On a min-max theorem on bipartite graphs

Zoltán Szigeti*

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Abstract

Frank, Sebő and Tardos [4] proved that for any connected bipartite graft (G, T), the minimum size of a T-join is equal to the maximum value of a partition of A, where A is one of the two colour classes of G. Their proof consists of constructing a partition of A of value $|F|$, by using a minimum T-join F. That proof depends heavily on the properties of distances in graphs with conservative weightings. We follow the dual approach, that is starting from a partition of A of maximum value k, we construct a T-join of size k. Our proof relies only on Tutte's theorem on perfect matchings.

It is known [5] that the results of Lovász on 2-packing of T-cuts, of Seymour on packing of T-cuts in bipartite graphs and in grafts that cannot be T-contracted onto $(K_4, V(K_4))$, and of Sebő on packing of T-borders are implied by this theorem of Frank et al.

The main contribution of the present paper is that all of these results can be derived from Tutte's theorem.

1 Introduction

This paper concerns matchings and T-joins. Since T-joins are generalizations of matching, the minimum weight T-join problem contains the minimum weight perfect matching problem. On the other hand, Edmonds and Johnson [2] showed that the former problem can be reduced to the latter one. Thus, these problems are - in fact - equivalent.

In matching theory lots of min-max results are known. Concerning matchings, in fact, we shall consider Tutte's theorem [11] on the existence of perfect matchings in general graphs, and not the min-max version, the Tutte-Berge formula. Concerning T-joins, we mention the following min-max theorems: The results of Edmonds-Johnson [2], Lovász [7] on 2-packing of T-cuts, of Seymour [9], [10] on packing of T-cuts in bipartite graphs and in grafts that cannot be T-contracted onto $(K_4, V(K_4))$, of Sebő [8] on packing of T-borders and a generalization of Seymour's theorem due to Frank, Sebő and Tardos [4]. (For the definitions and the theorems see [3] or [5].) There are some easy known implications between these results, some others can be found in [5], where we showed that the result of Frank et al. [4] implies all of these results, including the Tutte theorem.

Our aim in this paper is to demonstrate a new (surprising) implication, namely, Tutte's theorem implies the result of Frank et al. [4], and consequently, all of these min-max results can be derived from Tutte's theorem.

2 Definitions, notation

In this paper $H = (V, E)$ denotes a graph where V is the set of vertices and E is the set of edges. $G = (A, B; E)$ denotes always a bipartite connected graph and $T \subseteq A \cup B$ a subset of

*Equipe Combinatoire, Université Paris 6, 75252 Paris, Cedex 05, France.

vertices of even cardinality. The pair (G,T) is called a bipartite graft. An edge set $F \subseteq E$ is a **T-join** if $T = \{v \in A \cup B : d_F(v) \text{ is odd}\}$. The minimum size of a T-join is denoted by $\boldsymbol{\tau(G,T)}$. We mention that a bipartite graft (G,T) contains always a T-join.

For a bipartite graft $(G=(A,B;E),T)$ let us introduce an auxiliary graph $G_A := (T, E_A)$ on the vertex set T, where for $u,v \in T$, $uv \in E_A$ if at least one of u and v belongs to A and there exists a path in G connecting u and v of length one or two.

Let K be a vertex set in G. Then $\delta(K)$ denotes the set of edges connecting K and $(A \cup B) - K$. $G[K]$ denotes the subgraph induced by K. b_K^T is defined to be 0 or 1 depending on the parity of $|T \cap K|$. K is called **T-odd** if $b_K^T = 1$ and **T-even** if $b_K^T = 0$. For a subgraph K of G, $\overline{K} = G[V(G) - V(K)]$.

We shall need the following operation applied for grafts. For a connected subgraph K of G, by **T-contracting** K we mean the graft (G',T') obtained from (G,T) where $G' = G/K$ (that is K is contracted into one vertex v_K) and $T' = T - V(K)$ if $b_K^T = 0$ and $T' = T - V(K) + \{v_K\}$ if $b_K^T = 1$.

In what follows a **component** of a graph means a connected component. For $X \subseteq V(G)$, $\mathcal{K}(G-X)$ denotes the set of components of $G-X$ and $\mathcal{K}_T(G-X)$ denotes the set of T-odd components of $G-X$. Let $q_T(G-X) = |\mathcal{K}_T(G-X)|$.

We denote by $\mathcal{P}_A := \{u : u \in A\}$ the partition of A where the elements of $\mathcal{P}_A$ are the vertices in A as singletons. The value of a (sub)partition $\mathcal{P} = \{A_1, \dots, A_k\}$ of A is defined to be

$$val(\mathcal{P}) = \sum \{q_T(G-A_i) : A_i \in \mathcal{P}\}, \tag{1}$$

in other words,

$$val(\mathcal{P}) = \sum \{b_K^T : K \in \bigcup_{A_i \in \mathcal{P}} \mathcal{K}(G-A_i)\}. \tag{2}$$

The theorem of Frank et al. [4] that generalizes all the min-max results mentioned in the Introduction is as follows.

Theorem 1 (Frank, Sebő, Tardos) *If (G,T) is a bipartite graft with $G=(A,B;E)$, then*

$$\tau(G,T) = \max\{val(\mathcal{P}) : \mathcal{P} \textit{ is a partition of } A\}. \tag{3}$$

In order to be able to prove Theorem 1 by induction we will have to prove a slightly stronger result than Theorem 1. To present it we need some definitions. An edge set C of a connected graph G is called **bicut** if $G-C$ has exactly two connected components. Note that each edge of a tree is a bicut. Let $\mathcal{P} = \{A_1, \dots, A_k\}$ be a partition of A and let $\mathcal{Q} = \{B_1, \dots, B_l\}$ be a partition of B. Then $\mathcal{P} \cup \mathcal{Q}$ is called a **bi-partition** of $A \cup B$ in G. Let us denote by $\boldsymbol{G/(\mathcal{P} \cup \mathcal{Q})}$ the bipartite graph obtained from G by identifying the vertices in R for every member $R \in \mathcal{P} \cup \mathcal{Q}$ and by taking the underlying simple graph. A bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ is called **admissible** if

(i) $F := G/(\mathcal{P} \cup \mathcal{Q})$ is a tree, and

(ii) for each edge e of F, the edge set of G that corresponds to e forms a bicut of G.

By Claim 4, for any bipartite graft there exists an admissible bi-partition.

Theorem 2 *If (G,T) is a bipartite graft with $G=(A,B;E)$, then*

$$\tau(G,T) = \max\{val(\mathcal{P}) : \mathcal{P} \cup \mathcal{Q} \textit{ is an admissible bi-partition of } A \cup B\}. \tag{4}$$

The proof of Frank et al. [4] for Theorem 1 consists of constructing a partition of A of value $|F|$, by using a minimum T-join F. That proof depends heavily on the properties of distances in graphs with conservative weightings. We follow the dual approach, that is starting from a bi-partition of $A \cup B$ of maximum value k, we construct a T-join of size k. Our proof applies induction. Taking a special optimal admissible bi-partition either we can use induction for some contracted graphs (and here we need admissibility of the bi-partition) or we can apply Tutte's theorem on perfect matchings, namely a graph H has a perfect matching if and only if $q_V(H-X) \leq |X|$ for every vertex set X of $V(H)$.

We must mention two papers on this topic. Kostochka [6] and Ageev and Kostochka [1] proved results similar to Theorem 2. Their proof technique is different from the present one.

3 Preliminary results

Claim 3 *Let $(G=(A,B;E),T)$ be a bipartite graft.*

(a) Then the bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ satisfies (i) where $\mathcal{P} := \{a : a \in A\}$ and $\mathcal{Q} := \{B\}$.

(b) If $X \subseteq A$, then the bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ satisfies (i) where $\mathcal{P} := \{a : a \in A - X\} \cup \{X\}$ and $\mathcal{Q} := \{K \cap B : K \in \mathcal{K}(G-X)\}$.

□

The following claim (whose proof is left for the reader) shows that for any bipartite graft there exists an admissible bi-partition.

Claim 4 *Let $(G=(A,B;E),T)$ be a bipartite graft.*

(a) If there is no cut vertex in A then $\mathcal{P} \cup \mathcal{Q}$ is an admissible bi-partition of $A \cup B$, where $\mathcal{P} := \{a : a \in A\}$ and $\mathcal{Q} := \{B\}$.

(b) If there is a cut vertex $v \in A$, that is G can be decomposed into two connected bipartite subgraphs $G_1 = (A_1, B_1; E_1)$ and $G_2 = (A_2, B_2; E_2)$ with exactly one vertex in common, namely v, then let us denote by (G_1, T_1) and (G_2, T_2) the two grafts obtained from (G,T) by T-contracting $V(G_2)$ and $V(G_1)$. If for $i = 1, 2$, $\mathcal{P}_i \cup \mathcal{Q}_i$ is an admissible bi-partition of $A_i \cup B_i$ and $v \in A'_i$ then $\mathcal{P} \cup \mathcal{Q}$ is an admissible bi-partition of $A \cup B$, where $\mathcal{P} := (\mathcal{P}_1 - A'_1) \cup (\mathcal{P}_2 - A'_2) \cup \{A'_1 \cup A'_2\}$ and $\mathcal{Q} := \mathcal{Q}_1 \cup \mathcal{Q}_2$.

□

The definition of an admissible bi-partition implies at once the following claim.

Claim 5 *Let $\mathcal{P} \cup \mathcal{Q}$ be an admissible bi-partition of $A \cup B$.*

(a) $K \in \mathcal{K}_T(G - A_i)$ for some $A_i \in \mathcal{P}$ if and only if $\overline{K} \in \mathcal{K}_T(G - B_j)$ for some $B_j \in \mathcal{Q}$.

(b) $val(\mathcal{P}) = val(\mathcal{Q})$.

□

Claim 6 *Let $\mathcal{P}$ be a partition of A and F a T-join in a bipartite graft $(G=(A,B;E),T)$.*

(a) Then $val(\mathcal{P}) \leq |F|$.

(b) Moreover, if $val(\mathcal{P}) = |F|$, then for every component K of $G - A_i$ for any $A_i \in \mathcal{P}$, $|\delta(K) \cap F| = b_K^T$.

Proof. Let $\mathcal{R} := \bigcup_{A_i \in \mathcal{P}} \mathcal{K}(G - A_i)$. By parity, for each $K \in \mathcal{R}$,

$$b_K^T \leq |\delta(K) \cap F|.$$

Since for $K_1, K_2 \in \mathcal{R}, \delta(K_1) \cap \delta(K_2) = \emptyset$, we have

$$val(\mathcal{P}) = \sum_{K \in \mathcal{R}} b_K^T \leq \sum_{K \in \mathcal{R}} |\delta(K) \cap F| \leq |F|.$$

□

Claim 7 *For every partition $\mathcal{P}$ of A in a bipartite graft $(G=(A,B;E),T)$,*

$$val(\mathcal{P}) \equiv |T \cap A| \pmod 2.$$

Proof. Since $|T|$ is even, for each $A_i \in \mathcal{P}, q_T(G - A_i) \equiv |T \cap A_i| \pmod 2$. Thus

$$val(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} q_T(G - A_i) \equiv \sum_{A_i \in \mathcal{P}} |T \cap A_i| = |T \cap A|.$$

□

We shall deal with some bi-partitions along the proofs. The admissibility of these bi-partitions can always be easily verified. The following easy fact may be useful.

Claim 8 *Let X be a subset of vertices of a connected graph H. Let K be a component of $H-X$. If X is contained in one of the components of $H-K$, then $H-K$ is connected.* □

Claim 9 *Let H be a connected graph with $|V(H)|$ even. If X is a minimal vertex set with $q_V(H-X) > |X|$, then for every component K of $H-X$, $H-K$ is connected.*

Proof. By assumption, using the usual parity argument, $q_V(H-X) \geq |X|+2$. Let K be any component of $H-X$. Then at least one component N of $H-K$ contains more odd components of $H-X$ than vertices in X, that is $q_V(H-(N \cap X)) > |N \cap X|$. Then, by the minimality of X, $N \cap X = X$, that is, by Claim 8, $H-K$ is connected. □

Claim 10 *Let $(G=(A,B;E),T)$ be a bipartite graft. If the auxiliary graph G_A has a perfect matching M then G contains a T-join of cardinality $|T \cap A|$.*

Proof. For every edge $uv \in M$ there exists an (u,v)-path in G of length at most two. Since M is a matching these paths are edge disjoint. The union F of these paths is a T-join of G because M covers all the vertices of T. By construction, $|F| = |T \cap A|$. □

4 The proof of Theorem 2

Let (G,T) be a counterexample with minimum number of vertices in G. By Claim 6(a), for any admissible bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$, $val(\mathcal{P}) \leq \tau(G,T)$, so $val(\mathcal{P}) < \tau(G,T)$.

Lemma 11 *G is 2-connected.*

Proof. Suppose that G contains a cut vertex v, by symmetry we may suppose that $v \in A$. We use the notation of Claim 4. For $i=1,2$, (G_i,T_i) is a bipartite graft and $|A_i \cup B_i| < |A \cup B|$ so there exists an admissible bi-partition $\mathcal{P}_i \cup \mathcal{Q}_i$ of $A_i \cup B_i$ with

$$\tau(G_i,T_i) = val(\mathcal{P}_i). \tag{5}$$

Clearly,

$$\tau(G,T) = \tau(G_1,T_1) + \tau(G_2,T_2). \tag{6}$$

Let $\mathcal{P} \cup \mathcal{Q}$ be the admissible bi-partition of $A \cup B$ defined in Claim 4(b). Note that

$$val(\mathcal{P}) = val(\mathcal{P}_1) + val(\mathcal{P}_2). \tag{7}$$

Then, by (6), (5) and (7), $\tau(G,T) = val(\mathcal{P})$ showing that (G,T) is not a counterexample. □

Let us denote by MAX the maximum value of an admissible bi-partition of $A \cup B$. Observe that MAX$\geq |T \cap A|$ and MAX$\geq |T \cap B|$. The first comes from the admissible bi-partition $\mathcal{P} = \{v : v \in A\}, \mathcal{Q} = \{B\}$, the other one from $\mathcal{P} = \{A\}, \mathcal{Q} = \{v : v \in B\}$. These bi-partitions are admissible by Claim 4(a).

CASE 1. First suppose that MAX$= |T \cap A|$ (or MAX$= |T \cap B|$).

Lemma 12 *If the auxiliary graph G_A has no perfect matching then there exists an admissible bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ with $val(\mathcal{P}) > |T \cap A|$.*

Proof. By Tutte's Theorem, there exists a set $X \subset T$ so that $q_T(G_A - X) > |X|$. Let us take a minimal such set.

We claim that $X \cap B = \emptyset$. Suppose that $a \in X \cap B$. Suppose that a is connected to two odd components K_1 and K_2 of $G_A - X$. Then, by the definition of G_A, there is an edge between K_1 and K_2, that is they cannot be different components of $G_A - X$. Thus a is connected to at most one odd component of $G_A - X$. Hence $q_T(G_A - (X - a)) \geq q_T(G_A - X) - 1 \geq |X| > |X - a|$, contradicting the minimality of X.

Let us denote by B_1 the set of vertices in $B - T$ that has at least one neighbour in $A \cap T$ and let $B_2 := B - T - B_1$. Let $G_1 := G[T \cup B_1]$ and $G_2 := G[(A - T) \cup B_2]$. Note that by the definition of G_A there is a bijection between the components of $G_A - X$ and the components of $G_1 - X$ different from isolated vertices in B_1. Moreover, the T parity of the corresponding components are the same. Let $\mathcal{R} = \mathcal{K}(G_2)$. Note that if $R \in \mathcal{R}$ then there is no edge between $R \cap B_2$ and $A \cap T$. We distinguish two cases.

Case I. First suppose that $X = \emptyset$, that is $q_T(G_1) \geq 1$, in other words $q_T(G - (A - T)) \geq 1$. Let $\mathcal{R}_1 \subseteq \mathcal{R}$ be a minimal subset of $\mathcal{R}$ so that $q_T(G - A') \geq 1$, where $A' := \bigcup\{R \cap A : R \in \mathcal{R}_1\}$. Let $\mathcal{P} = \{u : u \in A - A'\} \cup \{A'\}$ and let $\mathcal{Q} = \{R \cap B : R \in \mathcal{K}(G - A')\}$. By Claim 3(b), $\mathcal{P} \cup \mathcal{Q}$ satisfies (i). Since $A' \subseteq A - T$, $|(V(G) - A') \cap T|$ is even so $q_T(G - A') \geq 2$ and, by the minimality of $\mathcal{R}_1$, each such component has at least one neighbour in every $R \in \mathcal{R}_1$. Since G is 2-connected and for every $R \in \mathcal{R}_1, G[R]$ is connected, it follows that for every $D \in \mathcal{K}(G - A')$, $G - D$ is connected, that is (ii.) is also satisfied, so $\mathcal{P} \cup \mathcal{Q}$ is an admissible bi-partition and

$$val(\mathcal{P}) = \sum_{A_i \in \mathcal{P}} q_T(G - A_i) \geq \sum_{t \in A - A'} b_t^T + q_T(G - A') \geq |T \cap A| + 2.$$

Case II. Secondly suppose that $X \neq \emptyset$. By the minimality of X, $X \subset V(G')$ where $G' \in \mathcal{K}(G_1)$. Let $\mathcal{R}_1 \subseteq \mathcal{R}$ be a minimal subset of $\mathcal{R}$ so that all the components of $G' - X$ rest in different components of $G - A'' - X$, where $A'' := \bigcup\{R \cap A : R \in \mathcal{R}_1\}$. Let $\mathcal{P} := \{X \cup A''\} \cup \{u : u \in A - (X \cup A'')\}$ and let $\mathcal{Q} = \{R \cap B : R \in \mathcal{K}(G - X - A'')\}$. By Claim 3(b), $\mathcal{P} \cup \mathcal{Q}$ satisfies (i). For each $R \in \mathcal{R}_1$, $G[R]$ is connected and, by the minimality of $\mathcal{R}_1$, R has neighbours in at least two different components of $G - X - A''$. Moreover, by Claim 9, for each $K \in \mathcal{K}(G' - X)$, $G' - K$ is connected, hence $(G - \bigcup\{R : R \in \mathcal{R}_1\}) - K'$ is connected, where $K' \in \mathcal{K}(G - X - A'')$ that contains K. It follows that $X \cup A''$ is contained in one of the components of $G - K'$. Thus, by Claim 8 and by 2-connectivity, $\mathcal{P} \cup \mathcal{Q}$ is an admissible bi-partition of $A \cup B$ and

$$\begin{aligned} val(\mathcal{P}) &= \sum_{A_i \in \mathcal{P}} q_T(G - A_i) = \sum_{t \in A - X - A''} b_t^T + q_T(G - (X \cup A'')) \\ &= |A \cap T| - |X| + q_T(G_A - X) > |T \cap A|. \end{aligned}$$

□

By Lemma 12, G_A (G_B, resp.) has a perfect matching and thus, by Claim 10, G contains a T-join of cardinality $|T \cap A|$ ($|T \cap B|$, resp.). By Claim 6, the proof of the theorem is complete.

CASE 2. Secondly suppose that MAX$> |T \cap A|$ and MAX$> |T \cap B|$. Then, by Lemma 11, every optimal admissible bi-partition contains a set A_i with $1 < |A_i| < |A|$. Let us choose an optimal admissible bi-partition $\mathcal{P} \cup \mathcal{Q}$ of $A \cup B$ so that such a set A_i of $\mathcal{P}$ is as large as possible. Let $K \in \mathcal{K}(G - A_i)$ so that $|V(K)| \geq 2$. (Since $|A_i| < |A|$ such a set exists.) Then, by Claim 5, $\overline{K} \in \mathcal{K}(G - B_j)$ for some $B_j \in \mathcal{Q}$ and $|V(\overline{K})| \geq 2$. Let us denote by (G_1, T_1) and (G_2, T_2) the two bipartite grafts obtained from (G, T) by T-contracting the connected subgraphs K and $\overline{K}$, respectively. The colour classes of G_r will be denoted by A^r and B^r, while the contracted vertex of G_r is denoted by v_r for $r = 1, 2$. Let $\mathcal{P}_1 := \{A_k \in \mathcal{P} : A_k \subseteq A^1\}$ and $\mathcal{Q}_1 := \{B_l \in \mathcal{Q} : B_l \subseteq B^1\} \cup \{v_1\}$. Let $\mathcal{P}_2 := \{A_k \in \mathcal{P} : A_k \subseteq A^2\} \cup \{v_2\}$ and $\mathcal{Q}_2 := \{B_l \in \mathcal{Q} : B_l \subseteq B^2\}$. The admissibility of the bi-partition $\mathcal{P} \cup \mathcal{Q}$ implies the following Claim.

Claim 13 *(a) $\mathcal{P}_r \cup \mathcal{Q}_r$ is an admissible bi-partition of $A^r \cup B^r$ in $G_r, r = 1, 2$.*

(b) $val_{(G,T)}(\mathcal{P}) = val_{(G_1,T_1)}(\mathcal{P}_1) - b^{T_1}_{v_1} + val_{(G_2,T_2)}(\mathcal{P}_2)$. □

Lemma 14 *For $r = 1, 2$, $\mathcal{P}_r \cup \mathcal{Q}_r$ is an optimal admissible bi-partition of $A^r \cup B^r$ in (G_r, T_r).*

Proof. By Claim 13(a), only the optimality must be verified. By symmetry, it is enough to prove it for $r = 2$. Suppose that $\mathcal{P}' \cup \mathcal{Q}'$ is an admissible bi-partition of $A^2 \cup B^2$ in G_2 with $val_{(G_2,T_2)}(\mathcal{P}') > val_{(G_2,T_2)}(\mathcal{P}_2)$. Let us denote by X that member of $\mathcal{P}'$ that contains v_2. Since $\mathcal{P}_1 \cup \mathcal{Q}_1$ and $\mathcal{P}' \cup \mathcal{Q}'$ are admissible bi-partitions and $\overline{K}$ is connected, $\mathcal{P}'' := (\mathcal{P}_1 - A_i) \cup (\mathcal{P}' - X) \cup \{(X - v_2) \cup A_i\}$, $\mathcal{Q}'' = (\mathcal{Q}_1 - \{v_1\}) \cup \mathcal{Q}'$ is an admissible bi-partition of $A \cup B$ in G. By Claim 13(b),

$$\begin{aligned} val_{(G,T)}(\mathcal{P}'') &= val_{(G_1,T_1)}(\mathcal{P}_1) - b^{T_1}_{v_1} + val_{(G_2,T_2)}(\mathcal{P}') \\ &> val_{(G_1,T_1)}(\mathcal{P}_1) - b^{T_1}_{v_1} + val_{(G_2,T_2)}(\mathcal{P}_2) = val_{(G,T)}(\mathcal{P}), \end{aligned}$$

a contradiction. □

Lemma 15 *If K is T-odd, then for every edge v_2u of G_2, $\mathcal{P}_2 \cup \mathcal{Q}_2$ is an optimal admissible bi-partition of $A^2 \cup B^2$ in (G_2, T_2') of value $val_{(G_2,T_2)}(\mathcal{P}_2) - 1$, where $T_2' := T_2 \oplus \{v_2, u\}$.*

Proof. By Claim 13(a), only the optimality must be verified. $val_{(G_2,T_2')}(\mathcal{P}_2) = val_{(G_2,T_2)}(\mathcal{P}_2) - 1$ because for a component L of $G_2 - R$ with $R \in \mathcal{P}_2 - \{v_2\}$, $|L \cap T_2| \equiv |L \cap T_2'|$ $(mod\ 2)$ and the unique component K of $G_2 - v_2$ becomes T_2'-even. Suppose that $\mathcal{P}' \cup \mathcal{Q}'$ is an admissible bi-partition of $A^2 \cup B^2$ in (G_2, T_2') with $val_{(G_2,T_2')}(\mathcal{P}') > val_{(G_2,T_2)}(\mathcal{P}_2) - 1$. By Claim 7, $val_{(G_2,T_2')}(\mathcal{P}') \geq val_{(G_2,T_2)}(\mathcal{P}_2) + 1$. Note that since K is T-odd, $b^{T_1}_{v_1} = 1$. Let us denote by X that member of $\mathcal{P}'$ that contains v_2. Since K and $\overline{K}$ are connected, $\mathcal{P}'' := (\mathcal{P}_1 - A_i) \cup (\mathcal{P}' - X) \cup \{(X - v_2) \cup A_i\}$, $\mathcal{Q}'' = (\mathcal{Q}_1 - \{v_1\}) \cup \mathcal{Q}'$ is an admissible bi-partition of $A \cup B$ in G.

If $X = v_2$ then, by Claim 13(b),

$$\begin{aligned} val_{(G,T)}(\mathcal{P}'') &= val_{(G_1,T_1)}(\mathcal{P}_1) + val_{(G_2,T_2')}(\mathcal{P}') \\ &\geq val_{(G_1,T_1)}(\mathcal{P}_1) + val_{(G_2,T_2)}(\mathcal{P}_2) + 1 > val_{(G,T)}(\mathcal{P}), \end{aligned}$$

a contradiction.

If $X \neq v_2$, then, by Claim 13(b),

$$\begin{aligned} val_{(G,T)}(\mathcal{P}'') &\geq (val_{(G_1,T_1)}(\mathcal{P}_1) - 1) + (val_{(G_2,T_2')}(\mathcal{P}') - 1) \\ &\geq val_{(G_1,T_1)}(\mathcal{P}_1) - 1 + val_{(G_2,T_2)}(\mathcal{P}_2) = val_{(G,T)}(\mathcal{P}), \end{aligned}$$

that is $\mathcal{P}'' \cup \mathcal{Q}''$ is an optimal admissible bi-partition of $A \cup B$ in G, but $|(X - v_2) \cup A_i| > |A_i|$, contradicting the maximality of A_i. □

By induction ($|V(G_1)| < |V(G)|$ because $|V(K)| > 2$) and by Lemma 14, there exists a T_1-join F_1 in G_1 with $|F_1| = val(\mathcal{P}_1)$.

First suppose that K is a T-even component of $G - A_i$. By induction ($|V(G_2)| < |V(G)|$ because $|V(\overline{K})| \geq 2$) and by Lemma 14, there exists a T_2-join F_2 in G_2 with $|F_2| = val(\mathcal{P}_2)$. Then, by Claim 6, $|F_1 \cap \delta(K)| = 0 = |F_2 \cap \delta(K)|$, hence $F := F_1 \cup F_2$ is a T-join and, by Claim 13(b), $|F| = |F_1| + |F_2| = val(\mathcal{P}_1) + val(\mathcal{P}_2) = val(\mathcal{P})$. By Claim 6, we are done.

Now suppose that K is a T-odd component of $G - A_i$. Then, by Claim 14, $|F_1 \cap \delta(K)| = 1$. This edge corresponds to an edge v_2u in G_2. By induction ($|V(G_2)| < |V(G)|$ because $|V(\overline{K})| \geq 2$) and by Lemma 15 with edge v_2u, there exists a T_2'-join F_2 in G_2 with $|F_2| = val(\mathcal{P}_2) - 1$. Then $F := F_1 \cup F_2$ is a T-join and, by Claim 13(b), $|F| = |F_1| + |F_2| = val(\mathcal{P}_1) + val(\mathcal{P}_2) - 1 = val(\mathcal{P})$. By Claim 6, we are done. □

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