Directed hypergraph connectivity augmentation by hyperarc reorientations

Moritz Mühlenthaler, Benjamin Peyrille, Zoltán Szigeti

May 2, 2023

Abstract

The orientation theorem of Nash-Williams states that an undirected graph admits a k-arc-connected orientation if and only if it is $2k$-edge-connected. Recently, Ito et al. [8] showed that any orientation of an undirected $2k$-edge-connected graph can be transformed into a k-arc-connected orientation by reorienting one arc at a time without decreasing the arc-connectivity at any step, thus providing an algorithmic proof of Nash-Williams' theorem. We generalize their result to hypergraphs and therefore provide an algorithmic proof of the characterization of hypergraphs with a k-hyperarc-connected orientation originally given by Frank et al. [6]. We prove that any orientation of an undirected (k, k)-partition-connected hypergraph can be transformed into a k-hyperarc-connected orientation by reorienting one hyperarc at a time without decreasing the hyperarc-connectivity in any step. Furthermore, we provide a simple combinatorial algorithm for computing such a transformation in polynomial time.

1 Introduction

An undirected (resp., directed) graph is *k-edge-connected* (resp., *k-arc-connected*) if every ordered pair of vertices is connected by at least k edge-disjoint (resp., arc-disjoint) paths. For any undirected graph we may obtain an *orientation* of that graph by orienting each of its edges in one of the two possible ways. In a classic theorem, Nash-Williams [11] showed that an undirected graph admits a k-arc-connected orientation if and only if it is $2k$-edge-connected. Frank [4] introduced reorientations of undirected graphs and proved that any two k-arc-connected orientations of a given undirected graph can be transformed into one another by reorienting directed cycles and directed paths. Recently, Ito et al. [8] showed that any orientation of an undirected $2k$-edge-connected graph can be transformed into a k-arc connected orientation by changing the orientation of a single arc in each step. Furthermore, the arc-connectivity of each orientation is at least as high as that of the previous orientations. Their result gives an algorithmic proof of the non-trivial direction of Nash-Williams' theorem. In fact, Ito et al. showed such a transformation of length kn^3 can be computed in polynomial time, where n is the number of vertices of the input graph.

We propose a generalization of the result of Ito et al. to hypergraphs. An undirected hypergraph is (k, k)-partition connected if for any partition $\mathcal{P} = \{V_1, \dots, V_t\}$ of the vertex set, the number of hyperedges intersecting at least two members of $\mathcal{P}$ is at least kt (see Section 2 for other definitions). Frank et al. [6] generalized Nash-Williams' theorem by showing that an undirected hypergraph admits a k-hyperarc-connected orientation if and only if it is (k, k)-partition connected. We prove that any orientation of an undirected (k, k)-partition-connected hypergraph can be transformed into a k-hyperarc-connected orientation by reorienting one hyperarc at a time without decreasing the hyperarc-connectivity at any step. Furthermore, such

a transformation of length at most kn^3 can be computed in time polynomial in the size of the input hypergraph. This provides an algorithmic proof for the characterization of hypergraphs that admit a k-hyperarc-connected orientations by Frank et al. [6]. To our knowledge, our result is the first efficient algorithm for computing a k-hyperarc-connected orientation of a hypergraph if one exists.

Our approach is similar to that of Ito et al. [8]: Starting from any orientation of the (k, k)-partition-connected input hypergraph, we repeatedly reorient hyperarcs along carefully chosen hyperpaths until a k-hyperarc-connected orientation is obtained. Our approach generalizes the work of Ito et al. [8], provides a simpler proof of their result for graphs and explicitly gives an algorithm.

2 Definitions

Let $\mathcal{H} := (V, \mathcal{E})$ be a *hypergraph*, where $\mathcal{E}$ is a multiset of subsets of V. The elements of $\mathcal{E}$ are called *hyperedges*. The *degree* $d_{\mathcal{H}}(X)$ of a set X of vertices of $\mathcal{H}$ is the number of hyperedges Z such that Z intersects both X and $V - X$. The hypergraph $\mathcal{H}$ is *k-hyperedge-connected* if for any non-empty proper subset X of vertices, we have $d_{\mathcal{H}}(X) \geq k$. The *hyperedge-connectivity* of $\mathcal{H}$, denoted by $\lambda(\mathcal{H})$, is the maximum integer k such that $\mathcal{H}$ is k-hyperedge-connected.

We can orient a hyperedge $X \in \mathcal{E}$ towards $v \in X$ to obtain a *hyperarc* $(X - v, v)$, where $X - v$ is the set of *tail vertices* of the hyperarc and v is the *head vertex* of the hyperarc. By orienting each edge of $\mathcal{H}$ we obtain an *orientation* $\vec{\mathcal{H}} = (V, \mathcal{A})$ of $\mathcal{H}$ which is a *directed hypergraph*. Our definition of a directed hypergraph coincides with the B-hypergraph of Gallo et al. in [7]. Let $X \subsetneq V$ with $X \neq \emptyset$. The *in-degree* $d_{\vec{\mathcal{H}}}^-(X)$ of X is the number of hyperarcs (Y, v) of $\vec{\mathcal{H}}$ such that the head vertex v is in X and at least one tail vertex of Y is not contained in X, that is, $Y - X \neq \emptyset$. The *out-degree* $d_{\vec{\mathcal{H}}}^+(X)$ of X is the number of hyperarcs (Y, v) of $\vec{\mathcal{H}}$ such that $v \notin X$ and $X \cap Y \neq \emptyset$. We use $d_{\vec{\mathcal{H}}}^-(v)$ for $d_{\vec{\mathcal{H}}}^-(\{v\})$ and $d_{\vec{\mathcal{H}}}^+(v)$ for $d_{\vec{\mathcal{H}}}^+(\{v\})$. Note that $d_{\vec{\mathcal{H}}}^-(X) = d_{\vec{\mathcal{H}}}^+(V - X)$. We say that $\vec{\mathcal{H}}$ is *k-hyperarc-connected* if for any non-empty proper subset X of vertices, we have $d_{\vec{\mathcal{H}}}^+(X) \geq k$. The *hyperarc-connectivity* of $\vec{\mathcal{H}}$, denoted by $\lambda(\vec{\mathcal{H}})$, is the maximum integer k such that $\vec{\mathcal{H}}$ is k-hyperarc-connected. When it is clear from the context what hypergraph and what orientation we are considering, we will omit the mention of $\mathcal{H}$ and $\vec{\mathcal{H}}$ from the degree functions.
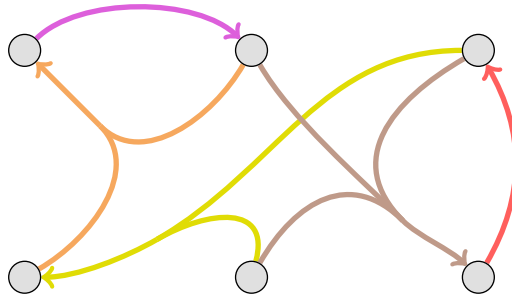

Figure 1: Directed hypergraph with 5 hyperarcs

For $s, t \in V$, a subset X of V that contains s but not t is called an *(s, t)-separator*. Let $\mathcal{P}$ be a partition of V. We denote by $e_{\mathcal{H}}(\mathcal{P})$ the number of hyperedges of $\mathcal{H}$ that intersect at least two members of $\mathcal{P}$. The hypergraph $\mathcal{H}$ is *(k, k)-partition-connected* if for any partition $\mathcal{P}$ of V, we have $e_{\mathcal{H}}(\mathcal{P}) \geq k|\mathcal{P}|$. Applying (k, k)-partition-connectivity to any partition of size 2 shows that (k, k)-partition-connectivity implies $2k$-hyperedge-connectivity. The converse is not true:
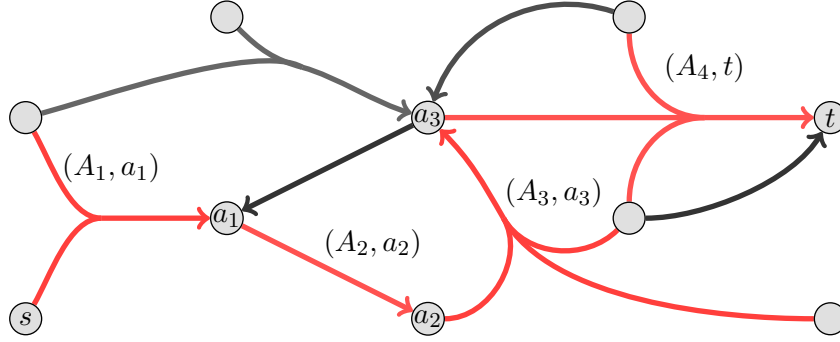
2

Figure 2: (s, t)-Hyperpath in red

consider the hypergraph with vertex set $\{a, b, c\}$ and two identical hyperedges $\{a, b, c\}$. This graph is 2-hyperedge-connected, but not $(1, 1)$-partition-connected.

Let $P = (A_1, a_1), (A_2, a_2), \ldots, (A_\ell, a_\ell)$ be a finite sequence of hyperarcs of $\vec{\mathcal{H}}$. Then P is an (s, t)-hyperpath if $s \in A_1$, $t = a_\ell$, $a_i \in A_{i+1}$ for $i \in 1, \ldots, \ell - 1$, and the vertices a_i are all distinct. An example of an (s, t)-hyperpath is shown in Figure 2. If $s = t$ then the (s, t)-hyperpath is called a *directed hypercycle*. The *trimming* of an (s, t)-hyperpath P is a directed (s, t)-path P' whose vertices are in the order $s, a_1, a_2, \ldots, a_{\ell-1}, t$. We *reorient* a hyperarc (X, v) of $\vec{\mathcal{H}}$ towards $u \in X$, $u \neq v$, by replacing (X, v) by the hyperarc $(X - u + v, u)$.

We say that a function f is *submodular* if for any two vertex sets X, Y, we have $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$. Remark that each presented degree function is submodular (see [5]). Two vertex sets X, Y are said to be *crossing* if $X \cap Y \neq \emptyset$, $X \cup Y \neq V$ and $X - Y \neq \emptyset \neq Y - X$.

3   Previous work

The following fundamental result on orientations of undirected graphs is due to Nash-Williams [11]. The original proof of Nash-Williams is not algorithmic, but meanwhile many different proofs are known, including algorithmic ones [10].

Theorem 1. *[11] An undirected graph G has a k-edge-connected orientation if and only if G is $2k$-edge-connected.*

Since then, orientations of undirected graphs have been studied widely [1, 12, 3].

In his seminal work [4], Frank considered reorientations of k-arc-connected directed graphs.

Theorem 2. *[4, Theorem 1] If $\vec{G}$ and $\vec{G}'$ are two k-arc-connected orientations of an undirected graph G then there is a sequence $\vec{G} = \vec{G}_0, \vec{G}_1, \ldots, \vec{G}_\ell = \vec{G}'$ of k-arc-connected orientations of G such that each $\vec{G}_i$ arises from $\vec{G}_{i-1}$ by reversing one directed cycle or directed path.*

In [8], Ito et al. noticed that given two k-arc-connected orientations $\vec{G}$ and $\vec{G}'$ of an undirected graph G, it is not always possible to get $\vec{G}'$ from $\vec{G}$ by reorienting arcs one by one such that each intermediate orientation to be k-arc-connected. For an example, see the two 1-arc-connected orientations of a cycle of length 3. However, for a $(2k + 2)$-edge-connected undirected graph, they proved that such a reorientation is possible, using Theorem 2 and the following result as a subroutine. They mentioned that this result also provides a new algorithmic proof of Theorem 1.

Theorem 3. *[8, Theorem 1.1] Let $G = (V, E)$ be a $2k$-edge-connected graph and $\vec{G}$ an orientation of G. Then there exists a sequence $\vec{G} = \vec{G}_0, \vec{G}_1, \vec{G}_2, \ldots, \vec{G}_\ell$ of orientations of G such that for $1 \leq i \leq \ell$, the orientation $\vec{G}_i$ is obtained from $\vec{G}_{i-1}$ by reorienting a single arc of $\vec{G}_{i-1}$, $\lambda(\vec{G}_{i-1}) \leq \lambda(\vec{G}_i)$, and $\lambda(\vec{G}_\ell) = k$. Furthermore, such a sequence with $\ell \leq (k - \lambda(\vec{G}))|V|^3$ can be computed in time polynomial in the size of G.*

Using Theorems 2 and 3, the following result is given by Ito et al. [8].

Theorem 4. *[8, Theorem 1.2] If $\vec{G}$ and $\vec{G}'$ are two k-arc-connected orientations of a $(2k+2)$-edge-connected graph G then there exists a sequence $\vec{G} = \vec{G}_0, \vec{G}_1, \ldots, \vec{G}_\ell = \vec{G}'$ of k-arc-connected orientations of G such that each $\vec{G}_i$ arises from $\vec{G}_{i-1}$ by reorienting a single arc.*

In this paper, we study reorientations of hypergraphs. The following characterization of hypergraphs that admit a k-hyperarc-connected orientation is due to Frank, Király, and Király [6].

Theorem 5. *[6, Corollary 3.5] A hypergraph $\mathcal{H}$ admits a k-hyperarc-connected orientation if and only if $\mathcal{H}$ is (k, k)-partition-connected.*

Remark that not only Theorem 5 is the hypergraphic analogue of Nash-Williams' theorem but also a generalization of it.

4   New results

Our aim is to generalize Theorem 3 to hypergraphs, providing a combinatorial algorithmic proof for Theorem 5 as well as an explicit algorithm for connectivity-augmentation of directed hypergraphs by reorienting hyperarcs. The main result of the present paper is the following theorem.

Theorem 6. *Let $\mathcal{H} = (V, \mathcal{E})$ be a (k, k)-partition-connected undirected hypergraph and $\vec{\mathcal{H}}$ an orientation of $\mathcal{H}$. Then there exists a sequence $\vec{\mathcal{H}} = \vec{\mathcal{H}}_0, \vec{\mathcal{H}}_1, \ldots, \vec{\mathcal{H}}_\ell$ of orientations of $\mathcal{H}$ such that for $1 \leq i \leq \ell$, the orientation $\vec{\mathcal{H}}_i$ is obtained from $\vec{\mathcal{H}}_{i-1}$ by reorienting a single hyperarc of $\vec{\mathcal{H}}_{i-1}$, $\lambda(\vec{\mathcal{H}}_{i-1}) \leq \lambda(\vec{\mathcal{H}}_i)$, and $\lambda(\vec{\mathcal{H}}_\ell) = k$. Furthermore, such a sequence with $\ell \leq (k - \lambda(\vec{\mathcal{H}}))|V|^3$ can be computed in time polynomial in the size of $\mathcal{H}$.*

Theorem 6 provides the proof of sufficiency in Theorem 5. Notice that the necessity in Theorem 5 is straight-forward. Indeed, let $\vec{\mathcal{H}}$ be a k-hyperarc-connected orientation of $\mathcal{H}$. Then, for any partition $\mathcal{P}$ of V, we have $e_{\mathcal{H}}(\mathcal{P}) = \sum_{X \in \mathcal{P}} d^-_{\vec{\mathcal{H}}}(X) \geq k|\mathcal{P}|$ and hence $\mathcal{H}$ is (k, k)-partition-connected.

Theorem 6 is obtained by applying $k - \lambda(\vec{\mathcal{H}})$ times Theorem 7.

Theorem 7. *Let $\mathcal{H} = (V, \mathcal{E})$ be a $(k+1, k+1)$-partition-connected hypergraph and $\vec{\mathcal{H}}$ a k-hyperarc-connected orientation of $\mathcal{H}$. Then there exists a sequence $\vec{\mathcal{H}} = \vec{\mathcal{H}}_0, \vec{\mathcal{H}}_1, \vec{\mathcal{H}}_2, \ldots, \vec{\mathcal{H}}_\ell$ of orientations of $\mathcal{H}$ such that for $1 \leq i \leq \ell$, the orientation $\vec{\mathcal{H}}_i$ is obtained from $\vec{\mathcal{H}}_{i-1}$ by reorienting a single hyperarc of $\vec{\mathcal{H}}_{i-1}$, $\lambda(\vec{\mathcal{H}}_{i-1}) \leq \lambda(\vec{\mathcal{H}}_i)$, and $\lambda(\vec{\mathcal{H}}_\ell) = k + 1$. Furthermore, such a sequence with $\ell \leq |V|^3$ can be computed in time polynomial in the size of $\mathcal{H}$.*

The proof of Theorem 7 can be found in Sections 5 to 9 and 11.

We can also generalize Theorem 2 to hypergraphs as follows.

Theorem 8. *If $\vec{\mathcal{H}}$ and $\vec{\mathcal{H}}'$ are two k-hyperarc-connected orientations of an undirected hypergraph $\mathcal{H}$ then there is a sequence $\vec{\mathcal{H}} = \vec{\mathcal{H}}_0, \vec{\mathcal{H}}_1, \ldots, \vec{\mathcal{H}}_\ell = \vec{\mathcal{H}}'$ of k-hyperarc-connected orientations of $\mathcal{H}$ such that each $\vec{\mathcal{H}}_i$ arises from $\vec{\mathcal{H}}_{i-1}$ by reversing one directed hypercycle or hyperpath.*

The proof of Theorem 8 is similar to that of Frank in [3] for Theorem 2. We briefly explain what must be changed in that proof to apply it for hypergraphs. Frank first considers the case when $d_{\vec{G}}^-(v) = d_{\vec{G}'}^-(v)$ for all vertices of G. He calls *red* the arcs of $\vec{G}$ that are oriented differently in $\vec{G}$ and in $\vec{G}'$. Notice that in the directed graph D^r induced by the red arcs, the in-degree is equal to the out-degree for every vertex. Frank then decomposes D^r into directed cycles and reorients each. For hypergraphs, we call *red* the hyperedges of $\mathcal{H}$ that are oriented differently in $\vec{\mathcal{H}}$ and in $\vec{\mathcal{H}}'$. We define a directed graph D^r on the vertices of $\mathcal{H}$. For each red hyperedge X, we add the arc uv to D^r where u and v are the head vertices of the corresponding hyperarc $(X - v, v)$ of $\vec{\mathcal{H}}$ and hyperarc $(X - u, u)$ of $\vec{\mathcal{H}}'$. We then proceed to decompose D^r into directed cycles and reorient the corresponding directed hypercycles. The rest of the proof is identical.

Finally, we generalize Theorem 4 to hypergraphs.

Theorem 9. *If $\vec{\mathcal{H}}$ and $\vec{\mathcal{H}}'$ are two k-hyperarc-connected orientations of a $(k+1, k+1)$-partition-connected hypergraph $\mathcal{H}$ then there exists a sequence $\vec{\mathcal{H}} = \vec{\mathcal{H}}_0, \vec{\mathcal{H}}_1, \ldots, \vec{\mathcal{H}}_\ell = \vec{\mathcal{H}}'$ of k-hyperarc-connected orientations of $\mathcal{H}$ such that each $\vec{\mathcal{H}}_i$ arises from $\vec{\mathcal{H}}_{i-1}$ by reorienting a single hyperarc.*

Similarly to how Theorem 4 is obtained in [8] from Theorems 2 and 3, Theorem 9 is obtained by using Theorems 6 and 8.

5 Tight and dangerous sets

For the remainder of this article we fix a $(k+1, k+1)$-partition-connected hypergraph $\mathcal{H} = (V, \mathcal{E})$. In order to avoid the verification of $X \cup Y = V$ for whether two sets $X, Y \subseteq V$ are crossing, we fix an arbitrary vertex $r \in V$ in the following definitions. To prove Theorem 7, we follow the paper of Ito et al. [8] and introduce the following sets with respect to an arbitrary k-hyperarc-connected orientation $\vec{\mathcal{H}}$ of $\mathcal{H}$, which will be clear from the context.

$$
\begin{aligned}
\mathcal{T}_- &= \{X \subseteq V - r : d^-(X) = k\} \cup \{V\} \\
\mathcal{T}_+ &= \{X \subseteq V - r : d^+(X) = k\} \cup \{V\} \\
\mathcal{M}_- &= \text{inclusion-wise minimal members of } \mathcal{T}_- \\
\mathcal{M}_+ &= \text{inclusion-wise minimal members of } \mathcal{T}_+ \\
\mathcal{M} &= \text{inclusion-wise minimal members of } \mathcal{M}_- \cup \mathcal{M}_+ \\
\mathcal{D}_- &= \{X \subseteq V - r : d^-(X) = k + 1\} \\
\mathcal{D}_+ &= \{X \subseteq V - r : d^+(X) = k + 1\} \\
\mathcal{R} &= \{R \subseteq V - r : R \text{ is inclusion wise minimal such that} \\
&\qquad \text{either } R \in \mathcal{T}_- \text{ and } R \text{ contains a member of } \mathcal{T}_+ \\
&\qquad \text{or } R \in \mathcal{T}_+ \text{ and } R \text{ contains a member of } \mathcal{T}_-\}
\end{aligned}
$$

We present briefly some properties of these sets that follow from submodularity of the degree functions.

Claim 1. *Let X, Y be two crossing sets in V.*
(a) If $X, Y \in \mathcal{T}_-$ then $X \cup Y$ and $X \cap Y \in \mathcal{T}_-$.
(b) If $X, Y \in \mathcal{T}_+$ then $X \cup Y$ and $X \cap Y \in \mathcal{T}_+$.
(c) If $X \in \mathcal{T}_-$ and $Y \in \mathcal{T}_+$ then $X - Y \in \mathcal{T}_-$ and $Y - X \in \mathcal{T}_+$.
(d) If $X \in \mathcal{M}_-$ and $Y \in \mathcal{T}_+ \cup \mathcal{D}_+$ then $Y \in \mathcal{D}_+$ and $Y - X \in \mathcal{T}_+$.
(e) If $X \in \mathcal{T}_- \cup \mathcal{D}_-$ and $Y \in \mathcal{M}_+$ then $X \in \mathcal{D}_-$ and $X - Y \in \mathcal{T}_-$.

5

Proof. (a) By $X, Y \in \mathcal{T}_-$, submodularity of d^-, $X \cap Y \neq \emptyset$, $X \cup Y \neq V$, and $\lambda(\vec{\mathcal{H}}) = k$, we have

$$k + k = d^-(X) + d^-(Y) \geq d^-(X \cup Y) + d^-(X \cap Y) \geq k + k.$$

Hence equality holds everywhere, so we have $X \cup Y, X \cap Y \in \mathcal{T}_-$.

(b) can be proved in the same way as (a) by replacing d^- by d^+.

(c) By $X \in \mathcal{T}_-$, $Y \in \mathcal{T}_+$, $d^-(V - Y) = d^+(Y)$, submodularity of d^-, $X - Y \neq \emptyset \neq Y - X$, and $\lambda(\vec{\mathcal{H}}) = k$, we have

$$\begin{aligned}
k + k &= d^-(X) + d^-(V - Y) \\
&\geq d^-(X \cap (V - Y)) + d^-(X \cup (V - Y)) \\
&= d^-(X - Y) + d^+(Y - X) \\
&\geq k + k.
\end{aligned}$$

Hence equality holds everywhere, so we have $X - Y \in \mathcal{T}_-$ and $Y - X \in \mathcal{T}_+$.

(d) By $X \in \mathcal{M}_-$, $Y \in \mathcal{T}_+ \cup \mathcal{D}_+$, the submodularity of d^-, $\lambda(\vec{\mathcal{H}}) = k$, $X - Y \neq \emptyset \neq Y - X$, and minimality of X, we obtain

$$\begin{aligned}
k + (k + 1) &\geq d^-(X) + d^+(Y) \\
&= d^-(X) + d^-(V - Y) \\
&\geq d^-(X - Y) + d^-(V - (Y - X)) \\
&= d^-(X - Y) + d^+(Y - X) \\
&\geq (k + 1) + k.
\end{aligned}$$

It follows that $d^+(Y) = k + 1$ and that $d^+(Y - X) = k$.

(e) can be proved similarly as (d). ∎

By Claim 1, each of the families $\mathcal{M}_-, \mathcal{M}_+$ and $\mathcal{M}$ is a subpartition of V.

6 Main algorithm

We follow the framework presented by [8] and adapt it for hypergraphs.

For $S \in \mathcal{M}_-$, a vertex $s \in S$ is a *safe source* in S if

(a) for every $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$,

(b) for every $s \in X \in \mathcal{D}_+$ such that $S - X \neq \emptyset$, there exists $Y \in \mathcal{T}_+$ such that $s \notin Y \subsetneq X$.

For $T \in \mathcal{M}_+$, a vertex $t \in T$ is a *safe sink* in T if

(a) for every $t \in X \in \mathcal{T}_-$, we have $T \subsetneq X$,

(b) for every $t \in X \in \mathcal{D}_-$ such that $T - X \neq \emptyset$, there exists $Y \in \mathcal{T}_-$ such that $t \notin Y \subsetneq X$.

Let P be a directed (s, t)-hyperpath in $\vec{\mathcal{H}}$ and R a set in $\mathcal{R}$. We say that P is *admissible* in R if

(a) s is a safe source in a set $S \in \mathcal{M}_-$, t is a safe sink in a set $T \in \mathcal{M}_+$, S and T are contained in R,

(b) reorienting all hyperarcs in P one by one, from start to end if $R \in \mathcal{R} \cap \mathcal{T}_+$ and from end to start if $R \in \mathcal{R} \cap \mathcal{T}_-$, never decreases the hyperarc-connectivity of the current directed hypergraph,

(c) after all hyperarcs of P have been reoriented, the new orientation's $\mathcal{M}'$ set has either less elements than $\mathcal{M}$, or the same number of elements as $\mathcal{M}$ but covers more vertices than $\mathcal{M}$.

Algorithm 1 aims to compute a $(k+1)$-hyperarc-connected orientation of $\mathcal{H}$ from a given k-hyperarc-connected orientation $\vec{\mathcal{H}}$ of $\mathcal{H}$. The algorithm starts by selecting an arbitrary vertex r of $\mathcal{H}$, which will allow us to compute the sets $\mathcal{R}, \mathcal{M}_-, \mathcal{M}_+$ with respect to $\vec{\mathcal{H}}$ and r. If $\mathcal{M}_- = \mathcal{M}_+ = \{V\}$ then for all $X \subseteq V - r$, we have $d_{\vec{\mathcal{H}}}^-(X) \geq k+1$ and $d_{\vec{\mathcal{H}}}^+(X) \geq k+1$, this implies that we have a $(k+1)$-hyperarc-connected orientation and we are done. Otherwise, we pick an arbitrary set R in $\mathcal{R}$ and find an admissible (s,t)-hyperpath P in R. Depending on whether R is in $\mathcal{T}_+$ or $\mathcal{T}_-$, we reorient one by one each hyperarc of P either from start to end or from end to start such that we do not decrease the hyperarc-connectivity of $\vec{\mathcal{H}}$ in each reorientation step. We repeat the process with the new orientation until $|\mathcal{M}| = \{V\}$. Since $\mathcal{M}$ is a subpartition of V and by the definition of an admissible hyperpath, the case $|\mathcal{M}'| = |\mathcal{M}|$ can only occur at most $|V|$ times until $|\mathcal{M}'| < |\mathcal{M}|$, which itself can only happen at most $|V|$ times. Thus, since no two hyperarcs in a hyperpath have the same head vertex, we reorient at most $|V|^3$ hyperarcs and hence the algorithm terminates. It then suffices to show the existence of an admissible path and that reorienting its hyperarcs one by one does not decrease the hyperarc-connectivity to prove Theorem 7.

Algorithm 1: Hypergraph connectivity augmentation algorithm

Input: A k-hyperarc-connected orientation $\vec{\mathcal{H}}$ of a $(k+1, k+1)$-partition-connected hypergraph $\mathcal{H}$.

Output: A $(k+1)$-hyperarc-connected orientation $\vec{\mathcal{H}}'$ of $\mathcal{H}$.

1 Take an arbitrary vertex r of $\mathcal{H}$;
2 Let $\vec{\mathcal{H}}' := \vec{\mathcal{H}}$;
3 Compute the sets $\mathcal{R}, \mathcal{M}_-, \mathcal{M}_+$ of $\vec{\mathcal{H}}'$;
4 **if** $\mathcal{M}_- = \mathcal{M}_+ = \{V\}$ **then**
5   Return $\vec{\mathcal{H}}'$;
6 Take a set $R \in \mathcal{R}$;
7 Find an admissible (s,t)-hyperpath $P = (A_1, a_1), (A_2, a_2), \ldots, (A_\ell, a_\ell)$ in R, using Algorithm 2 if $R \in \mathcal{T}_-$ and Algorithm 3 if $R \in \mathcal{T}_+$;
8 **if** $R \in \mathcal{T}_-$ **then**
9   **for** $i = \ell, \ell-1, \ldots, 2$ **do**
10   Reorient (A_i, a_i) towards a_{i-1};
11   Reorient (A_1, a_1) towards s;
12 **else**
13   Reorient (A_1, a_1) towards s;
14   **for** $i = 2, 3, \ldots, \ell$ **do**
15   Reorient (A_i, a_i) towards a_{i-1};
16 **go to** line 3

---

7 Existence of safe sources and safe sinks

We first show that any $S \in \mathcal{M}_-$ contains a safe source and any $T \in \mathcal{M}_+$ contains a safe sink. Notice that the proofs are not the same and, because of asymmetry of directed hypergraphs, the proof for the existence of safe sources cannot be used for the existence of safe sinks and vice versa. We start with the existence of a safe source whose proof is inspired by a proof of Lovász [9].

Lemma 10. *For $S \in \mathcal{M}_-$, there exists a safe source s in S.*

Proof. Let $S \in \mathcal{M}_-$. If $S = V$ then $s = r$ is a safe source, so assume that $S \neq V$. Let

$\mathcal{X} = \{X_1, X_2, \cdots, X_\ell\}$ be a family of vertex sets such that $X_i \subseteq S$, $X_i \in \mathcal{T}_+ \cup \mathcal{D}_+$ and $\mathcal{X}$ covers as many vertices of S as possible and under this condition minimizes ℓ.

Claim 10.1. *The family $\mathcal{X}$ does not cover all vertices of S.*

Proof. Suppose for a contradiction that $\bigcup_{X \in \mathcal{X}} X = S$. Since $S \in \mathcal{T}_-$, $S \neq V$, and $\mathcal{H}$ is $(k+1, k+1)$-partition-connected, we have $d^+(S) \geq 2(k+1) - d^-(S) = k+2$. Thus, $S \notin \mathcal{T}_+ \cup \mathcal{D}_+$ and $\ell \geq 2$. We also have

$$2 \leq d^+(S) - d^-(S) = d^-(\bar{S}) - d^+(\bar{S}). \tag{1}$$

For every $1 \leq i \leq \ell$, let $Y_i = X_i - \bigcup_{i \neq j} X_j$. By the minimality of ℓ, we have $Y_i \neq \emptyset$. Let $X_0 = Y_0 = \bar{S} \neq \emptyset$. We show that

$$\sum_{i=0}^{\ell} d^-(Y_i) \leq \sum_{i=0}^{\ell} d^+(X_i). \tag{2}$$

Indeed, since $\{Y_1, \ldots, Y_\ell\}$ is a subpartition of V, every hyperarc contributes at most 1 to the left-hand side. Moreover, if a hyperarc (Z, v) contributes 1 to the left-hand side then $v \in Y_i$ for some $i \in \{0, 1, \ldots, \ell\}$ and there is some tail $u \in Z$ such that $u \in \bar{Y}_i = \bigcup_{j \neq i} X_j$. Therefore, $u \in X_j$ for some $j \neq i$, so (Z, v) contributes at least 1 to the right-hand side and (2) follows.

Furthermore, for $1 \leq i \leq \ell$, by $\ell \geq 2$, $X_i \in \mathcal{T}_+ \cup \mathcal{D}_+$, $\lambda(\vec{\mathcal{H}}) = k$, and $\emptyset \neq Y_i \subsetneq S \in \mathcal{M}_-$, we have

$$d^+(X_i) \leq k + 1 \leq d^-(Y_i). \tag{3}$$

By equations (1)–(3), we obtain

$$2 \leq d^-(\bar{S}) - d^+(\bar{S}) = d^-(Y_0) - d^+(X_0) \leq \sum_{i=1}^{\ell}(d^+(X_i) - d^-(Y_i)) \leq 0,$$

a contradiction that proves Claim 10.1. □

By Claim 10.1, there exists a vertex $s \in S - \bigcup_{i=1}^{\ell} X_i$. We aim to show s is a safe source in S. Suppose that $s \in X \in \mathcal{T}_+ \cup \mathcal{D}_+$. By the definition of s and X, we have $X - S \neq \emptyset$. Suppose now that $S - X \neq \emptyset$. By Claim 1(d), we have $d^+(X) = k + 1$ and $d^+(X - S) = k$. Then, for every $s \in X \in \mathcal{T}_+$, we have $S \subsetneq X$, and for every $s \in X \in \mathcal{D}_+$ such that $S - X \neq \emptyset$, we have $X - S \in \mathcal{T}_+$. Therefore, s is a safe source in S. ∎

We next show the existence of a safe sink whose proof is similar to the one found in [8].

Lemma 11. *For $T \in \mathcal{M}_+$, there exists a safe sink t in T.*

Proof. Let $T \in \mathcal{M}_+$. If $T = V$ then $t = r$ is a safe sink, so we consider $T \neq V$. Let $\mathcal{Y} = \{Y_1, \ldots, Y_\alpha\}$ be the inclusionwise maximal vertex sets that are in $\mathcal{T}_-$ and contained in T. Using Claim 1, if two members of $\mathcal{Y}$ did cross then their union would be in $\mathcal{T}_-$, thus the family $\mathcal{Y}$ is a subpartition.

Let $\mathcal{Z} = \{Z_1, \ldots, Z_\beta\}$ be the inclusionwise maximal vertex sets that are in $\mathcal{D}_-$ and contained in $T - \bigcup_{Y \in \mathcal{Y}} Y$. We show that $\mathcal{Z}$ is a subpartition; suppose for a contradiction that this is not the case. Then there exist distinct indices $i \neq j$ such that $Z_i \cap Z_j \neq \emptyset$. If $d^-(Z_i \cap Z_j) = k$ then $Z_i \cap Z_j \subseteq Y_\ell \in \mathcal{Y}$ for some ℓ, contradicting that $Z_i \subseteq T - \bigcup_{Y \in \mathcal{Y}} Y$. Similarly, if $d^-(Z_i \cup Z_j) = k$ then $Z_i \cup Z_j \subseteq Y_\ell \in \mathcal{Y}$ for some ℓ, contradicting that $Z_i \subseteq T - \bigcup_{Y \in \mathcal{Y}} Y$. Hence, by the submodularity of d^- and $\lambda(\vec{\mathcal{H}}) = k$,

$$2(k + 1) = d^-(Z_i) + d^-(Z_j) \geq d^-(Z_i \cap Z_j) + d^-(Z_i \cup Z_j) \geq 2(k + 1),$$

so $d^-(Z_i \cup Z_j) = k + 1$, which contradicts the maximality of Z_i.

Claim 11.1. *The family $\mathcal{Y} \cup \mathcal{Z}$ does not cover all vertices of T.*

Proof. Suppose this is not the case. Then $\mathcal{P} = \mathcal{Y} \cup \mathcal{Z} \cup \{\bar{T}\}$ is a partition of V. Notice that $d^-(\bar{T}) = d^+(T) = k$ and $e_{\mathcal{H}}(\mathcal{P}) = \sum_{X \in \mathcal{P}} d^-(X)$. Then, by $(k+1, k+1)$-partition-connectivity of $\mathcal{H}$, we have a contradiction: $\alpha k + \beta(k+1) + k = e_{\mathcal{H}}(\mathcal{P}) \geq |\mathcal{P}|(k+1) = (\alpha + \beta + 1)(k+1)$. $\square$

By Claim 11.1, there exists a vertex $t \in T - \bigcup_{Y \in \mathcal{Y}} Y - \bigcup_{Z \in \mathcal{Z}} Z$. We first show that any set $X \in \mathcal{T}_-$ that contains t also contains T. If $X \subseteq T$ then $X \subseteq Y_i$ for some i, contradicting the choice of t. We hence suppose that $X - T \neq \emptyset$. If $T - X \neq \emptyset$ then by Claim 1(c), we have $T - X \in \mathcal{T}_+$ which contradicts the minimality of T. This proves $T \subsetneq X$.

We now show that any set $X \in \mathcal{D}_-$ such that $t \in X$ and $T - X \neq \emptyset$ contains a set $Y \subseteq X - t$ such that $Y \in \mathcal{T}_-$. First, suppose $X \subsetneq T$. Since $t \in X \subsetneq T$, $X \in \mathcal{D}_-$, and $t \notin \bigcup_{Z \in \mathcal{Z}} Z$, we get that X is not contained in $T - \bigcup_{Y \in \mathcal{Y}} Y$. Thus, X must intersect a set Y_i for some i. By the submodularity of d^-, $\lambda(\vec{\mathcal{H}}) = k$, and the maximality of Y_i, we have

$$(k+1) + k = d^-(X) + d^-(Y_i) \geq d^-(X \cap Y_i) + d^-(X \cup Y_i) \geq k + (k+1).$$

Therefore, $X \cap Y_i \in \mathcal{T}_-$. Finally, suppose that $X - T \neq \emptyset$. Then, by the minimality of T and Claim 1(e), we have $X - T \in \mathcal{T}_-$, proving t is a safe sink in T. $\blacksquare$

For $v \in V$, Q_-^v denotes the inclusion-wise minimal set of $\mathcal{T}_-$ containing v and Q_+^v denotes the inclusion-wise minimal set of $\mathcal{T}_+$ containing v. Notice that there is no ambiguity since, by Claim 1(a), the sets Q_-^v et Q_+^v are unique.

Lemma 12 (see [8, Lemma 4.12])**.**

(a) *For every $s \in V$ and $t \in Q_+^s$, there exists an (s,t)-hyperpath P that does not leave Q_+^s.*

(b) *For every $t \in V$ and $s \in Q_-^t$, there exists an (s,t)-hyperpath P that does not leave Q_-^t.*

Proof. (a) Suppose that the claim does not hold, so there exist $s \in V$ and $t \in Q_+^s$ such that any (s,t)-hyperpath leaves Q_+^s. Then there exists a vertex set $s \in Z \subseteq Q_+^s - t$ such that any hyperarc that leaves Z also leaves Q_+^s. Therefore, we have $k \geq d^+(Q_+^s) \geq d^+(Z) \geq k$, which implies that $Z \in \mathcal{T}_+$, contradicting the minimality of Q_+^s.

(b) can be proved similarly as (a). Suppose that the claim does not hold, so there exist $t \in V - r$ and $s \in Q_-^t$ such that any (s,t)-hyperpath leaves Q_-^t. Then there exists a vertex set $t \in Z \subseteq Q_-^t - s$ such that any hyperarc that enters Z also enters Q_-^t. Therefore, we have $k \geq d^-(Q_-^t) \geq d^-(Z) \geq k$, which shows that $Z \in \mathcal{T}_-$, contradicting the minimality of Q_-^t. $\blacksquare$

Lemma 13. *Let $R \in \mathcal{R}, S \in \mathcal{M}_-, T \in \mathcal{M}_+$ such that $S, T \subseteq R$. Furthermore, let s be a safe source in S, and t a safe sink in T. Then for all sets $X \subseteq V - r$ such that $s \in X, t \notin X$, we have $d^+(X) \geq k+1$, and for all sets $X \subseteq V - r$ such that $t \in X, s \notin X$, we have $d^-(X) \geq k+1$.*

Proof. If the claim does not hold then, by $\lambda(\vec{\mathcal{H}}) = k$, there exists $X \subseteq V - r$ such that either (a) $s \in X, t \notin X$ and $X \in \mathcal{T}_+$ or (b) $t \in X, s \notin X$ and $X \in \mathcal{T}_-$.

(a) By $S \in \mathcal{T}_-$, $s \in X \in \mathcal{T}_+$ and since s is a safe source, we have $S \subsetneq X$. As $t \in R - X$ and $R \in \mathcal{R}$, we have that $X \nsubseteq R$ which implies $X - R \neq \emptyset$.

   - If $R \in \mathcal{T}_-$ then by $t \in R - X$, $X \in \mathcal{T}_+$ and Claim 1 on X and R, we get that $R - X \in \mathcal{T}_-$. If $X \cap T \neq \emptyset$, then, by Claim 1, we get that $X \cap T \in \mathcal{T}_+$, contradicting the minimality of T. Hence $T \subseteq R - X$. This, by $T \in \mathcal{T}_+$ and $R - X \in \mathcal{T}_-$, contradicts that $R \in \mathcal{R}$.

9

- If $R \in \mathcal{T}_+$, by $X \in \mathcal{T}_+$ and Claim 1, we get that $X \cap R \in \mathcal{T}_+$. This, by $S \in \mathcal{T}_-$, $S \subseteq R \cap X$ and $t \in R - X$, contradicts that $R \in \mathcal{R}$.

(b) By $T \in \mathcal{T}_+, t \in X \in \mathcal{T}_-$ and since t is a safe sink, we have $T \subsetneq X$. As $s \in R - X$ and $R \in \mathcal{R}$, we have that $X \not\subseteq R$ which implies $X - R \neq \emptyset$.

- If $R \in \mathcal{T}_-$ then by $X \in \mathcal{T}_-$ and Claim 1, we get that $X \cap R \in \mathcal{T}_-$. This, by $T \in \mathcal{T}_+$, $T \subseteq X \cap R$ and $t \in R - X$, contradicts that $R \in \mathcal{R}$.

- If $R \in \mathcal{T}_+$, then by $s \in R - X$, $X \in \mathcal{T}_-$ and Claim 1 on X and R, we get that $R - X \in \mathcal{T}_+$. If $X \cap S \neq \emptyset$, then, by Claim 1, we get that $X \cap S \in \mathcal{T}_-$, contradicting the minimality of S. Hence $S \subseteq R - X$. This, by $S \in \mathcal{T}_-$ and $R - X \in \mathcal{T}_+$, contradicts that $R \in \mathcal{R}$.

■

8 Finding an admissible hyperpath

In this section we present two algorithms to find an admissible hyperpath in a set R of $\mathcal{R}$. We use Algorithm 2 when R is in $\mathcal{T}_-$ and Algorithm 3 when R is in $\mathcal{T}_+$.

Algorithm 2 finds an (s,t)-hyperpath P such that its trimming P' neither leaves R nor any set of $\mathcal{T}_+$ it enters. The algorithm works as follows. First, we choose a minimal set S contained in R (possibly R itself), such that $S \in \mathcal{M}_-$ as well as a safe source s in S. We keep track of a set Z of vertices we already explored (that contains s initially), a *search s-out-arborescence F* and a set V' of vertices that will potentially be explored and which is initialized to R. While there is a hyperarc (X, v) leaving Z and whose head v is in V', we pick an arbitrary vertex $u \in X \cap Z$. This vertex u will serve as the tail of the trimmed hyperarc. Note that u exists by the choice of (X, v). We mark the head vertex v as explored, trim the hyperarc (X, v) into the arc uv, and add it to F. We ensure that if our search enters a set in $\mathcal{T}_+$, we do not leave that set (see Line 11 of Algorithm 2). We let T be the last set in $\mathcal{T}_+$ entered. We will see that $T \in \mathcal{M}_+$ and we find a safe sink t in T. Finally, we let P' be the unique directed (s,t)-path in F denoted by $F[s,t]$. Notice that P' is the trimming of an (s,t)-hyperpath P and the vertex set of P' is contained in R. The algorithm returns S, T, s, t, P and terminates.

Algorithm 2: Admissible hyperpath in $R \in \mathcal{T}_-$

Input: A set $R \in \mathcal{R} \cap \mathcal{T}_-$.
Output: A safe source s in $S \in \mathcal{M}_-$ with $S \subseteq R$, a safe sink t in $T \in \mathcal{M}_+$ with $T \subseteq R$,
         and an (s,t)-hyperpath P that does not leave R.

1 Take a set $S \in \mathcal{M}_-$ with $S \subseteq R$;
2 Take a safe source s in S;
3 $Z := \{s\}$;
4 $F := (Z, \emptyset)$;
5 $V' := R$;
6 **while** *an (X, v) hyperarc exists in $\vec{\mathcal{H}}$ such that $v \in V' - Z$ and $X \cap Z \neq \emptyset$* **do**
7   Take an arbitrary vertex $u \in X \cap Z$;
8 $Z := Z \cup \{v\}$;
9 $F := F + uv$;
10   **if** $Q_+^v \subsetneq V'$ **then**
11 $V' := Q_+^v$;
12 $T := V'$;
13 Take a safe sink t in T;
14 $P' := F[s,t]$;
15 Let P be the hyperpath in $\vec{\mathcal{H}}$ corresponding to P';
16 Return S, T, s, t, P;

We apply Algorithm 2 to prove the following lemma that we need in order to show the existence of an admissible path in case of $R \in \mathcal{R} \cap \mathcal{T}_-$.

Lemma 14. *If $R \in \mathcal{R} \cap \mathcal{T}_-$ then there exist $S \in \mathcal{M}_-$, $T \in \mathcal{M}_+$, a safe source s in S, a safe sink t in T, and an (s,t)-hyperpath P such that $S \subseteq R$, $T \subsetneq R$, P does not leave R, $Q_-^t = R$ and $\delta_{P'}^+(X) = \emptyset$ for every $X \in \mathcal{T}_+$ with $s, t \notin X$ where P' is the unique directed (s,t)-path obtained from P by trimming.*

Proof. The algorithm starts by taking a safe source in S. Such a safe source exists by Lemma 10.

We show that Algorithm 2 provides the required hyperpath P. Let $T' \in \mathcal{M}_+$ with $T' \subsetneq R$ and t' a safe sink in T' which exists by Lemma 11. Then, by the definition of a safe sink and Claim 1(a), $T' \subsetneq Q_-^{t'} \subseteq R$. Thus, by minimality of R, we have $Q_-^{t'} = R$. Hence, by Lemma 12(b), t' is reachable from s by a hyperpath P that does not leave $Q_-^{t'}$. This implies that the algorithm will eventually enter a set $T \in \mathcal{M}_+$, containing a safe sink t by Lemma 11. Thus, the algorithm terminates with a safe source s in $S \in \mathcal{M}_-$ with $S \subseteq R$, a safe sink t in $T \in \mathcal{M}_+$ with $T \subseteq R$, and an (s,t)-hyperpath P that does not leave R.

Recall that $P' = F[s,t]$. It remains to prove that for all $X \in \mathcal{T}_+$ such that $s, t \notin X$, $\delta_{P'}^+(X) = \emptyset$. Suppose that $\delta_{P'}^+(X) \neq \emptyset$. Since $s \notin X$, the path P' enters X. Let uv be the first arc of P' that enters X. Let $Y = Q_+^v$ and note that P' enters Y. As $v \in X \cap Y$, $X, Y \in \mathcal{T}_+$, we get, using Claim 1(b) on X and Y and the minimality of Y, that $Y \subseteq X \subsetneq V - t$. As $Y \in \mathcal{T}_+$ and $t \in T \in \mathcal{M}_+$, we get, by Claim 1(b) and the minimality of T, that Y and T are disjoint. If $Y - R \neq \emptyset$ then, by $Y \in \mathcal{T}_+$, $R \in \mathcal{T}_-$, $t \in R - Y$ and by Claim 1(c), we obtain that $R - Y \in \mathcal{T}_-$. Since $T \subseteq R - Y$, this contradicts $R \in \mathcal{R}$. Hence, $Q_+^v = Y \subsetneq R$. Then, since $Q_+^v \subseteq Q_+^u$, by the *if* condition on line 10 of Algorithm 2, we have that F and hence P' contains no arc leaving Q_+^v and thus leaving X. This contradicts $d_{P'}^+(X) \neq \emptyset$, completing the proof. ∎

Algorithm 3 finds an (s,t)-hyperpath P such that its trimming P' neither leaves R nor enters any set of $\mathcal{T}_-$ that does not contain neither s nor t.

Algorithm 3: Admissible hyperpath in $R \in \mathcal{T}_+$

Input: A set $R \in \mathcal{R} \cap \mathcal{T}_+$

Output: A safe source s in $S \in \mathcal{M}_-$ with $S \subseteq R$, a safe sink t in $T \in \mathcal{M}_+$ with $T \subseteq R$, and an (s,t)-hyperpath P that does not leave R.

1 Take a set $T \in \mathcal{M}_+$ with $T \subseteq R$;

2 Take a safe sink t in T;

3 $Z := \{t\}$;

4 $F := (Z, \emptyset)$;

5 $V' := R$;

6 **while** *an (X, v) hyperarc exists in $\vec{\mathcal{H}}$ such that $v \in Z$ and $(X \cap V') - Z \neq \emptyset$* **do**

7   **while** $(X \cap V') - Z \neq \emptyset$ **do**

8   Take u in $(X \cap V') - Z$;

9 $Z := Z \cup \{u\}$;

10 $F := F + uv$;

11   **if** $Q_-^u \subsetneq V'$ **then**

12 $V' := Q_-^u$;

13 $S := V'$;

14 Take a safe source s in S;

15 $P' := F[s, t]$;

16 Let P be the hyperpath in $\vec{\mathcal{H}}$ corresponding to P';

17 Return S, T, s, t, P;

Algorithm 3 works as follows. First, we choose a minimal set T contained in R (possibly R itself), such that $T \in \mathcal{M}_+$ as well as a safe sink t in T. We keep track of a set Z of vertices we already explored (that contains t initially), a *search t-in-arborescence* F and a set V' of vertices that will potentially be explored and which is initialized to R. While there is a hyperarc (X, v) entering Z and whose tail vertex set X contains unexplored vertices in V', we consider all possible trimmings of the hyperarc (X, v) into (u, v), where $u \in (X \cap V') - Z$. We mark the tail vertex u as explored, trim the hyperarc (X, v) into the arc uv, and add it to F. We ensure that if our search enters a set in $\mathcal{T}_-$, we do not leave that set (see Line 11 of Algorithm 3). We will see that the last set in $\mathcal{T}_-$ entered by the search is in $\mathcal{M}_-$ and we let S be that set. We find a safe source s in S. Finally, we let P' be the unique directed (s,t)-path $F[s, t]$. Notice that P' is the trimming of an (s,t)-hyperpath P and the vertex set of P' is contained in R. The algorithm returns S, T, s, t, P and terminates.

The proof of Lemma 15 is similar to that of Lemma 14.

Lemma 15. *If $R \in \mathcal{R} \cap \mathcal{T}_+$, then there exist $S \in \mathcal{M}_-$, $T \in \mathcal{M}_+$, a safe source s in S, a safe sink t in T, and an (s,t)-hyperpath P such that $S \subsetneq R$, $T \subseteq R$, P does not leave R, $Q_+^s = R$ and $\delta_{P'}^-(X) = \emptyset$ for every $X \in \mathcal{T}_-$ with $s, t \notin X$ where P' is the unique directed (s,t)-path obtained from P by trimming.*

Proof. The algorithm starts by taking a safe sink in T. Such a safe sink exists by Lemma 11.

We show that Algorithm 3 provides the required hyperpath P. Let $S' \in \mathcal{M}_-$ with $S' \subsetneq R$ and s' a safe sink in S' by Lemma 10. Then, by Claim 1(a), $S' \subsetneq Q_+^{s'} \subseteq R$. Thus, by minimality of R, $Q_+^{s'} = R$. Then, by Lemma 12(a), t is reachable from s' by a hyperpath P that does not leave $Q_+^{s'}$. This implies that the algorithm will eventually enter a set $S \in \mathcal{M}_-$, containing a safe source s by Lemma 10. Thus, the algorithm terminates with a safe source s in $S \in \mathcal{M}_-$ with $S \subsetneq R$, a safe sink t in $T \in \mathcal{M}_+$ with $T \subseteq R$, and an (s,t)-hyperpath P that does not leave R.

Recall that $P' = F[s,t]$. It remains to prove that for all $X \in \mathcal{T}_-$ such that $s, t \notin X$, $\delta_{P'}^-(X) = \emptyset$. Suppose that $\delta_{P'}^-(X) \neq \emptyset$. Since $t \notin X$, the path P' leaves X. Let uv be the first arc of P' that leaves X. Let $Y = Q_-^u$ and note that P' leaves Y. As $u \in X \cap Y, X, Y \in \mathcal{T}_-$, we get, by Claim 1(a) on X and Y and the minimality of Y, that $Y \subseteq X \subsetneq V - t$. As $Y \in \mathcal{T}_-$ and $s \in S \in \mathcal{M}_-$, we get, by Claim 1(a), that Y and S are disjoint. If $Y - R \neq \emptyset$, then, by $Y \in \mathcal{T}_-, R \in \mathcal{T}_+, s \in R - Y$ and by Claim 1(c), we obtain that $R - Y \in \mathcal{T}_+$. Since $S \subseteq R - Y$, this contradicts $R \in \mathcal{R}$. Hence, $Q_-^u = Y \subsetneq R$. Then, since $Q_-^u \subseteq Q_-^v$, by the *if* condition on line 12 of the Algorithm 3, we have that F and hence P' contains no arc entering Q_-^u and thus entering X. This contradicts $\delta_{P'}^-(X) \neq \emptyset$, completing the proof. ∎

9   Proving the admissibility of the hyperpath

In this section, we prove that Algorithms 2 and 3 provide an admissible hyperpath.

Let $R \in \mathcal{R}$ and $\mathcal{M} \neq \{V\}$. By Lemmas 14 and 15, there exist S in $\mathcal{M}_-$, T in $\mathcal{M}_+$, with $S, T \subseteq R$, a safe source s in S, a safe sink t in T, and an (s,t)-hyperpath P that does not leave R. Let P' be the trimming of P found by either Algorithm 2 or Algorithm 3. Furthermore, if $R \in \mathcal{T}_-, \delta_{P'}^+(X) = \emptyset$ for all $X \in \mathcal{T}_+$ with $s, t \notin X$. Similarly, if $R \in \mathcal{T}_+, \delta_{P'}^-(X) = \emptyset$ for all $X \in \mathcal{T}_-$ with $s, t \notin X$. It is clear, from the search technique of the algorithm, that $\ell := |\mathcal{A}(P)| < |V|$. For $i = 0, 1, \ldots, \ell$, let $\vec{\mathcal{H}}^i$ be obtained from $\vec{\mathcal{H}}$ by reorienting the last (resp. first) i hyperarcs (A_j, a_j) of P towards a_{j-1} if $R \in \mathcal{T}_-$ (resp. $R \in \mathcal{T}_+$).

We first prove that the hyperarc-connectivity never decreases when we reorient the hyperarcs of P.

Lemma 16. *For $i = 1, \ldots, \ell$, $\lambda(\vec{\mathcal{H}}^i) \geq k$.*

Proof. Suppose that there exists i and $X \subseteq V - r$ such that either (a) $d_{\vec{\mathcal{H}}^i}^+(X) < k$ or (b) $d_{\vec{\mathcal{H}}^i}^-(X) > k$. We take the first of such i.

(a) – If $R \in \mathcal{T}_-$ then, by $\lambda(\vec{\mathcal{H}}^{i-1}) \geq k$, we get that $t \notin X \in \mathcal{T}_+$ and $d_{P'}^+(X) \neq \emptyset$. By the definition of P and Lemma 14, we have $s \in X$. Thus, by Lemma 13, since we are reorienting hyperarcs along a path, we have a contradiction: $k + 1 \leq d_{\vec{\mathcal{H}}^0}^+(X) \leq d_{\vec{\mathcal{H}}^i}^+(X) + 1 \leq k$.

   – If $R \in \mathcal{T}_+$ then, by $\lambda(\vec{\mathcal{H}}^{i-1}) \geq k$, we get $s \in X \in \mathcal{T}_+$. By $s \in R \in \mathcal{T}_+$ and Claim 1(b), we get that $X \cap R \in \mathcal{T}_+$. By Lemma 15, we have $R = Q_+^s$ and $\delta_{P'}^+(X) \neq \emptyset$. Hence we have $R = Q_+^s \subseteq X \cap R \subsetneq R$, a contradiction.

(b) – If $R \in \mathcal{T}_-$ then, by $\lambda(\vec{\mathcal{H}}^{i-1}) \geq k$, we get $t \in X \in \mathcal{T}_-$. By $t \in R \in \mathcal{T}_-$ and Claim 1(a), we get that $X \cap R \in \mathcal{T}_-$. By Lemma 14, we have $R = Q_-^t$ and $\delta_{P'}^-(X) \neq \emptyset$. Hence we have $R = Q_-^t \subseteq X \cap R \subsetneq R$, a contradiction.

   – If $R \in \mathcal{T}_+$, by $\lambda(\vec{\mathcal{H}}^{i-1}) \geq k$, we get that $s \notin X \in \mathcal{T}_-$ and $d_{P'}^-(X) \neq \emptyset$. By the definition of P and Lemma 15, we have $t \in X$. Thus, by Lemma 13, since we are reorienting hyperarcs along a path, we have a contradiction: $k + 1 \leq d_{\vec{\mathcal{H}}^0}^-(X) \leq d_{\vec{\mathcal{H}}^i}^-(X) + 1 \leq k$.

∎

Let $\vec{\mathcal{H}}' = \vec{\mathcal{H}}^\ell$. We define the sets $\mathcal{T}_-', \mathcal{T}_+'$ and $\mathcal{M}'$ for $\vec{\mathcal{H}}'$ as the sets $\mathcal{T}_-, \mathcal{T}_+$ and $\mathcal{M}$ were defined for $\vec{\mathcal{H}}$. Remark that if R is in $\mathcal{T}_-$ (resp. $\mathcal{T}_+$) then by Lemma 14 (resp. Lemma 15) we get that R is in $\mathcal{T}_-'$ (resp. $\mathcal{T}_+'$). For P to be admissible, it remains to show the following.

13

Lemma 17. *Either (a) $|\mathcal{M}'| < |\mathcal{M}|$ or (b) $|\mathcal{M}'| = |\mathcal{M}|$ and $\sum_{X \in \mathcal{M}'} |X| > \sum_{X \in \mathcal{M}} |X|$ holds.*

Proof. If $T \neq R \neq S$ then, by Claim 1(c) and the minimality of S and T, we get that S and T are disjoint. Hence, since $s \in S$ and $t \in T$, the path P' leaves S and enters T. If $S = R$ then, since s is a safe source in R, $T \in \mathcal{T}_+$ and $T \subsetneq R$, we have $s \notin T$. Hence, since $t \in T$, the path P' enters T. If $T = R$, then, since t is a safe sink in R, $S \in \mathcal{T}_-$ and $S \subsetneq R$, we have $t \notin S$. Hence, since $s \in S$, the path P' leaves S. Since $\mathcal{H}$ is $(k+1, k+1)$-partition-connected, we have $d^-_{\overline{\mathcal{H}'}}(T) = d_{\mathcal{H}}(T) - d^+_{\overline{\mathcal{H}'}}(T) \geq (2k+2) - (k+1) = k+1$ if P' enters T and $d^+_{\overline{\mathcal{H}'}}(S) = d_{\mathcal{H}}(S) - d^-_{\overline{\mathcal{H}'}}(S) \geq (2k+2) - (k+1) = k+1$ if P' leaves S. Thus, $S \in \mathcal{M} - (\mathcal{T}'_- \cup \mathcal{T}'_+)$ if P' leaves S, and $T \in \mathcal{M} - (\mathcal{T}'_- \cup \mathcal{T}'_+)$ if P' enters T. We now characterize every element of $\mathcal{M}' - \mathcal{M}$ in the following two claims.

Claim 17.1. *If $X \in (\mathcal{T}_- \cup \mathcal{T}_+) \cap (\mathcal{M}' - \mathcal{M})$, then $X = R$.*

Proof. Since $X \in (\mathcal{T}_- \cup \mathcal{T}_+) \cap (\mathcal{T}'_- \cup \mathcal{T}'_+)$, either $s, t \notin X$ or $s, t \in X$. In the former case, $X \in \mathcal{M}'$ implies that $X \in \mathcal{M}$ which is a contradiction. Hence, we consider $s, t \in X$. We distinguish some cases.

(a) If $X \in \mathcal{T}_-$ then since t is a safe sink in T, we get $T \subsetneq X$.

- If $R \in \mathcal{T}_-$ then we know, by Claim 1(a), that $s, t \in X \cap R \in \mathcal{T}_-$, so we have $X \cap R \in \mathcal{T}'_-$. Thus, by $X \in \mathcal{M}'$, we obtain that $X = X \cap R \subseteq R$. Hence, by $T \in \mathcal{T}_+$, $X \in \mathcal{T}_-$ and $T \subsetneq X \subseteq R \in \mathcal{R}$, we have $X = R$.
- If $R \in \mathcal{T}_+$ then if $R - X \neq \emptyset \neq X - R$, by Claim 1(c), we get $X - R \in \mathcal{T}'_-$ which contradicts $X \in \mathcal{M}'$. Hence, either $R \subseteq X$ or $X \subseteq R$. If $R = X$, we have a contradiction from the $(k+1, k+1)$-partition-connectivity of $\mathcal{H}$. Hence, in the former case, we get $X \notin \mathcal{M}'$ and in the latter case, by $T \in \mathcal{T}_+$, $X \in \mathcal{T}_-$ and $T \subsetneq X \subsetneq R$, we get $R \notin \mathcal{R}$. In both cases, we have a contradiction.

(b) If $X \in \mathcal{T}_+$ then since s is a safe source in S, we get $S \subsetneq X$.

- If $R \in \mathcal{T}_-$ then if $R - X \neq \emptyset \neq X - R$, by Claim 1(c), we get $X - R \in \mathcal{T}'_+$ which contradicts $X \in \mathcal{M}'$. Hence, either $R \subseteq X$ or $X \subseteq R$. If $R = X$, we have a contradiction from the $(k+1, k+1)$-partition-connectivity of $\mathcal{H}$. Hence, in the former case, we get $X \notin \mathcal{M}'$ and in the latter case, by $S \in \mathcal{T}_-$, $X \in \mathcal{T}_+$ and $S \subsetneq X \subsetneq R$, we get $R \notin \mathcal{R}$. In both cases, we have a contradiction.
- If $R \in \mathcal{T}_+$ then we know, by Claim 1(b), that $s, t \in X \cap R \in \mathcal{T}_+$, so we have $X \cap R \in \mathcal{T}'_+$. Thus, by $X \in \mathcal{M}'$, we obtain that $X = X \cap R \subseteq R$. Hence, by $S \in \mathcal{T}_-$, $X \in \mathcal{T}_+$ and $S \subsetneq X \subseteq R \in \mathcal{R}$, we have $X = R$.

$\square$

Claim 17.2. *If $X \in \mathcal{M}' - (\mathcal{T}_- \cup \mathcal{T}_+)$, then $S \subsetneq X \subsetneq R$ or $T \subsetneq X \subsetneq R$.*

Proof. Since $X \in \mathcal{M}' - (\mathcal{T}_- \cup \mathcal{T}_+)$, either (a) $s \notin X, t \in X \in \mathcal{D}_-$ or (b) $t \notin X, s \in X \in \mathcal{D}_+$ holds.

(a) We first show that $T \subsetneq X$. Indeed, if $T - X \neq \emptyset$, then by the fact that t is a safe sink, there exists $Y \subseteq X - t$ with $Y \in \mathcal{T}_-$. Since $s, t \notin Y$, $Y \in \mathcal{T}'_-$ contradicting $X \in \mathcal{M}'$.

- If $R \in \mathcal{T}_-$ then $R \in \mathcal{T}'_-$, so by Claim 1(a) and $X \in \mathcal{M}'_-$, we get that $X \subseteq R$. Since $s \in R - X$, we obtain that $X \subsetneq R$.

14

- If $R \in \mathcal{T}_+$ then, by $X \in \mathcal{M}' \cap \mathcal{T}'_-$, $R \in \mathcal{T}'_+$ and Claim 1(c) on X and R, we get that X and R are not crossing. Thus, by $s \in R - X$, $t \in X \cap R$ and $r \in V - (X \cup R)$, we have $X \subsetneq R$.

(b) We first show that $S \subsetneq X$. Indeed, if $S - X \neq \emptyset$, then by the fact that s is a safe source, there exists $Y \subseteq X - s$ with $Y \in \mathcal{T}_+$. Since $s, t \notin Y$, $Y \in \mathcal{T}'_+$ contradicting $X \in \mathcal{M}'$.

- If $R \in \mathcal{T}_-$ then, by $X \in \mathcal{M}' \cap \mathcal{T}'_+$, $R \in \mathcal{T}'_-$ and Claim 1(c) on X and R, we get that X and R are not crossing. Thus, by $s \in R - X$, $t \in X \cap R$ and $r \in V - (X \cup R)$, we have $X \subsetneq R$.

- If $R \in \mathcal{T}_+$ then $R \in \mathcal{T}'_+$, so by Claim 1(b) and $X \in \mathcal{M}'_+$, we get that $X \subseteq R$. Since $t \in R - X$, we obtain that $X \subsetneq R$.

$\square$

By Claims 17.1 and 17.2, every element of $\mathcal{M}' - \mathcal{M}$ contains at least an element of $\mathcal{M} - \mathcal{M}'$ strictly (S or T). Furthermore, by Claim 1, the elements of $\mathcal{M}' - \mathcal{M}$ are disjoint. Thus, $|\mathcal{M}' - \mathcal{M}| \leq |\mathcal{M} - \mathcal{M}'|$, and if $|\mathcal{M}' - \mathcal{M}| = |\mathcal{M} - \mathcal{M}'|$, we have $\sum_{X \in \mathcal{M}'} |X| > \sum_{X \in \mathcal{M}} |X|$. $\blacksquare$

10 Minimum (s,t)-separators in directed hypergraphs

For directed graphs, it is well-known that the max flow-min cut algorithm of Edmonds-Karp [2] provides a minimum out-degree (s,t)-separator that is contained in all minimum out-degree (s,t)-separators. By submodularity, for directed hypergraphs $\vec{\mathcal{H}}$ also holds that for every $s, t \in V$, there exists a minimum out-degree (s,t)-separator that is contained in all minimum out-degree (s,t)-separators. We denote that unique set by $S_{s,t}^{\vec{\mathcal{H}}}$. Similarly, there exists a minimum in-degree (t,s)-separator that is contained in all minimum in-degree (t,s)-separators. We denote that unique set by $T_{t,s}^{\vec{\mathcal{H}}}$. The question is how to find them in polynomial time. For this purpose we provide a simple reduction to the same problem on directed graphs. In a similar way, a reduction for general directed capacitated hypergraphs can be obtained. For $\vec{\mathcal{H}} = (V, \mathcal{A})$, we consider a directed graph that is obtained from its directed bipartite incidence graph $\vec{G} = (V', A')$ as follows. We let $V' = V \cup \{w_{(X,v)} : (X, v) \in \mathcal{A}\}$. The arc set A' contains for every $(X, v) \in \mathcal{A}$, the arc $w_{(X,v)}v$ and $|\mathcal{A}| + 1$ parallel arcs $xw_{(X,v)}$ for every $x \in X$. Note that $|V'| = |V| + |\mathcal{A}|$ and $|A'| \leq |V||\mathcal{A}|(|\mathcal{A}| + 1)$. An example of this construction is shown in Figure 3. Let $\overleftarrow{G}$ be the directed graph obtained from $\vec{G}$ by changing the orientation of all arcs.
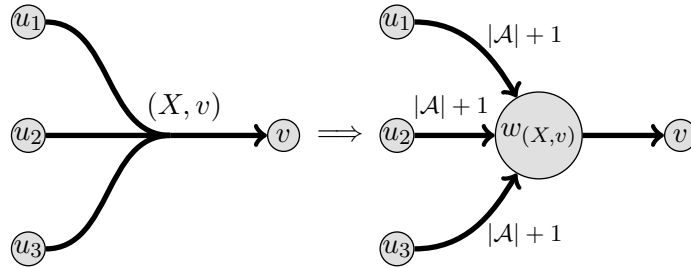

Figure 3: Arcs in $\vec{G}$ incident to $w_{(X,v)}$ resulting from a hyperedge (X, v) of $\vec{\mathcal{H}}$

Lemma 18. $S_{s,t}^{\vec{\mathcal{H}}} = V \cap S_{s,t}^{\vec{G}}$ and $T_{t,s}^{\vec{\mathcal{H}}} = V \cap S_{t,s}^{\overleftarrow{G}}$.

15

Proof. Let $Z' = S_{s,t}^{\vec{G}}$, $Z = V \cap Z'$ and $X = S_{s,t}^{\vec{\mathcal{H}}}$. Since Z' is an (s,t)-separator in $\vec{G}$, we have $s \in Z \subseteq V - t$. For a set $W \subseteq V$, let $\varphi(W) = W \cup \{w_{(Y,v)} : (Y,v) \in \mathcal{A}, W \cap Y \neq \emptyset\}$. Note that, by the definition of Z, we have $\varphi(Z) \subseteq Z'$. Let $X' = \varphi(X)$. Note that if an arc leaves X' in $\vec{G}$ then there exists a directed hyperedge that leaves X in $\vec{\mathcal{H}}$. Note also that if a directed hyperedge leaves Z in $\vec{\mathcal{H}}$ then there exists an arc that leaves Z' in $\vec{G}$. Then, by the definitions of Z' and X, we have $d_{\vec{G}}^+(Z') \leq d_{\vec{G}}^+(X') \leq d_{\vec{\mathcal{H}}}^+(X) \leq d_{\vec{\mathcal{H}}}^+(Z) \leq d_{\vec{G}}^+(Z')$. It follows that equality holds everywhere, in particular we get that X' is a minimum out-degree (s,t)-separator in $\vec{G}$ and Z is a minimum out-degree (s,t)-separator in $\vec{\mathcal{H}}$. Then we have $Z' \subseteq X'$ and $X \subseteq Z$. Thus, by $\varphi(Z) \subseteq Z'$, we get $X' = \varphi(X) \subseteq \varphi(Z) \subseteq Z' \subseteq X'$. It follows that equality holds everywhere, in particular we get that $\varphi(X) = \varphi(Z)$ and hence $X = Z$.

Let $U' = S_{t,s}^{\overleftarrow{G}}$, $U := V \cap U'$ and $X = T_{t,s}^{\vec{\mathcal{H}}}$. Since U' is a (t,s)-separator in $\overleftarrow{G}$, we have that $t \in U \subseteq V - s$. For a set $W \subseteq V$, let $\phi(W) = W \cup \{w_{(Y,v)} : (Y,v) \in \mathcal{A}, Y \cup \{v\} \subseteq W\}$. Note that, by the definition of U, we have $\phi(U) \subseteq U'$. Let $X' = \phi(X)$. Note that if an arc leaves X' in $\overleftarrow{G}$ then there exists a directed hyperedge that enters X in $\vec{\mathcal{H}}$. Note also that if a directed hyperedge enters U in $\vec{\mathcal{H}}$ then there exists an arc that leaves U' in $\overleftarrow{G}$. Then, by the definitions of U' and X, we have $d_{\overleftarrow{G}}^+(U') \leq d_{\overleftarrow{G}}^+(X') \leq d_{\vec{\mathcal{H}}}^-(X) \leq d_{\vec{\mathcal{H}}}^-(U) \leq d_{\overleftarrow{G}}^+(U')$. It follows that equality holds everywhere, in particular we get that X' is a minimum out-degree (t,s)-separator in $\overleftarrow{G}$ and U is a minimum in-degree (t,s)-separator in $\vec{\mathcal{H}}$. Then we have $U' \subseteq X'$ and $X \subseteq U$. Thus, by $\phi(U) \subseteq U'$, we get $X' = \phi(X) \subseteq \phi(U) \subseteq U' \subseteq X'$. It follows that equality holds everywhere, in particular we get that $\phi(X) = \phi(U)$ and hence $X = U$. ∎

We can hence apply the algorithm of Edmonds-Karp [2] to compute in polynomial time the sets $S_{s,t}^{\vec{G}}$ and $S_{t,s}^{\overleftarrow{G}}$ and obtain, by Lemma 18, the sets $S_{s,t}^{\vec{\mathcal{H}}}$ and $T_{t,s}^{\vec{\mathcal{H}}}$.

11 Complexity of the algorithms

In order to compute the sets $\mathcal{R}$, $\mathcal{M}_-$, and $\mathcal{M}_+$ used in Algorithm 1, we apply the algorithms of Section 10.

Lemma 19. *There is an algorithm that computes the sets $\mathcal{R}$, $\mathcal{M}_-$, $\mathcal{M}_+$ in time polynomial in the size of $\mathcal{H}$.*

Proof. Let $\mathcal{Q} = \bigcup_{v \in V}\{Q_-^v\} \cup \{Q_+^v\}$. We may obtain $\mathcal{Q}$ in polynomial time by applying the algorithms of Section 10 to compute $S_{u,v}^{\vec{\mathcal{H}}}$ and $T_{u,v}^{\vec{\mathcal{H}}}$ for every pair (u,v) of vertices of $\vec{\mathcal{H}}$.

We show that $\mathcal{R}, \mathcal{M}_-, \mathcal{M}_+ \subseteq \mathcal{Q}$. If $v \in S \in \mathcal{M}_-$ then $Q_-^v = S$ and if $v \in T \in \mathcal{M}_+$ then $Q_+^v = T$, so $\mathcal{M}_-, \mathcal{M}_+ \in \mathcal{Q}$. Let $R \in \mathcal{R}$. If $R \in \mathcal{T}_-$ then R contains $T \in \mathcal{M}_+$ and by Lemma 11 there exists a safe sink t in T. We have seen in the proof of Algorithm 2 that $Q_-^t = R$. Similarly, if $R \in \mathcal{T}_+$ then R contains $S \in \mathcal{M}_-$ and by Lemma 10 there exists a safe source s in S. We have seen in the proof of Algorithm 3 that $Q_+^t = R$. Therefore, $\mathcal{R} \subseteq \mathcal{Q}$.

Finally, we can iteratively eliminate the sets from $\mathcal{Q}$ not in $\mathcal{R}, \mathcal{M}_-$ and $\mathcal{M}_+$ in polynomial time. ∎

Lemma 20. *Algorithms 2 and 3 run in time polynomial in the size of the input $\mathcal{H}$.*

Proof. Both algorithms execute a simple search which runs in polynomial time in the number of hyperedges. It remains to show we can find a safe source and a safe sink in polynomial time. By symmetry, we only show how to find a safe source in a given $S \in \mathcal{M}_-$. For $u, v \in S$, let $X_v^u = S_{u,v}^{\vec{\mathcal{H}}}$.

Claim 20.1. *The vertex u is not a safe source in S if and only if there exists a vertex $v \in S - u$ such that either $X_v^u \in \mathcal{T}_+$ or $X_v^u \in \mathcal{D}_+$ and X_v^u does not contain a set $Z \in \mathcal{T}_+$.*

Proof. The sufficiency follows from the definition of a safe source. To prove the necessity, we suppose that u is not a safe source in S and $X_v^u \notin \mathcal{T}_+$ for all $v \in S - u$. Then there exists by definition, for some vertex $v \in S - u$, a minimum out-degree (u, v)-separator Y_v^u such that $Y_v^u \in \mathcal{D}_+$ where there is no set $Z \in \mathcal{T}_+$ such that $Z \subseteq Y_v^u - u$. Since $X_u^v \subseteq Y_u^v$, we get that X_v^u does not contain a set $Z \in \mathcal{T}_+$. $\square$

To compute whether u is a safe source in S or not, it then suffices, for all $v \in S - u$, to check whether $X_v^u \in \mathcal{T}_+$ or whether $X_v^u \in \mathcal{D}_+$ contains a set $Z \in \mathcal{T}_+$, which can be done in polynomial time. By Lemma 10, S contains a safe source, thus there exists at least one vertex $s \in S$ that does not satisfy the conditions above which can be checked in polynomial time. $\blacksquare$

It follows from the previous lemmas that Algorithm 1 runs in polynomial time.

12   Concluding remarks

We generalized the results of Ito et al. [8] to hypergraphs and gave an algorithmic proof of Theorem 5. Several interesting questions remain open. We do not know if the $|V|^3$ bound on the number of hyperarc reorientations is tight. Generalizing this result to general directed hypergraphs with tail vertex sets and head vertex sets for each hyperarc would also provide a general characterization of k-hyperarc-connected orientable hypergraphs, which is not known to our knowledge.

References

[1] Attila Bernáth, Satoru Iwata, Tamás Király, Zoltán Király, and Zoltán Szigeti. Recent results on well-balanced orientations. *Discrete Optimization*, 5(4):663–676, 2008.

[2] Jack Edmonds and Richard M Karp. Theoretical improvements in algorithmic efficiency for network flow problems. *Journal of the ACM (JACM)*, 19(2):248–264, 1972.

[3] András Frank. On the orientation of graphs. *Journal of Combinatorial Theory, Series B*, 28(3):251–261, 1980.

[4] András Frank. A note on k-strongly connected orientations of an undirected graph. *Discrete Mathematics*, 39(1):103–104, 1982.

[5] András Frank. *Connections in combinatorial optimization*, volume 38. Oxford University Press Oxford, 2011.

[6] András Frank, Tamás Király, and Zoltán Király. On the orientation of graphs and hypergraphs. *Discrete Applied Mathematics*, 131(2):385–400, 2003.

[7] Giorgio Gallo, Giustino Longo, Stefano Pallottino, and Sang Nguyen. Directed hypergraphs and applications. *Discrete Applied Mathematics*, 42(2-3):177–201, 1993.

[8] Takehiro Ito, Yuni Iwamasa, Naonori Kakimura, Naoyuki Kamiyama, Yusuke Kobayashi, Shun-ichi Maezawa, Yuta Nozaki, Yoshio Okamoto, and Kenta Ozeki. Monotone edge flips to an orientation of maximum edge-connectivity à la Nash-Williams. *ACM Transactions on Algorithms (TALG)*, 19(1), 2023.

[9] László Lovász. Connectivity in digraphs. *Journal of Combinatorial Theory, Series B*, 15(2):174–177, 1973.

[10] László Lovász. *Combinatorial problems and exercises*. North Holland, 1979.

[11] Crispin St. J. A. Nash-Williams. On orientations, connectivity and odd-vertex-pairings in finite graphs. *Canadian Journal of Mathematics*, 12:555–567, 1960.

[12] Zoltán Szigeti. Orientations of graphs. *Matemática Contemporânea*, 39:179–188, 2010.