

A note on 2-vertex-connected orientations

Florian Hörsch

TU Ilmenau, Weimarer Straße 25, Ilmenau, Germany, 98693.

Zoltán Szigeti

Univ. Grenoble Alpes, Grenoble INP, CNRS, G-SCOP, 46 Avenue Félix Viallet, Grenoble, France, 38000.

Abstract

We consider two possible extensions of a theorem of Thomassen characterizing the graphs admitting a 2-vertex-connected orientation. First, we show that the problem of deciding whether a mixed graph has a 2-vertex-connected orientation is NP-hard. This answers a question of Bang-Jensen, Huang and Zhu. For the second part, we call a directed graph $D = (V, A)$ $2T$ -connected for some $T \subseteq V$ if D is 2-arc-connected and $D - v$ is strongly connected for all $v \in T$. We deduce a characterization of the graphs admitting a $2T$ -connected orientation from the theorem of Thomassen.

1. Introduction

In this article, we deal with two possible extensions of a theorem of Thomassen characterizing graphs having a 2-vertex-connected orientation. All undefined notions can be found in Section 2.

During the history of graph orientations, the question of characterizing graphs having orientations with certain connectivity properties has played a central role. The following fundamental theorem of Robbins [8] dates back to 1939.

Theorem 1. *A graph has a strongly connected orientation if and only if it is 2-edge-connected.*

For higher arc-connectivity, this theorem was later generalized by Nash-Williams [7].

Theorem 2. *Let G be a graph and k a positive integer. Then G has a k -arc-connected orientation if and only if G is $2k$ -edge-connected.*

The analogous problem for vertex-connectivity turns out to be much more complicated. The following conjecture was proposed by Frank in [6].

Email addresses: florian.hoersch@tu-ilmenau.de (Florian Hörsch),
Zoltan.Szigeti@grenoble-inp.fr (Zoltán Szigeti)

Conjecture 1. *Let $G = (V, E)$ be a graph and k a positive integer. Then G has a k -vertex-connected orientation if and only if $|V| \geq k + 1$ and $G - X$ is $2(k - |X|)$ -edge-connected for all $X \subseteq V$ with $|X| \leq k - 1$.*

Although Conjecture 1 remained open for a long time, little progress was made on it. Finally, Conjecture 1 was proven for $k = 2$ by Thomassen [11]. More explicitly, he proved the following theorem.

Theorem 3. *A graph G has a 2-vertex-connected orientation if and only if G is 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in V$.*

On the other hand, Conjecture 1 was disproven for every $k \geq 3$ by Durand de Gevigney [3]. Moreover, he proved the following result which makes a good characterization of the graphs admitting a k -vertex-connected orientation for any $k \geq 3$ seem out of reach.

Theorem 4. *The problem of deciding whether a given graph has a k -vertex-connected orientation is NP-hard for any $k \geq 3$.*

It remains interesting to search for some big class of graphs that admit highly vertex-connected orientations. The following conjecture was proposed by Thomassen [10].

Conjecture 2. *There is a function $f : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ such that every $f(k)$ -vertex-connected graph has a k -vertex-connected orientation for all $k \in \mathbb{Z}_+$.*

Conjecture 2 remains open for all $k \geq 3$.

In this article, we deal with two possible extensions of Theorem 3. In the first part, we deal with a possible generalization of Theorem 3 to the case when some of the edges are pre-oriented. The following is the first important result on orientations of mixed graphs satisfying connectivity properties. It was proven by Boesch and Tindell [2].

Theorem 5. *A mixed graph $G = (V, A \cup E)$ has a strongly connected orientation if and only if $d_A^-(X) + \frac{1}{2}d_E(X) \geq 1$ for every nonempty $X \subsetneq V$.*

For general arc-connectivity, this problem has been solved by Frank [5] who obtained a pretty technical characterization of mixed graphs admitting a k -arc-connected orientation for all $k \in \mathbb{Z}_+$ using the theory of generalized polymatroids. For higher vertex-connectivity, the possibility of a good characterization of the mixed graphs admitting a k -vertex-connected orientation has been ruled out by Theorem 4 for any $k \geq 3$. However, the case of $k = 2$ remained open.

The first main contribution of this work is to show that there is also no hope to find a good characterization for this problem. More formally, we consider the following algorithmic problem:

2-vertex-connected orientation of mixed graphs (2VCOMG):

Input: A mixed graph $G = (V, A \cup E)$.

Question: Does G have a 2-vertex-connected orientation?

The question of determining the complexity of this problem was first hinted at by Thomassen in [11] and then asked explicitly by Bang-Jensen, Huang and Zhu [1]. Our main contribution is the following answer to this problem.

Theorem 6. *2VCOMG is NP-hard.*

Our reduction that proves Theorem 6 is inspired by the one used by Durand de Gevigney when proving Theorem 4.

In the second part, we deal with a connectivity property that generalizes both 2-vertex-connectivity and 2-arc-connectivity and was introduced by Durand de Gevigney and the second author in [4]. Namely a given digraph $D = (V, A)$ is called *2T-connected* for some $T \subseteq V$ if D is 2-arc-connected and $D - v$ is strongly connected for all $v \in T$. We prove the following theorem characterizing the graphs $G = (V, E)$ admitting a 2T-connected orientation for some given $T \subseteq V$.

Theorem 7. *Let G be a graph and $T \subseteq V(G)$. Then G has a 2T-connected orientation if and only if G is 4-edge-connected and $G - v$ is 2-edge-connected for all $v \in T$.*

Observe that Theorem 7 implies both Theorem 3 and Theorem 2 for $k = 2$ as 2T-connectivity corresponds to 2-arc-connectivity for $T = \emptyset$ and to 2-vertex-connectivity for $T = V$. The proof of Theorem 7 works by a rather simple deduction from Theorem 3. It would be nice to find a proof of Theorem 7 that does not use Theorem 3 and hence to get a transparent proof of Theorem 3.

The rest of this article is structured as follows: In Section 2, we give some more formal definitions and some preliminary results. In Section 3, we give the reduction that proves Theorem 6. In Section 4, we prove Theorem 7. Finally, in Section 5, we conclude our work.

2. Preliminaries

We first give some basic notation in graph theory. A *mixed graph* consists of a vertex set V , an arc set A and an edge set E . If $A = \emptyset$, then G is a graph and if $E = \emptyset$, then G is a digraph. For a single vertex v , we often use v instead of $\{v\}$. For some mixed graph $G = (V, A \cup E)$ and some $X \subseteq V$, we use $d_A^-(X)$ for the number of arcs in A whose tail is in $V - X$ and whose head is in X , $d_A^+(X)$ for $d_A^-(V - X)$ and $d_E(X)$ to denote the number of edges in E that have exactly one endvertex in X . For some $u, v \in V$, an *uv-path* in G is a sequence of vertices v_1, \dots, v_t such that $u = v_0, v = v_t$ and for all $i = 0, \dots, t-1$ either $v_i v_{i+1} \in E$ or $v_i v_{i+1} \in A$. Two *uv-paths* are called *internally disjoint* if they share no vertices apart from u and v . For a vertex set $X \subseteq V$ and a vertex $v \in V - X$, a *(v, X)-path* is a path from v to a vertex of X . Similarly, a *(X, v)-path* is a path from a vertex of X to v . Further, for some $X \subseteq V$, G is called *k-vertex-connected in X* if $|V| \geq k + 1$ and there are k internally disjoint *uv-paths* for any $u, v \in X$. Also, G is called *k-vertex-connected* if G is *k-vertex-connected in V*. For some $X \subseteq V$, we denote by $G[X]$ the subgraph of G induced on X .

A graph $G = (V, E)$ is called *k-edge-connected* for some positive integer k if $d_E(X) \geq k$ for every nonempty $X \subsetneq V$. A digraph $D = (V, A)$ is called *k-arc-connected* for some positive integer k if $d_A^-(X) \geq k$ for every nonempty $X \subsetneq V$. If D is 1-arc-connected then we say that it is *strongly connected*.

A connected graph with every vertex of degree 2 is called a *cycle* and a *double cycle* is obtained from a cycle by duplicating every edge. A strongly connected orientation of a cycle is called a *circuit*. A digraph $D = (V, A)$ whose underlying graph does not contain a cycle is called an *r-in-arborescence* (*r-out-arborescence*) if $r \in V$ and D contains a path from v to r (from r to v) for every $v \in V$.

Given two graphs G and H and a vertex v of G , *blowing up* v into H means that we replace v by H and we replace every edge wv incident to v in G by an edge wu for some vertex u in H .

We now give one basic result on vertex-connectivity in digraphs.

Proposition 1. *Let $D = (V, A)$ be a digraph, $X \subseteq V$ such that D is 2-vertex-connected in X and $v \in V - X$. If D contains two (v, X) -paths whose vertex sets only intersect in v and D contains two (X, v) -paths whose vertex sets only intersect in v , then D is 2-vertex-connected in $X \cup v$.*

We also need one property on edge-connectivity in graphs.

Proposition 2. *Given two graphs G and H and a vertex v of G , if G and H are k -edge-connected then so is the graph obtained from G by blowing up v into H .*

The algorithmic problem we need for our reduction is MNAE3SAT.

Monotone not-all-equal-3SAT (MNAE3SAT)

Input: A set X of boolean variables, a formula consisting of a set \mathcal{C} of clauses each containing 3 distinct variables, none of which are negated.

Question: Is there a truth assignment to the variables of X such that every clause in \mathcal{C} contains at least one true and at least one false literal?

An assignment satisfying the above condition will be called *feasible*.

This problem will be used in the reduction which is justified by the following result due to Schaefer [9].

Theorem 8. *MNAE3SAT is NP-complete.*

3. The reduction

Let $\Phi = (X, \mathcal{C})$ be an instance of MNAE3SAT. The set of pairs (x, C) such that $x \in C \in \mathcal{C}$ is denoted by $P(\Phi)$. In the following, we first create an instance of 2VCOMG and then show that it is a positive instance if and only if Φ is a positive instance of MNAE3SAT.

We construct a mixed graph $G = (V, A \cup E)$ as follows. First, let V contain a set Q of three vertices p, q and r . Further, V contains a set Z containing one vertex z_C for every $C \in \mathcal{C}$. Finally, for every $(x, C) \in P(\Phi)$, V contains a set R_C^x of 4 vertices $\{t_C^x, u_C^x, w_C^x, y_C^x\}$. First, let A contain the arcs pq, qp, pr, rp, qr, rq . Further, for every $C \in \mathcal{C}$, A contains the arcs pz_C and z_Cq . Finally, for every $(x, C) \in P(\Phi)$, A contains the arcs of the path $p, t_C^x, u_C^x, y_C^x, w_C^x, q$. First, let E contain an edge $z_C u_C^x$ for every $(x, C) \in P(\Phi)$. Now for every $x \in X$, let $C_1, \dots, C_{\mu(x)}$ be an arbitrary ordering of the clauses in \mathcal{C} containing x . Let

$b_1^x = r$ and for $i = 1, \dots, \mu(x)$, let $b_{3i-1}^x = y_{C_i}^x$, $b_{3i}^x = w_{C_i}^x$ and $b_{3i+1}^x = t_{C_i}^x$. We add the edges of the cycle $B^x = b_1^x, b_2^x, \dots, b_{3\mu(x)+1}^x, b_1^x$ to E . This finishes the construction of G . Note that the size of G is clearly polynomial in the size of Φ . A drawing can be found in Figure 1.

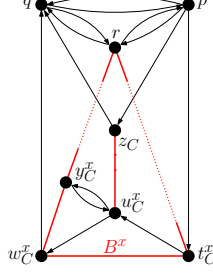


Figure 1: A schematic drawing of G containing Q and R_C^x and z_C for some $(x, C) \in P(\Phi)$.

For some $x \in X$, we will refer to the circuit $b_1^x, b_2^x, \dots, b_{3\mu(x)+1}^x, b_1^x$ as $\overrightarrow{B^x}$ and to the circuit $b_1^x, b_{3\mu(x)+1}^x, \dots, b_2^x, b_1^x$ as $\overleftarrow{B^x}$.

To show that G is a positive instance of 2VCOMG if and only if Φ is a positive instance of MNAE3SAT we need the following lemma.

Lemma 1. *An orientation $\vec{G} = (V, A \cup \vec{E})$ of G is 2-vertex-connected if and only if*

$$\vec{G}[B^x] = \overrightarrow{B^x} \text{ or } \vec{G}[B^x] = \overleftarrow{B^x} \quad \text{for every } x \in X, \quad (1)$$

$$u_C^x z_C \in \vec{E} \text{ if and only if } \vec{G}[B^x] = \overrightarrow{B^x} \quad \text{for every } (x, C) \in P(\Phi), \quad (2)$$

$$u_C^{x_1} z_C, z_C u_C^{x_2} \in \vec{E} \text{ for some } x_1, x_2 \in C \quad \text{for every } C \in \mathcal{C}. \quad (3)$$

Proof First suppose that \vec{G} is 2-vertex-connected.

Since for every $(x, C) \in P(\Phi)$, the vertices t_C^x, w_C^x and y_C^x have one arc entering in A , one arc leaving in A and two edges entering in E , (1) follows.

Let $(x, C) \in P(\Phi)$. For some i , we have $y_C^x = b_{3i-1}^x$. Since $\vec{G} - t_C^x$ is strongly connected, $\{u_C^x, w_C^x, y_C^x\}$ has no arc entering in A and two edges entering in E , at least one of $z_C u_C^x$ and $b_{3i-2}^x b_{3i-1}^x$ exists in \vec{E} . Since $\vec{G} - w_C^x$ is strongly connected, $\{u_C^x, y_C^x\}$ has no arc leaving in A and two edges entering in E , at least one of $u_C^x z_C$ and $b_{3i-1}^x b_{3i-2}^x$ exists in \vec{E} . We obtain that $u_C^x z_C \in \vec{E}$ if and only if $b_{3i-2}^x b_{3i-1}^x \in \vec{E}$. Now (1) yields (2).

Since for every $C \in \mathcal{C}$, the vertex z_C has one arc entering in A , one arc leaving in A and three edges entering in E , (3) follows.

Now suppose that (1), (2) and (3) hold.

We first show that \vec{G} is 2-vertex-connected in $Q \cup R_C^x$ for every $(x, C) \in P(\Phi)$. We fix some $(x, C) \in P(\Phi)$ and for convenience, we denote $z_C, t_C^x, u_C^x, w_C^x, y_C^x$ by z, t, u, w, y , respectively. Note that $\vec{G}[Q]$ is 2-vertex-connected. We distinguish two cases depending on the orientation of B^x in \vec{G} . By (1), we have either $\vec{G}[B^x] = \overrightarrow{B^x}$ or $\vec{G}[B^x] = \overleftarrow{B^x}$.

Case 1. $\vec{G}[B^x] = \vec{B}^x$. Observe that $\vec{G}[B_x]$ consists of a path S_1 from r to y disjoint from $\{t, w\}$, of the arcs yw, wt and of a path S_2 from t to r disjoint from $\{y, w\}$. By (2), we have $uz \in \vec{E}$. Let F_1 be the r -out-arborescence consisting of S_1 and the arcs yu, yw and wt . Let F_2 be the p -out-arborescence consisting of the arcs pt, tu, uw and uy . Then F_1 and F_2 contain two (Q, v) -paths whose vertex sets only intersect in v for every vertex v in R_C^x . Let F_3 be the r -in-arborescence consisting of S_2 and the arcs yw, uw and wt . Let F_4 be the q -in-arborescence consisting of the arcs tu, yu, uz, zq and wq . Then F_3 and F_4 contain two (v, Q) -paths whose vertex sets only intersect in v for every vertex v in R_C^x . An illustration can be found in Figure 2.

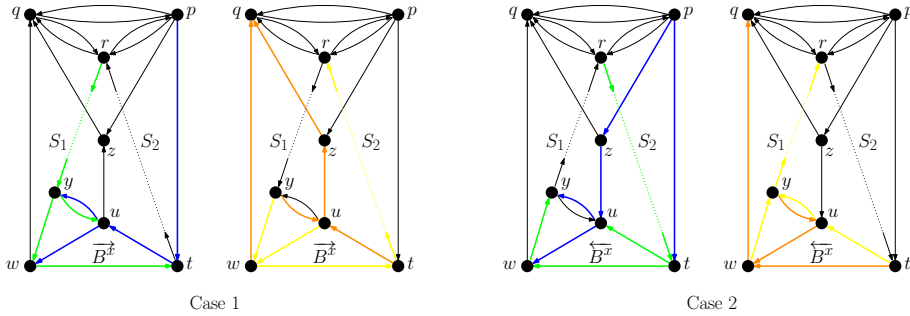


Figure 2: An illustration for the two cases in the proof of Lemma 1. The out-arborescences F_1, F_2 and the in-arborescences F_3, F_4 are depicted in green, blue, yellow and orange, respectively.

Case 2. $\vec{G}[B^x] = \overleftarrow{B}^x$. Observe that $\vec{G}[B_x]$ consists of a path S_1 from y to r disjoint from $\{t, w\}$, of the arcs tw, wy and of a path S_2 from r to t disjoint from $\{y, w\}$. By (2), we have $zu \in \vec{E}$. Let F_1 be the r -out-arborescence consisting of S_2 and the arcs tu, tw and wy . Let F_2 be the p -out-arborescence consisting of the arcs pt, pz, zu, uw and uy . Then F_1 and F_2 contain two (Q, v) -paths whose vertex sets only intersect in v for every vertex v in R_C^x . Let F_3 be the r -in-arborescence consisting of S_1 and the arcs tu, uy and wy . Let F_4 be the q -in-arborescence consisting of the arcs tw, uw, yu and wq . Then F_3 and F_4 contain two (v, Q) -paths whose vertex sets only intersect in v for every vertex v in R_C^x . An illustration can be found in Figure 2.

In either case, we obtain by Proposition 1, that \vec{G} is 2-vertex-connected in $Q \cup R_C^x$. As (x, C) was chosen arbitrarily, we in fact obtained that \vec{G} is 2-vertex-connected in $V - Z$.

To finish the proof we consider some $C \in \mathcal{C}$. By (3), $u_C^{x_1} z_C, z_C u_C^{x_2} \in \vec{E}$ for some $x_1, x_2 \in C$. Further, $z_C q, p z_C \in A$. Then Proposition 1 yields that \vec{G} is 2-vertex-connected in $(V - Z) \cup z_C$. As C was chosen arbitrarily, the proof of Lemma 1 is finished. ■

Lemma 2. *There exists a feasible truth assignment for Φ if and only if G has a 2-vertex-connected orientation.*

Proof First suppose that there exists a feasible truth assignment $f : X \rightarrow \{\text{true}, \text{false}\}$ for Φ . We create an orientation \vec{G} of G in the following way: for

every $x \in X$, we orient B^x as $\overrightarrow{B^x}$ if $f(x) = true$ and as $\overleftarrow{B^x}$ if $f(x) = false$. Further, for every $(x, C) \in P(\Phi)$, we orient $z_C u_C^x \in E$ from u_C^x to z_C if $f(x) = true$ and from z_C to u_C^x if $f(x) = false$. Observe that (1) and (2) hold. Since f is feasible for Φ , (3) also holds. Then, by Lemma 1, \vec{G} is 2-vertex-connected.

Now suppose that G has a 2-vertex-connected orientation \vec{G} . Then, by Lemma 1, (1), (2) and (3) hold. For every $x \in X$, by (1), we have $\vec{G}[B^x] = \overrightarrow{B^x}$ or $\vec{G}[B^x] = \overleftarrow{B^x}$. We can hence define a truth assignment f as follows: we set $f(x) = true$ if $\vec{G}[B^x] = \overrightarrow{B^x}$ and $false$ if $\vec{G}[B^x] = \overleftarrow{B^x}$. For every $C \in \mathcal{C}$, by (3), there exist arcs $u_C^{x_1} z_C$ and $z_C u_C^{x_2}$ for some $x_1, x_2 \in C$. By (2), we have $\vec{G}[B^{x_1}] = \overrightarrow{B^{x_1}}$ and $\vec{G}[B^{x_2}] = \overleftarrow{B^{x_2}}$. We obtain that $f(x_1) = true$ and $f(x_2) = false$. This implies that f is feasible for Φ . ■

By Lemma 2 and Theorem 8, the proof of Theorem 6 is finished.

4. Orientations for $2T$ -connectivity

This section is dedicated to proving Theorem 7.

Proof(of Theorem 7) Necessity is evident.

To prove the sufficiency, let H be obtained from $G = (V, E)$ by blowing up every vertex $v \in V - T$ into a double cycle C_v on a vertex set of size $\max\{3, \lceil \frac{d_G(v)}{2} \rceil\}$ such that every new vertex is incident to a set F_v of at most 2 edges not belonging to C_v .

Claim 1. H is 4-edge-connected and $H - w$ is 2-edge-connected for all $w \in V(H)$.

Proof Since G and C_v for all $v \in V - T$ are 4-edge-connected, so is H by Proposition 2.

Now let $w \in V(H)$. If $w \in T$, then since $G - w$ and C_v for all $v \in V - T$ are 2-edge-connected, so is $H - w$ by Proposition 2. Otherwise, $w \in V(C_u)$ for some $u \in V - T$. Note that $G' = G - F_u$ is 2-edge-connected because G is 4-edge-connected. Further, $C_u - u$ is 2-edge-connected. Observe that $H - w$ is the graph obtained from G' by blowing up every vertex $v \in (V - u) - T$ into C_v and then blowing up u into $C_u - u$. It follows, by Proposition 2, that $H - w$ is 2-edge-connected. ■

By Claim 1 and Theorem 6, we obtain that H has a 2-vertex-connected orientation \vec{H} . Now let \vec{G} be obtained from contracting $V(C_v)$ into v for all $v \in V(G) - T$. We will show that \vec{G} is $2T$ -connected. Since \vec{H} is 2-vertex-connected, we obtain that \vec{H} is also 2-arc-connected. As \vec{G} is obtained from \vec{H} through contractions, we obtain that \vec{G} is also 2-arc-connected. Now let $v \in T$. Since \vec{H} is 2-vertex-connected, we obtain that $\vec{H} - v$ is strongly connected. As $\vec{G} - v$ is obtained from $\vec{H} - v$ through contractions, we obtain that $\vec{G} - v$ is also strongly connected. ■

5. Conclusion

We show that the problem of deciding whether a mixed graph has a 2-vertex-connected orientation is NP-hard and give a characterization for the graphs admitting a $2T$ -connected orientation. The first result closes the dichotomy for the problem of finding k -vertex-connected orientations of mixed graphs.

In the spirit of Conjecture 2, we pose the following problem.

Conjecture 3. *There is a function $f : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ such that every $f(k)$ -vertex-connected mixed graph has a k -vertex-connected orientation for all $k \in \mathbb{Z}_+$.*

Clearly, for any fixed $k \geq 3$, Conjecture 3 implies Conjecture 2. It would be interesting to see whether Conjecture 3 is tractable more easily for $k = 2$.

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