$. A *packing of reachability R-arborescences* is a set $\{\vec{T}_1, \ldots, \vec{T}_t\}$ of pairwise arc-disjoint reachability r_i-arborescences $\vec{T}_i$ in $\vec{G}$.

Kamiyama, Katoh, Takizawa [9] provided a nice extension of Theorem 3 for packing reachability arborescences.

Theorem 4 (Kamiyama, Katoh, Takizawa [9]). *There exists a packing of reachability R-arborescences in a rooted digraph $(\vec{G}, R)$ if and only if* (10) *holds.*

Note that Theorem 4 reduces to Theorem 3 when each vertex in $\vec{G}$ is reachable from each element of R: in this case a packing of reachability R-arborescences becomes a packing of spanning R-arborescences.

We mention here that Fujishige [8] proposed a seemingly more general result on packing arborescences spanning convex sets, however Cs. Király showed in [11] that it is in fact equivalent to Theorem 4.

4 Matroid-rooted digraphs

A *matroid-rooted digraph*, denoted by $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$, is a digraph $\vec{G}$ on a matroid-rooted vertex-set $(V, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$.

4.1 Packing matroid-rooted arborescences

A *matroid-rooted arborescence* is a pair $(\vec{T}, \mathsf{s})$ where $\vec{T}$ is an r-arborescence for some vertex r and s is one of the elements of S mapped to r. We say that s is the *root* of the matroid-rooted arborescence $(\vec{T}, \mathsf{s})$. A *matroid-based packing of matroid-rooted arborescences* of $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\vec{T}_1, \mathsf{s}_1), \ldots, (\vec{T}_t, \mathsf{s}_t)\}$ of pairwise arc-disjoint matroid-rooted arborescences such that for each $v \in V$, the set of roots s_i of the matroid-rooted arborescences $(\vec{T}_i, \mathsf{s}_i)$ which contain the vertex v forms a base of the matroid $\mathcal{M}$, that is $\{\mathsf{s}_i \in \mathsf{S} : v \in V(\vec{T}_i)\}$ is a base of S. Note that each element s_i of S must be the root of one of the matroid-rooted arborescences $(\vec{T}_i, \mathsf{s}_i)$ in the packing.

Durand de Gevigney, Nguyen, Szigeti [3] provided an extension of Edmonds' theorem in an other direction.

Theorem 5 (Durand de Gevigney, Nguyen, Szigeti [3])**.** *Let $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted digraph. There exists a matroid-based packing of matroid-rooted arborescences in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\rho_A(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \qquad \text{for all non-empty } X \subseteq V. \tag{11}$$

Note that Theorem 5 is an extension of Theorem 3 if $\mathsf{S} = R$, $\mathcal{M}$ is the free matroid on S and each vertex in $\vec{G}$ is reachable from each element of R. In this case, (11) is equivalent to (10) and since $\mathcal{M}$ is the free matroid, each vertex must belong to all the arborescences in the packing, that is a matroid-based packing of matroid-rooted arborescences is a packing of spanning R-arborescences, hence Theorem 5 reduces to Theorem 3.

4.2 Reachability-based packing of matroid-rooted arborescences

Let $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted digraph where $\vec{G} = (V, A)$ and $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$. A *reachability-based packing of matroid-rooted arborescences* of $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\vec{T}_1, \mathsf{s}_1), \ldots, (\vec{T}_t, \mathsf{s}_t)\}$ of pairwise arc-disjoint matroid-rooted arborescences such that for each $v \in V$, the set of roots s_i of the matroid-rooted arborescences $(\vec{T}_i, \mathsf{s}_i)$ which contain the vertex v forms a base of the set of the elements in S which are mapped to the set $P_A(v)$ of vertices from which v is reachable in $\vec{G}$, that is $\{\mathsf{s}_i \in \mathsf{S} : v \in V(\vec{T}_i)\}$ is a base of $\mathsf{S}_{P_A(v)}$.

A common generalization of Theorem 4 and Theorem 5 was given by Cs. Király [11].

Theorem 6 (Cs. Király [11])**.** *Let $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted digraph. There exists a reachability-based packing of matroid-rooted arborescences in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\rho_A(X) \quad \geq \quad r_{\mathcal{M}}(\mathsf{S}_{P_A(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \text{ for all } X \subseteq V. \tag{12}$$

Note that Theorem 6 reduces to Theorem 4 when $\mathsf{S} = \{r_1, \ldots, r_t\}$ and $\mathcal{M}$ is the free matroid on S: in this case (12) is equivalent to (10) and $\{r_i \in \mathsf{S} : v \in V(\vec{T}_i)\}$ must be equal to $\mathsf{S}_{P_A(v)}$, that is if an element v was reachable from r_i in $\vec{G}$ then v must belong to the matroid-rooted arborescence $(\vec{T}_i, r_i)$ and hence a reachability-based packing of matroid-rooted arborescences is a packing of reachability arborescences. Note also that Theorem 6 reduces to Theorem 5 when $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$ satisfies (11): in this case (11) implies that $r_{\mathcal{M}}(\mathsf{S}_{P_A(v)}) = r_{\mathcal{M}}(\mathsf{S})$ for all $v \in V$ and hence (12) is equivalent to (11) and a reachability-based packing of matroid-rooted arborescences is a matroid-based packing of matroid-rooted arborescences.

We generalize Theorem 6 for matroid-rooted dypergraphs in Section 6.2.

5 Rooted dypergraphs

Let $\vec{\mathcal{G}} = (V, \mathcal{A})$ be a dypergraph. We say that a vertex w *can be reached* from a vertex u in $\vec{\mathcal{G}}$ if there exists an alternating sequence $v_1 = u, Z_1, v_2, \ldots, v_i, Z_i, v_{i+1}, \ldots, v_j = w$ of vertices and hyperarcs such that v_i is a tail of Z_i and v_{i+1} is the head of Z_i. For a set $X \subseteq V$, we denote by $\boldsymbol{P_{\mathcal{A}}(X)}$ the set of vertices from which X can be reached in $\vec{\mathcal{G}}$ and by $\boldsymbol{Q_{\mathcal{A}}(X)}$ the set of vertices that can be reached from X in $\vec{\mathcal{G}}$. Note that $P_{\mathcal{A}}(P_{\mathcal{A}}(X)) = P_{\mathcal{A}}(X)$. Let $\boldsymbol{R} = \{r_1, \ldots, r_t\}$ be a list of t not necessarily distinct vertices of $\vec{\mathcal{G}}$. We call the pair $(\vec{\mathcal{G}}, R)$ a *rooted dypergraph.* For $X \subseteq V$, we define $\boldsymbol{q^R_{\mathcal{A}}(X)}$ as the number of r_i's, which do not belong to X but from which X is reachable in $\vec{\mathcal{G}}$, in other words: $q^R_{\mathcal{A}}(X) = |\{i : r_i \notin X, Q_{\mathcal{A}}(r_i) \cap X \neq \emptyset\}|$.

Let $\vec{\mathcal{T}} = (U, \mathcal{B})$ be a subhypergraph of $\vec{\mathcal{G}}$ and U' the set of vertices in U whose in-degree in $\vec{\mathcal{T}}$ is not 0. We say that $\vec{\mathcal{T}}$ is an *r-hyperarborescence* if $r \in U$, $\rho_{\mathcal{B}}(r) = 0$, $\rho_{\mathcal{B}}(u) = 1$ for all $u \in U'$, $\rho_{\mathcal{B}}(X) \geq 1$ for all $X \subseteq V \setminus r, X \cap U' \neq \emptyset$ and each vertex $u \in U \setminus r$ belongs to a hyperarc in $\mathcal{B}$. Please note that in an r-hyperarborescence $\vec{\mathcal{T}}$ the vertices in $U \setminus (U' \cup r)$ can not be reached from r. We mention that $\vec{\mathcal{T}}$ is an r-hyperarborescence if and only if it can be trimmed to an r-arborescence. We call $\vec{\mathcal{T}}$ a *reachability r-hyperarborescence* in $\vec{\mathcal{G}}$ if $U' \cup r$ contains the set $Q_{\mathcal{A}}(r)$ of vertices that can be reached from r in $\vec{\mathcal{G}}$ (that is $U' = Q_{\mathcal{A}}(r) \setminus r$), in other words, if $Q_{\mathcal{B}}(r) = Q_{\mathcal{A}}(r)$. If all the vertices can be reached from r in $\vec{\mathcal{G}}$ (that is $Q_{\mathcal{A}}(r) = V$) then a reachability r-hyperarborescence is called *spanning*. Examples for a spanning hyperarborescence and for a reachability hyperarborescence can be found in Figure 2.

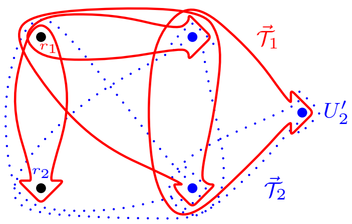

Figure 2: $\vec{\mathcal{T}}_1$ is a spanning r_1-hyperarborescence while $\vec{\mathcal{T}}_2$ is a reachability r_2-hyperarborescence of the dypergraph.

5.1 Packing spanning hyperarborescences

Let $(\vec{\mathcal{G}}, R)$ be a rooted dypergraph with $\vec{\mathcal{G}} = (V, \mathcal{A})$ and $R = \{r_1, \ldots, r_t\}$. A *packing of spanning R-hyperarborescences* is a set $\{\vec{\mathcal{T}}_1, \ldots, \vec{\mathcal{T}}_t\}$ of pairwise arc-disjoint spanning r_i-hyperarborescences $\vec{\mathcal{T}}_i$ in $\vec{\mathcal{G}}$.

Frank, T. Király, Kriesell [7] extended Edmonds' theorem for packing spanning hyperarborescences in dypergraphs. We present here the version with multiple roots. We should also cite here the paper Frank, T. Király, Z. Király [6].

Theorem 7 (Frank, T. Király, Kriesell [7])**.** *There exists a packing of spanning R-hyperarborescences in a rooted dypergraph $(\vec{\mathcal{G}}, R)$ if and only if each vertex in $\vec{\mathcal{G}}$ is reachable from each element of R and*

$$\rho_{\mathcal{A}}(X) \geq q_{\mathcal{A}}^{R}(X) \qquad \text{for all } X \subseteq V. \tag{13}$$

Note that Theorem 7 reduces to Theorem 3 when $\vec{\mathcal{G}}$ is a digraph.

5.2 Packing reachability hyperarborescences

Let $(\vec{\mathcal{G}}, R)$ be a rooted dypergraph with $\vec{\mathcal{G}} = (V, \mathcal{A})$ and $R = \{r_1, \ldots, r_t\}$. A *packing of reachability hyperarborescences* is a set $\{\vec{\mathcal{T}}_1, \ldots, \vec{\mathcal{T}}_t\}$ of pairwise arc-disjoint reachability r_i-hyperarborescences $\vec{\mathcal{T}}_i$ in $\vec{\mathcal{G}}$.

A common generalization of Theorem 7 and Theorem 4 was given by Bérczi, Frank [1].

Theorem 8 (Bérczi, Frank [1])**.** *There exists a packing of reachability hyperarborescences in a rooted dypergraph $(\vec{\mathcal{G}}, R)$ if and only if* (13) *holds.*

Note that Theorem 8 reduces to Theorem 7 when each vertex in $\vec{\mathcal{G}}$ is reachable from each element of R: in this case a packing of reachability R-hyperarborescences becomes a packing of spanning R-hyperarborescences. Note also that Theorem 8 reduces to Theorem 4 when $\vec{\mathcal{G}}$ is a digraph.

We generalize Theorem 8 in Section 6.2.

6 Matroid-rooted dypergraphs

A *matroid-rooted dypergraph*, denoted by $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$, is a dypergraph $\vec{\mathcal{G}} = (V, \mathcal{A})$ on a matroid-rooted vertex-set $(V, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathcal{M}$ is a matroid on $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$ with rank function $r_{\mathcal{M}}$ and π is a map from S to V.

6.1 Packing matroid-rooted hyperarborescences

We generalize the matroid-based packing of matroid-rooted arborescences theorem for matroid-rooted dypergraphs. A *matroid-rooted hyperarborescence* is a triple $(\vec{\mathcal{T}}, r, \mathsf{s})$ where $\vec{\mathcal{T}}$ is an r-hyperarborescence and s is an element of S mapped to r. We say that s is the *root* of the matroid-rooted hyperarborescence $(\vec{\mathcal{T}}, r, \mathsf{s})$. A *matroid-based packing of matroid-rooted hyperarborescences* of $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_t, r_t, \mathsf{s}_t)\}$ of pairwise arc-disjoint matroid-rooted hyperarborescences such that for each $v \in V$, the set of roots s_i of the matroid-rooted hyperarborescences $(\vec{\mathcal{T}}_i, r_i, \mathsf{s}_i)$ in which the vertex v can be reached from r_i forms a base of the matroid $\mathcal{M}$, that is $\{\mathsf{s}_i \in \mathsf{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ is a base of S.

Let us introduce the following function h_1, which is integer-valued and intersecting supermodular and satisfies $h_1(V) = 0 = h_1(\emptyset)$.

$$h_1(X) = \begin{cases} r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) & \text{if } X \neq \emptyset, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Theorem 1, applied for $\vec{\mathcal{H}}_1 = (V, \mathcal{A})$ and h_1, together with Theorem 5 provide the following result. This result can be found in the research project report of Léonard [12] that was written under the supervision of Szigeti.

Theorem 9. *Let $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted dypergraph. There exists a matroid-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\rho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) \qquad \text{for all non-empty } X \subseteq V. \tag{14}$$

Note that Theorem 9 reduces to Theorem 7 when $\mathsf{S} = R$, $\mathcal{M}$ is the free matroid on S and each vertex in $\vec{\mathcal{G}}$ is reachable from each element of R. In this case, (14) is equivalent to (13) and since $\mathcal{M}$ is the free matroid, each vertex must belong to all the hyperarborescences in the packing, that is a matroid-based packing of matroid-rooted hyperarborescences is a packing of spanning R-hyperarborescences. Note also that Theorem 9 reduces to Theorem 5 when $\vec{\mathcal{G}}$ is a digraph.

We generalize Theorem 9 in Section 6.2.

6.2 Reachability-based packing of matroid-rooted hyperarborescences

Let $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted dypergraph where $\vec{\mathcal{G}} = (V, \mathcal{A})$ and $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$. A *reachability-based packing of matroid-rooted hyperarborescences* of $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_t, r_t, \mathsf{s}_t)\}$ of pairwise arc-disjoint matroid-rooted hyperarborescences such that for each $v \in V$, the set B_v of roots s_i of the matroid-rooted hyperarborescences $(\vec{\mathcal{T}}_i, r_i, \mathsf{s}_i)$ in which the vertex v can be reached from r_i forms a base of the set of the elements of S which are mapped to the set $P_{\mathcal{A}}(v)$ of vertices from which v is reachable in $\vec{\mathcal{G}}$, that is $\mathsf{B}_v = \{\mathsf{s}_i \in \mathsf{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ is a base of $\mathsf{S}_{P_{\mathcal{A}}(v)}$.

The following theorem which is the main contribution of the present paper provides a common generalization of Theorems 6, 8 and 9.

Theorem 10. *Let $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted dypergraph where $\vec{\mathcal{G}} = (V, \mathcal{A})$. There exists a reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\rho_{\mathcal{A}}(X) \geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X) \qquad \text{for all } X \subseteq V. \tag{15}$$

Proof. First we prove the necessity. Let $\{(\vec{\mathcal{T}}_1, r_1, \mathsf{s}_1), \dots, (\vec{\mathcal{T}}_t, r_t, \mathsf{s}_t)\}$ be a reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$.

For any $v \in V$, since $\mathsf{S}_v \subseteq \mathsf{B}_v$ and B_v is independent in $\mathcal{M}$, so is S_v, and hence π is $\mathcal{M}$-independent.

Let now $X \subseteq V$ and $\mathsf{B} = \bigcup_{v \in X} \mathsf{B}_v$. Since $\mathrm{Span}_{\mathcal{M}}$ is monotone, B_v is a base of $\mathsf{S}_{P_{\mathcal{A}}(v)}$ and by definition of $P_{\mathcal{A}}(X)$, we have $\mathrm{Span}_{\mathcal{M}}(\mathsf{B}) \supseteq \bigcup_{v \in X} \mathrm{Span}_{\mathcal{M}}(\mathsf{B}_v) \supseteq \bigcup_{v \in X} \mathsf{S}_{P_{\mathcal{A}}(v)} = \mathsf{S}_{P_{\mathcal{A}}(X)}$. Then, since $r_{\mathcal{M}}$ is monotone, $(\star)$ $r_{\mathcal{M}}(\mathsf{B}) \geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)})$.

For each root $\mathsf{s}_i \in \mathsf{B} \setminus \mathsf{S}_X$, there exists a vertex $v \in X$ such that $\mathsf{s}_i \in \mathsf{B}_v$ and then since $\vec{\mathcal{T}}_i$ is an r_i-hyperarborescence and $v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i) \cap X$, there exists an hyperarc of $\vec{\mathcal{T}}_i$ that enters X. Since these matroid-rooted hyperarborescences are arc-disjoint, $r_{\mathcal{M}}$ is submodular and monotone and by $(\star)$, we have

$$\begin{aligned}
\rho_{\mathcal{A}}(X) &\geq |\mathsf{B} \setminus \mathsf{S}_X| \\
&\geq r_{\mathcal{M}}(\mathsf{B} \setminus \mathsf{S}_X) \\
&\geq r_{\mathcal{M}}(\mathsf{B} \cup \mathsf{S}_X) - r_{\mathcal{M}}(\mathsf{S}_X) \\
&\geq r_{\mathcal{M}}(\mathsf{B}) - r_{\mathcal{M}}(\mathsf{S}_X) \\
&\geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)
\end{aligned}$$

that is, (15) is satisfied.

The sufficiency follows from Theorem 6, by applying the trimming proof technique. Let us define the set function $h_{\mathcal{A}}$ as follows: $h_{\mathcal{A}}(X) = r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$ for all $X \subseteq V$. Unfortunately this function is not intersecting supermodular so we cannot apply Theorem 1 directly. On the other hand, as we will see, with an additional condition on the sets X and Y, one has the supermodular inequality for X and Y. The following claim for digraphs was proved by Cs. Király in [11], see also [5]. The same proof shows that it is also true for dypergraphs.

Claim 1 ([11])**.** *If $P_{\mathcal{A}}(Y) \subseteq P_{\mathcal{A}}(X \cap Y)$, then $h_{\mathcal{A}}$ satisfies the supermodular inequality for X and Y.*

Proof. Applying $P_{\mathcal{A}}(X) \subseteq P_{\mathcal{A}}(X \cup Y)$, $P_{\mathcal{A}}(Y) \subseteq P_{\mathcal{A}}(X \cap Y)$ and the monotonicity of $r_{\mathcal{M}}$, we get that $r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}()})$ satisfies the supermodular inequality for X and Y. Then, by the supermodularity of $-r_{\mathcal{M}}$, the claim follows. ■

We say that a set X is *tight* if $\rho_{\mathcal{A}}(X) = h_{\mathcal{A}}(X)$. Note that, for every vertex v, $P_{\mathcal{A}}(v)$ is a tight set, since $\rho_{\mathcal{A}}(P_{\mathcal{A}}(v)) = 0 = h_{\mathcal{A}}(P_{\mathcal{A}}(v))$.

Claim 2. *If X and Y are tight and $P_{\mathcal{A}}(Y) \subseteq P_{\mathcal{A}}(X \cap Y)$ then $X \cap Y$ is tight and $d_{\mathcal{A}}(X, Y) = 0$.*

Proof. By Claim 1, $h_{\mathcal{A}}$ satisfies the supermodular inequality (8) for X and Y, so

$$\begin{aligned}
\rho_{\mathcal{A}}(X) + \rho_{\mathcal{A}}(Y) &= h_{\mathcal{A}}(X) + h_{\mathcal{A}}(Y) \\
&\leq h_{\mathcal{A}}(X \cap Y) + h_{\mathcal{A}}(X \cup Y) \\
&\leq \rho_{\mathcal{A}}(X \cap Y) + \rho_{\mathcal{A}}(X \cup Y) \\
&= \rho_{\mathcal{A}}(X) + \rho_{\mathcal{A}}(Y) - d_{\mathcal{A}}(X, Y) \\
&\leq \rho_{\mathcal{A}}(X) + \rho_{\mathcal{A}}(Y)
\end{aligned}$$

and the claim follows. ■

Lemma 1. *If $\vec{\mathcal{G}}$ satisfies* (15)*, then $\vec{\mathcal{G}}$ can be trimmed to a digraph $\vec{G} = (V, A)$ that satisfies the following conditions.*

$$\rho_A(X) \geq h_A(X) \quad \textit{for all } X \subseteq V, \tag{16}$$

$$r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(v)}) = r_{\mathcal{M}}(\mathsf{S}_{P_A(v)}) \textit{ for all } v \in V. \tag{17}$$

Proof. We prove the lemma by induction on the sum of $|K|-2$ over all hyperarcs (K,k) of $\vec{\mathcal{G}}$. If this sum is zero, we are done. Otherwise, let (Z,z) be a hyperarc of $\vec{\mathcal{G}}$ of size at least three and u one of its tails. Let $\vec{\mathcal{G}}_u = (V, \mathcal{A}_u)$ where $\mathcal{A}_u = (\mathcal{A} \setminus (Z,z)) \cup (Z \setminus u, z)$. Note that $P_{\mathcal{A}_u}(X) \subseteq P_{\mathcal{A}}(X)$.

Claim 3. *If $\vec{\mathcal{G}}_u$ violates one of the following conditions*

$$\begin{aligned} \rho_{\mathcal{A}_u}(X) &\geq h_{\mathcal{A}_u}(X) \quad \textit{for all } X \subseteq V, & (18)\\ r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}_u}(v)}) &= r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(v)}) \ \textit{for all } v \in V, & (19)\end{aligned}$$

then there exists a tight set X_u in $\vec{G}$ such that

$$Z \setminus u \subseteq X_u \subseteq (V \setminus u) \cap P_{\mathcal{A}}(z). \tag{20}$$

Proof. If $\vec{\mathcal{G}}_u$ violates (18), then there exists a set X such that $\rho_{\mathcal{A}_u}(X) + 1 \leq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}_u}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)$. Then, by the monotonicity of $r_{\mathcal{M}}$ and (15), we have

$$\begin{aligned} \rho_{\mathcal{A}}(X) &\leq \rho_{\mathcal{A}_u}(X) + 1\\ &\leq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}_u}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)\\ &\leq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)\\ &\leq \rho_{\mathcal{A}}(X)\end{aligned}$$

so everywhere equality holds.

If $\vec{\mathcal{G}}_u$ violates (19), then there exists a vertex v such that $r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}_u}(v)}) + 1 \leq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(v)})$. Let $X = P_{\mathcal{A}_u}(v)$. Note that $\rho_{\mathcal{A}_u}(X) = 0$ and $P_{\mathcal{A}}(X) = P_{\mathcal{A}}(v)$. Then, by (15),

$$\begin{aligned} 1 &= \rho_{\mathcal{A}_u}(X) + 1\\ &\geq \rho_{\mathcal{A}}(X)\\ &\geq r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(X)}) - r_{\mathcal{M}}(\mathsf{S}_X)\\ &= r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(v)}) - r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}_u}(v)})\\ &\geq 1\end{aligned}$$

so everywhere equality holds.

In both cases it follows that X is tight and that $\rho_{\mathcal{A}_u}(X) + 1 = \rho_{\mathcal{A}}(X)$ that is $Z \setminus u \subseteq X \subseteq V \setminus u$.

Let $Y = P_{\mathcal{A}}(z)$. As mentioned above Y is tight. By $z \in X \cap Y$, we have $P_{\mathcal{A}}(Y) \subseteq P_{\mathcal{A}}(X \cap Y)$. By Claim 2, $X_u = X \cap Y$ is tight. By definition, $Z \setminus u \subseteq Y$, and we know that $Z \setminus u \subseteq X$, so $Z \setminus u \subseteq X_u$. Since $X \subseteq V \setminus u$ and $Y = P_{\mathcal{A}}(z)$, $X_u \subseteq (V \setminus u) \cap P_{\mathcal{A}}(z)$ so X_u satisfies (20) and Claim 3 is proved. ■

If $\vec{\mathcal{G}}_u$ satisfies both (18) and (19) then we are done by induction. So without loss of generality we may assume, by Claim 3, that there exists a tight set X_u that satisfies (20). By assumption, (Z,z) has at least two tails, let w be a tail of (Z,z) distinct of u. The same argument for w gives the set X_w. By (20) and $z \in X_w$, $P_{\mathcal{A}}(X_u) \subseteq P_{\mathcal{A}}(X_u \cap X_w)$. Then, by Claim 2, $d_{\mathcal{A}}(X_u, X_w) = 0$, which contradicts the existence of Z. This contradiction finishes the proof of Lemma 1. ■

Let $\vec{G} = (V, A)$ be the digraph given by Lemma 1. Since π is $\mathcal{M}$-independent and (16) is satisfied, Theorem 6 guarantees the existence of a reachability-based packing of matroid-rooted arborescences in $(\vec{G}, \mathcal{M}, \mathsf{S}, \pi)$. By (17), the corresponding matroid-rooted hyperarborescences form a reachability-based packing of matroid-rooted hyperarborescences in $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ and Theorem 10 is proved. ■

Note that Theorem 10 reduces to Theorem 6 when $\vec{\mathcal{G}}$ is a digraph. Note also that Theorem 10 reduces to Theorem 8 when $\mathsf{S} = \{r_1, \ldots, r_t\}$ and $\mathcal{M}$ is the free matroid on S : in this case $\{r_i \in \mathsf{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}}_i)}(r_i)\}$ must be equal to $\mathsf{S}_{P_{\mathcal{A}}(v)}$, that is if an element v was reachable from r_i in $\vec{G}$ then v must belong to the matroid-rooted hyperarborescence $(\vec{T}_i, r_i, \mathsf{s}_i)$ and hence a reachability-based packing of matroid-rooted hyperarborescences is a packing of reachability hyperarborescences. Finally, note that Theorem 10 reduces to Theorem 9 when $(\vec{\mathcal{G}}, \mathcal{M}, \mathsf{S}, \pi)$ satisfies (14): in this case (14) implies that $r_{\mathcal{M}}(\mathsf{S}_{P_{\mathcal{A}}(v)}) = r_{\mathcal{M}}(\mathsf{S})$ for all $v \in V$ and hence (15) is equivalent to (14) and a reachability-based packing of matroid-rooted hyperarborescences is a matroid-based packing of matroid-rooted hyperarborescences.

7 Rooted mixed graphs

Let $\boldsymbol{R} = \{r_1, \dots, r_t\}$ be a list of t not necessarily distinct vertices of V and $F = (V, E \cup A)$ a mixed graph where E is the set of edges and A is the set of arcs of F. We call the pair (F, R) a *rooted mixed graph*.

7.1 Packing spanning trees

Let $G = (V, E)$ be a graph. A subgraph T of G is called a *tree* if it is connected and it contains no cycle. Recall that T is a tree if and only if for any vertex r of T, T can be oriented to become an r-arborescence.

The undirected counterpart of Theorem 2 is the following result of Nash-Williams [13] and Tutte [14] on packing of spanning trees.

Theorem 11 (Nash-Williams [13], Tutte [14]). *There exist k edge-disjoint spanning trees in a graph $G = (V, E)$ if and only if*

$$e_E(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{21}$$

In this section we provide some generalizations of Theorem 11.

7.2 Packing mixed arborescences

We say that a mixed graph $F = (V, E \cup A)$ is a *mixed r-arborescence* if there exists an orientation of E such that F becomes an r-arborescence.

A common generalization of Theorem 11 and Theorem 2 was given by Frank [5].

Theorem 12 (Frank [5]). *There exist k (edge and arc)-disjoint spanning mixed r-arborescences in a mixed graph $F = (V, E \cup A)$ if and only if*

$$e_E(\mathcal{P}) \geq \sum_1^{\ell}(k - \rho_A(V_i)) \text{ for every partition } \mathcal{P} = \{r \in V_0, V_1, \dots, V_\ell\} \text{ of } V. \tag{22}$$

Note that Theorem 12 reduces to Theorem 11 when $A = \emptyset$ and to Theorem 2 when $E = \emptyset$.

We present here a common generalization of Theorem 12 and Theorem 3. Let (F, R) be a rooted mixed graph with $R = \{r_1, \dots, r_t\}$. A *packing of spanning mixed R-arborescences* is a set $\{T_1, \dots, T_t\}$ of pairwise arc-disjoint spanning mixed r_i-arborescences T_i in F.

Theorem 13. *There exists a packing of spanning mixed R-arborescences in a rooted mixed graph (F, R) where $F = (V, E \cup A)$ if and only if*

$$e_E(\mathcal{P}) \geq \sum_1^{\ell}(|R \setminus V_i| - \rho_A(V_i)) \text{ for every partition } \mathcal{P} = \{V_0, V_1, \dots, V_\ell\} \text{ of } V. \tag{23}$$

Note that Theorem 13 reduces to Theorem 12 when $R = \{r, \dots, r\}$ (k times) and to Theorem 3 when $E = \emptyset$. Theorem 13 is proved in Section 11.

8 Matroid-rooted mixed graphs

A *matroid-rooted graph* (respectively *mixed graph*) is a graph $G = (V, E)$ (respectively is a mixed graph $F = (V, E \cup A)$) on a matroid-rooted vertex-set $(V, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathcal{M}$ is a matroid on $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$ with rank function $r_\mathcal{M}$ and π is a map from S to V.

8.1 Packing matroid-rooted trees

A *matroid-rooted tree* is a pair (T, s) where T is a tree and s is an element of S mapped to a vertex of T. We say that s is the *root* of the matroid-rooted tree (T, s). A *matroid-based packing of matroid-rooted trees* of $(G, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(T_1, \mathsf{s}_1), \ldots, (T_t, \mathsf{s}_t)\}$ of pairwise edge-disjoint matroid-rooted trees such that for each $v \in V$, the set of roots s_i of the matroid-rooted trees $(\vec{T}_i, \mathsf{s}_i)$ which contain the vertex v forms a base of the matroid $\mathcal{M}$, that is $\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ is a base of S.

Kato,Tanigawa [10] provided the following elegant generalization of Tutte's and Nash-Williams' theorem for packing matroid-rooted trees.

Theorem 14 (Kato,Tanigawa [10])**.** *Let $(G, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted graph. There exists a matroid-based packing of matroid-rooted trees in $(G, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$e_E(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (r_\mathcal{M}(\mathsf{S}) - r_\mathcal{M}(\mathsf{S}_X)) \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{24}$$

Note that Theorem 14 reduces to Theorem 11 when $\mathsf{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_k\}$, $\mathcal{M}$ is the free matroid on S and π maps each s_i to r. In this case, (24) is equivalent to (21) and since $\mathcal{M}$ is the free matroid, each vertex must belong to all the matroid-rooted trees in the packing, that is a matroid-based packing of matroid-rooted trees is a packing of spanning trees.

8.2 Packing matroid-rooted mixed arborescences

A *matroid-rooted mixed arborescence* is a pair (T, s) where T is a mixed r-arborescence for some vertex r and s is an element of S mapped to r. We say that s is the *root* of the matroid-rooted mixed arborescence (T, s). A *matroid-based packing of matroid-rooted mixed arborescences* of $(F, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(T_1, \mathsf{s}_1), \ldots, (T_t, \mathsf{s}_t)\}$ of pairwise (edge and arc)-disjoint matroid-rooted mixed arborescences such that for each $v \in V$, the set of roots s_i of the matroid-rooted mixed arborescences $(\vec{T}_i, \mathsf{s}_i)$ which contain the vertex v forms a base of the matroid $\mathcal{M}$, that is $\{\mathsf{s}_i \in \mathsf{S} : v \in V(T_i)\}$ is a base of S.

A common generalization of Theorem 14 and Theorem 5 can be formulated as follows. This result can be found in the research project report of Léonard [12].

Theorem 15. *Let $(F, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted mixed graph where $F = (V, E \cup A)$. There exists a matroid-based packing of matroid-rooted mixed arborescences in $(F, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$e_E(\mathcal{P}) \geq \sum_1^\ell (r_\mathcal{M}(\mathsf{S}) - r_\mathcal{M}(\mathsf{S}_{V_i}) - \rho_A(V_i)) \text{ for every partition } \mathcal{P} = \{V_0, V_1, \ldots, V_\ell\} \text{ of } V. \tag{25}$$

Note that Theorem 15 reduces to Theorem 14 when $A = \emptyset$ and to Theorem 5 when $E = \emptyset$. Note also that Theorem 15 reduces to Theorem 13 when $\mathsf{S} = R$ and $\mathcal{M}$ is the free matroid. In this case, (25) is equivalent to (23) and since $\mathcal{M}$ is the free matroid, each vertex must belong to all the matroid-rooted mixed arborescences in the packing, that is a matroid-based packing of matroid-rooted mixed arborescences is a packing of spanning mixed r-arborescences. Theorem 15 is proved in Section 11.

9 Rooted mixed hypergraphs

Let $\boldsymbol{R} = \{r_1, \dots, r_t\}$ be a list of t not necessarily distinct vertices of V and $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ a mixed hypergraph where $\mathcal{E}$ is the set of hyperedges and $\mathcal{A}$ is the set of hyperarcs of $\mathcal{F}$. We call the pair $(\mathcal{F}, R)$ a *rooted mixed hypergraph*.

9.1 Packing spanning hypertrees

Let $\mathcal{G} = (V, \mathcal{E})$ be a hypergraph. We say that a subhypergraph $\mathcal{G}' = (V', \mathcal{E}')$ of $\mathcal{G}$ (that is $V' \subseteq V$ and $\mathcal{E}' \subseteq \mathcal{E}$) is a *hypertree* of $\mathcal{G}$ if there exists an orientation $\vec{\mathcal{G}'}$ of $\mathcal{G}'$ which is a hyperarborescence. A hypertree is called *spanning* if $|\mathcal{E}'| = |V| - 1$.

Frank, Király, Kriesell [7] provided the following generalization of Tutte's and Nash-Williams' theorem for packing spanning hypertrees, see Theorems 10.5.12 and 9.1.22 of [5].

Theorem 16 (Frank, Király, Kriesell [7]). *There exists a packing of k spanning hypertrees in a hypergraph* $\mathcal{G} = (V, \mathcal{E})$ *if and only if*

$$e_{\mathcal{E}}(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \qquad \textit{for every partition } \mathcal{P} \textit{ of } V. \tag{26}$$

Note that Theorem 16 reduces to Theorem 11 when $\mathcal{G}$ is a graph.

9.2 Packing spanning mixed hyperarborescences

We say that $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ is a *mixed r-hyperarborescence* if there exists an orientation of $\mathcal{E}$ such that $\mathcal{F}$ becomes an r-hyperarborescence.

A common generalization of Theorem 16 and Theorem 7 can be formulated as follows. Let $(\mathcal{F}, R)$ be a rooted mixed hypergraph with $R = \{r_1, \dots, r_t\}$. A *packing of spanning mixed R-hyperarborescences* is a set $\{T_1, \dots, T_t\}$ of pairwise (edge and arc)-disjoint spanning mixed r_i-hyperarborescences T_i in $\mathcal{F}$.

Theorem 17. *There exists a packing of spanning mixed R-hyperarborescences in a rooted mixed hypergraph* $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ *if and only if*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{1}^{\ell}(|R \setminus V_i| - \rho_{\mathcal{A}}(V_i)) \textit{ for every partition } \mathcal{P} = \{V_0, V_1, \dots, V_\ell\} \textit{ of } V. \tag{27}$$

Note that Theorem 17 reduces to Theorem 16 when $\mathcal{A} = \emptyset$, to Theorem 7 when $\mathcal{E} = \emptyset$ and to Theorem 13 when $\mathcal{F}$ is a mixed graph. Theorem 17 is proved in Section 11.

10 Matroid-rooted mixed hypergraphs

A *matroid-rooted hypergraph* (respectively *mixed hypergraph*) is a hypergraph $\mathcal{G} = (V, \mathcal{E})$ (respectively is a mixed hypergraph $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$) on a matroid-rooted vertex-set $(V, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathcal{M}$ is a matroid on $\mathsf{S} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$ with rank function $r_{\mathcal{M}}$ and π is a map from S to V.

10.1 Packing matroid-rooted hypertrees

A *matroid-rooted hypertree* is a triple $(\mathcal{T}, r, \mathsf{s})$ where $\mathcal{T}$ has an orientation $\vec{\mathcal{T}}$ that is an r-hyperarborescence and s is an element of S mapped to r. We say that s is the *root* of the matroid-rooted hypertree $(\mathcal{T}, r, \mathsf{s})$. A *matroid-based packing of matroid-rooted hypertrees* of $(\mathcal{G}, \mathcal{M}, \mathsf{S}, \pi)$ is a set $\{(\mathcal{T}_1, r_1, \mathsf{s}_1), \dots, (\mathcal{T}_t, r_t, \mathsf{s}_t)\}$ of pairwise edge-disjoint matroid-rooted hypertrees such that for each $v \in V$, the set of roots s_i of the matroid-rooted hyperarborescences $(\vec{T_i}, r_i, \mathsf{s}_i)$ in which the vertex v can be reached from r_i forms a base of the matroid $\mathcal{M}$, that is $\{\mathsf{s}_i \in \mathsf{S} : v \in Q_{\mathcal{A}(\vec{\mathcal{T}_i})}(r_i)\}$ is a base of S.

Theorem 14 can be generalized as follows.

Theorem 18. *Let $(\mathcal{G}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted hypergraph. There exists a matroid-based packing of matroid-rooted hypertrees in $(\mathcal{G}, \mathcal{M}, \mathsf{S}, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} (r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X)) \qquad \text{for every partition } \mathcal{P} \text{ of } V. \tag{28}$$

Note that Theorem 18 reduces to Theorem 14 when $\mathcal{G}$ is a graph.

10.2 Packing spanning matroid-rooted mixed hyperarborescences

A common generalization of Theorems 9, 18, 17 and 15 can be formulated as follows.

Theorem 19. *There exists a matroid-based packing of matroid-rooted mixed hyperarborescences in a matroid-rooted mixed hypergraph $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ where $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ if and only if π is $\mathcal{M}$-independent and*

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{1}^{\ell} (r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_{V_i}) - \rho_{\mathcal{A}}(V_i)) \qquad \text{for every partition } \mathcal{P} = \{V_0, V_1, \dots, V_\ell\} \text{ of } V. \tag{29}$$

Note that Theorem 19 reduces to Theorem 9 when $\mathcal{E} = \emptyset$, to Theorem 18 when $\mathcal{A} = \emptyset$, to Theorem 17 when $\mathsf{S} = R$, $\mathcal{M}$ is the free matroid on S, and to Theorem 15 when $\mathcal{F}$ is a mixed graph. Theorem 19 is proved in Section 11.

11 General orientation result on hypergraphs

The results on rooted mixed graphs, matroid-rooted mixed graphs, rooted mixed hypergraphs and matroid-rooted mixed hypergraphs can be proved by applying the following general orientation result on hypergraphs. The proof of Theorem 15.4.13 (the corresponding result for graphs) given in [5], with the necessary straightforward modifications, can be extended for hypergraphs. For the sake of completeness we provide the proof here. We mention that this result can also be obtained by using the techniques from [6].

Theorem 20. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and h an integer-valued, intersecting supermodular function (with possible negative values) such that $h(V) = 0$. There exists an orientation of $\mathcal{H}$ that covers h if and only if $\mathcal{H}$ is h-subpartition-connected.*

Proof. If such an orientation exists then, for every partition $\mathcal{P} = \{V_0, V_1, \dots, V_\ell\}$, we have by (6),

$$e_{\mathcal{E}}(\mathcal{P}) \geq \sum_{1}^{\ell} \rho_{\mathcal{A}}(V_i) \geq \sum_{1}^{\ell} h(V_i),$$

so (3) is satisfied.

To prove the sufficiency, let us suppose that (3) is satisfied. Let us introduce the following two integer-valued set functions.

$$b(X) = j_{\mathcal{E}}(X) \tag{30}$$
$$p(X) = h(X) + i_{\mathcal{E}}(X). \tag{31}$$

Since $j_{\mathcal{E}}$ is submodular, h is intersecting supermodular and $i_{\mathcal{E}}$ is supermodular, it follows that b is submodular and p is intersecting supermodular. Note that, by (30), (31) and $h(V) = 0$,

$$b(V) = j_{\mathcal{E}}(V) = |\mathcal{E}| \tag{32}$$
$$p(V) = h(V) + i_{\mathcal{E}}(V) = |\mathcal{E}|. \tag{33}$$

For every $Z \subseteq V$ and every partition $\{Z_1, \dots, Z_\ell\}$ of Z and $Z_0 = V \setminus Z$, we have, by (31), (3), (2), (1) and (30),

$$\begin{aligned}
\sum_1^\ell p(Z_i) &= \sum_1^\ell h(Z_i) + \sum_1^\ell i_{\mathcal{E}}(Z_i) \\
&\leq e_{\mathcal{E}}(\{Z_0, Z_1, \dots, Z_\ell\}) + \sum_1^\ell i_{\mathcal{E}}(Z_i) \\
&= j_{\mathcal{E}}(Z) \\
&= b(Z).
\end{aligned}$$

Then, by Theorem 12.2.2 in [5], there exists an integral vector m on V such that

$$p(X) \leq m(X) \leq b(X) \qquad \text{for every } X \subseteq V. \tag{34}$$

Inequalities (32), (33) and (34) provide

$$m(V) = |\mathcal{E}|. \tag{35}$$

By (35) and (7), we have

$$|\mathcal{E}| = m(V) = m(X) + m(V \setminus X) \qquad \text{for every } X \subseteq V. \tag{36}$$

By (30) and (34), we have

$$j_{\mathcal{E}}(V \setminus X) = b(V \setminus X) \geq m(V \setminus X) \qquad \text{for every } X \subseteq V, \tag{37}$$

and hence, by (1), (36) and (37),

$$(0 \leq)\ i_{\mathcal{E}}(X) \leq m(X) \qquad \text{for every } X \subseteq V. \tag{38}$$

Then, by (35), (38) and Theorem 9.4.2 in [5], there exists an orientation $\vec{\mathcal{H}} = (V, \mathcal{A})$ of $\mathcal{H}$ such that

$$\rho_{\mathcal{A}}(v) = m(v). \tag{39}$$

Then, for every $X \subseteq V$, we have by (39), (34) and (31),

$$\begin{aligned}
\rho_{\mathcal{A}}(X) &= \sum_{v \in X} \rho_{\mathcal{A}}(v) - i_{\mathcal{A}}(X) \\
&= \sum_{v \in X} m(v) - i_{\mathcal{E}}(X) \\
&= m(X) - i_{\mathcal{E}}(X) \\
&\geq p(X) - i_{\mathcal{E}}(X) \\
&= h(X)
\end{aligned}$$

that is $\vec{\mathcal{H}}$ covers h. ■

11.1 Orientations in rooted mixed graphs

Let (F, R) be a rooted mixed graph with $F = (V, E \cup A)$ and $R = \{r_1, \dots, r_t\}$. Let us introduce the following function h_2, which is integer-valued and intersecting supermodular and satisfies $h_2(V) = 0$.

$$h_2(X) = \begin{cases} |R \setminus X| - \rho_A(X) & \text{if } \emptyset \neq X \subseteq V, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Theorem 20, applied for $\mathcal{H}_2 = (V, E)$ and h_2, provides the following result.

Theorem 21. *There exists an orientation $\vec{F}$ of a rooted mixed graph (F, R) such that each vertex in $\vec{F}$ is reachable from each element of R and* (10) *holds if and only if* (23) *is satisfied.*

Note that, by Theorem 2, Theorem 13 and Theorem 21 are equivalent.

11.2 Orientations in matroid-rooted mixed graphs

Let $(F, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted mixed graph where $F = (V, E \cup A)$. Let us introduce the following function h_3, which is integer-valued and intersecting supermodular and satisfies $h_3(V) = 0$.

$$h_3(X) = \begin{cases} r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) - \rho_A(X) & \text{if } X \neq \emptyset, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Theorem 20, applied for $\mathcal{H}_3 = (V, E)$ and h_3, provides the following result.

Theorem 22. *There exists an orientation of a matroid-rooted mixed graph* $(F, \mathcal{M}, \mathsf{S}, \pi)$ *satisfying* (11) *if and only if* (25) *is satisfied.*

Note that Theorem 22 and Theorem 5 provide Theorem 15.

Note also that Theorem 22 reduces to Theorem 21 when $\mathsf{S} = R$ and $\mathcal{M}$ is the free matroid on S.

11.3 Orientations in rooted mixed hypergraphs

Let $(\mathcal{F}, R)$ be a rooted mixed hypergraph with $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$ and $R = \{r_1, \ldots, r_t\}$. Let us introduce the following function h_4, which is integer-valued and intersecting supermodular and satisfies $h_4(V) = 0$.

$$h_4(X) = \begin{cases} |R \setminus X| - \rho_{\mathcal{A}}(X) & \text{if } \emptyset \neq X \subseteq V, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Theorem 20, applied for $\mathcal{H}_4 = (V, \mathcal{E})$ and h_4, provides the following result.

Theorem 23. *There exists an orientation* $\vec{\mathcal{F}}$ *of a rooted mixed hypergraph* $(\mathcal{F}, R)$ *such that each vertex in* $\vec{\mathcal{F}}$ *is reachable from each element of* R *and* (13) *holds if and only if* (27) *is satisfied.*

Note that, by Theorem 7, Theorem 17 and Theorem 23 are equivalent.

Note that Theorem 23 reduces to Theorem 21 when $\mathcal{F}$ is a mixed graph.

11.4 Orientations in matroid-rooted mixed hypergraphs

Let $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ be a matroid-rooted mixed hypergraph where $\mathcal{F} = (V, \mathcal{E} \cup \mathcal{A})$. Let us introduce the following function h_5, which is integer-valued and intersecting supermodular and satisfies $h_5(V) = 0$.

$$h_5(X) = \begin{cases} r_{\mathcal{M}}(\mathsf{S}) - r_{\mathcal{M}}(\mathsf{S}_X) - \rho_{\mathcal{A}}(X) & \text{if } X \neq \emptyset, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

Theorem 20, applied for $\mathcal{H}_5 = (V, \mathcal{E})$ and h_5, provides the following result, which is a generalization of both Theorem 23 and Theorem 22.

Theorem 24. *There exists an orientation of a matroid-rooted mixed hypergraph* $(\mathcal{F}, \mathcal{M}, \mathsf{S}, \pi)$ *satisfying* (14) *if and only if* (29) *is satisfied.*

Note that Theorem 24 and Theorem 9 provide Theorem 19.

Note also that Theorem 24 reduces to Theorem 23 when $\mathsf{S} = R$ and $\mathcal{M}$ is the free matroid on S.

Finally, note that Theorem 24 reduces to Theorem 22 when $\mathcal{F}$ is a mixed graph.

12 Further remarks

We finish this paper by mentioning some remarks on other possible generalizations.

12.1 Packing reachability mixed arborescences

The first problem is about packing reachability mixed arborescences. We just mention the orientation version of the problem. Let $F = (V, E \cup A)$ be a mixed graph and $\{r_1, \ldots, r_t\}$ a list of t not necessarily distinct vertices of F. For a set $X \subseteq V$, we denote by $\boldsymbol{Q_{E\cup A}(X)}$ the set of vertices that can be reached from X in F and by $\boldsymbol{q_{E\cup A}(X)}$ the number of indices i such that $r_i \notin X$ and $Q_{E\cup A}(r_i) \cap X \neq \emptyset$. When does there exist an orientation $\vec{E}$ of E such that $(V, \vec{E} \cup A)$ covers $q_{E\cup A}$? Let us consider the following two conditions that are clearly necessary: for every partition $\mathcal{P} = \{V_0, V_1, \ldots, V_\ell\}$ of V,

$$e_E(\mathcal{P}) \geq \sum_1^\ell (q_{E\cup A}(V_i) - \rho_A(V_i)), \tag{40}$$

$$e_E(\mathcal{P}) \geq \sum_1^\ell (q_{E\cup A}(V \setminus V_i) - \rho_A(V \setminus V_i)). \tag{41}$$

The following example shows that conditions (40) and (41) are not sufficient. Let $F = (V, E \cup A)$ and $\{r_1, \ldots, r_t\}$ be defined as follows. $V = \{a, b, c, d\}$, $E = \{ab\}$, $A = \{ca, cb, ad, bd\}$, $r_1 = a$ and $r_2 = b$. It is easy to check that (40) and (41) are satisfied. However, the required orientation does not exist since the edge ab should be oriented in both directions.

12.2 Covering intersecting bi-set families under matroid constraints in dypergraphs

Finally, we mention that Bérczi, T. Király, Kobayashi [2] have provided an abstract result on covering intersecting bi-set families under matroid constraints that generalizes Theorem 6 and another result of Bérczi and Frank [1]. We do not want to go into details, we just mention that their proof also works for dypergraphs.

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