Minimal Connected τ-Critical Hypergraphs

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Abstract. A hypergraph $\mathscr{H}$ is τ-critical if $\tau(\mathscr{H} - \{E\}) < \tau(\mathscr{H})$ for every edge $E \in \mathscr{H}$, where $\tau(\mathscr{H})$ denotes the transversal number of $\mathscr{H}$. We show that if $\mathscr{H}$ is a connected τ-critical hypergraph, then $\mathscr{H} - \{E\}$ can be partitioned into $\tau(\mathscr{H}) - 1$ stars of size at least two, for every edge $E \in \mathscr{H}$. An immediate corollary is that a connected τ-critical hypergraph $\mathscr{H}$ has at least $2\tau(\mathscr{H}) - 1$ edges. This extends, in a very natural way, a classical theorem of Gallai on colour-critical graphs, and is equivalent to a theorem of Füredi on t-stable hypergraphs. We deduce a lower bound on the size of τ-critical hypergraphs of minimum degree at least two.

Key words. τ-critical hypergraphs, colour-critical graphs, t-stable hypergraphs

1. Introduction

Throughout this paper a *hypergraph* $\mathscr{H}$ is a finite family of finite non-empty sets called the *edges* of $\mathscr{H}$. The *vertices* of $\mathscr{H}$ are the elements of the set $V(\mathscr{H}) = \bigcup_{E \in \mathscr{H}} E$. Note that our definition of a hypergraph does not allow isolated vertices or multiple edges. A set $T \subseteq V(\mathscr{H})$ is a *transversal* (also *vertex cover* or *blocking set*) of $\mathscr{H}$ if $T \cap E \neq \emptyset$ for every $E \in \mathscr{H}$. The smallest cardinality of a transversal of $\mathscr{H}$ is the *transversal number* $\tau(\mathscr{H})$; if $\tau(\mathscr{H}) = 1$ we say $\mathscr{H}$ is a *star*. A hypergraph $\mathscr{H}$ is τ-*critical* if $\tau(\mathscr{H} - \{E\}) < \tau(\mathscr{H})$ for every $E \in \mathscr{H}$. A *path* is a sequence $(E_1, \ldots, E_k)$ of edges such that $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$, and a hypergraph is *connected* if every pair of its edges lies in a path.

A number of authors have studied τ-critical hypergraphs; see for example [1–3,5]. It is easy to see that every component of a τ-critical hypergraph must itself be τ-critical, so it seems natural to study *connected* τ-critical hypergraphs. The main result of this paper is the following.

Theorem 1. *If E is any edge of a connected τ-critical hypergraph $\mathscr{H}$, then $\mathscr{H} - \{E\}$ can be partitioned into $\tau(\mathscr{H}) - 1$ stars of size at least two.*

The proof of the theorem is given in Section 2. In the remainder of this section we will show how Theorem 1 can be used to deduce further results about τ-critical hypergraphs, as well as results about χ-vertex-critical graphs, t-stable hypergraphs and ν-stable graphs.

The *line graph* $L(\mathscr{H})$ of a hypergraph $\mathscr{H}$ is the graph on $\mathscr{H}$ such that two vertices are adjacent if and only if they intersect in $\mathscr{H}$. A partition of a hypergraph into 2-stars corresponds to a partition of the vertices of its line graph into 2-cliques, which is just a perfect matching. A graph G is *factor-critical* if $G - x$ has a perfect matching, for every vertex $x \in V(G)$. The following result follows immediately from Theorem 1.

Theorem 2. *If $\mathscr{H}$ is a connected τ-critical hypergraph, then $|\mathscr{H}| \geqslant 2\tau(\mathscr{H}) - 1$. If equality holds then $L(\mathscr{H})$ is a factor-critical graph.*

It is not hard to check that every odd cycle achieves equality in the above bound, and that these are the only extremal *graphs*. However, there are many other extremal hypergraphs, for example the hypergraph $\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,3,5\}\}$. The extremal hypergraphs turn out to be rather interesting objects; they are studied in detail in a forthcoming paper [11] (the extended abstract is published in [10]).

Theorem 2 can be used to prove the following result; the very easy proof is omitted.

Corollary 1. *Suppose $\mathscr{H}$ is a τ-critical hypergraph $\mathscr{H}$ of minimal degree at least two. Then $|\mathscr{H}| \geqslant \frac{3}{2}\tau(\mathscr{H})$, with equality if and only if $\mathscr{H}$ is the disjoint union of $\frac{1}{2}\tau(\mathscr{H})$ triangles.*

A graph G is *χ-vertex-critial* if $\chi(G - x) < \chi(G)$ for every $x \in V(G)$, where $\chi(G)$ denotes the chromatic number of G. There is a close link between χ-vertex-critical graphs and τ-critical hypergraphs, which we will now explain.

A hypergraph satisfies the *Helly property* if every intersecting partial hypergraph is a star. With any graph G we can associate a Helly hypergraph $\mathscr{A}^*(G)$ such that $L(\mathscr{A}^*(G)) = \overline{G}$, where $\overline{G}$ denotes the complement of G. Namely, let $\mathscr{A}^*(G)$ be the dual of the hypergraph $\mathscr{A}(G)$ formed with the maximal independent sets of G. Note that $|\mathscr{A}^*| = |V(G)|$, $\mathscr{A}^*(G)$ is connected if and only if G has a connected complement, and a colouring of the vertices of G corresponds to a partition of $\mathscr{A}^*(G)$ into stars. In particular, $\tau(\mathscr{A}^*(G)) = \chi(G)$ and $\mathscr{A}^*(G)$ is τ-critical if and only if G is χ-vertex-critical. The restriction of Theorem 1 to Helly hypergraphs is therefore equivalent to the following result of the author [9].

Theorem 3. *If x is any vertex of a χ-vertex-critical graph G with a connected complement, then $G - x$ has a $(\chi(G) - 1)$-colouring with all colour classes of size at least two.*

Similarly, the restriction of Theorem 2 to Helly hypergraphs is equivalent to the following classical result of Gallai [7], also proved in [8].

Theorem 4. *If G is a χ-vertex-critical graph and its complement is connected, then $|V(G)| \geqslant 2\chi(G) - 1$. If equality holds then $\overline{G}$ is a factor-critical graph.*

Füredi [4] defined the *expansion number* $t(\mathscr{H})$ of a hypergraph $\mathscr{H}$ as the maximum value of $|V(\mathscr{H}')| - |\mathscr{H}'|$ taken over all partial hypergraphs $\mathscr{H}' \subseteq \mathscr{H}$. A hypergraph is said to be *t-stable* if $t(\mathscr{H} - x) = t(\mathscr{H})$ holds for all vertices $x \in V(\mathscr{H})$, where $\mathscr{H} - x = \{E \in \mathscr{H} \mid x \notin E\}$. It can be shown that $t(\mathscr{H}) = |V(\mathscr{H})| - \rho(\mathscr{H})$, where $\rho(\mathscr{H})$ denotes the edge-covering number of $\mathscr{H}$, and therefore $t(\mathscr{H}) = |\mathscr{H}^*| - \tau(\mathscr{H}^*)$, where $\mathscr{H}^*$ denotes the dual of $\mathscr{H}$. With a little more work, we can show the following.

Proposition 1. *A hypergraph $\mathscr{H}$ is t-stable if and only if the dual $\mathscr{H}^*$ is τ-critical.*

Proof. Suppose $\mathscr{H}$ is t-stable and let X be any edge of $\mathscr{H}^*$. Then $\tau((\mathscr{H} - x)^*) < \tau(\mathscr{H}^*)$, where x denotes the vertex of $\mathscr{H}$ corresponding to X. Since $\mathscr{H}$ is t-stable it must be a clutter, so any transversal of $(\mathscr{H} - x)^*$ is a transversal of $\mathscr{H}^* - \{X\}$. Hence $\tau(\mathscr{H}^* - \{X\}) \leqslant \tau((\mathscr{H} - x)^*) < \tau(\mathscr{H}^*)$, and it follows that $\mathscr{H}^*$ is τ-critical.

Conversely, suppose $\mathscr{H}^*$ is τ-critical. Let x be any vertex of $\mathscr{H}$ and denote the corresponding edge of $\mathscr{H}^*$ by X. Then $\tau(\mathscr{H}^* - \{X\}) = \tau(\mathscr{H}^*) - 1$, and it can be shown that any minimal transversal of $\mathscr{H}^* - \{X\}$ is a minimal transversal of $(\mathscr{H} - x)^*$. Hence $\tau((\mathscr{H} - x)^*) = \tau(\mathscr{H}^*) - 1$, and it follows that $\mathscr{H}$ is t-stable. □

Theorem 1 can therefore be rephrased in terms of t-stable hypergraphs as follows.

Theorem 5. *If x is any vertex of a connected t-stable hypergraph $\mathscr{H}$, then $V(\mathscr{H}) - \{x\}$ can be partitioned into $|V(\mathscr{H})| - t(\mathscr{H}) - 1$ classes of size at least two, so that each class is a subset of some edge of $\mathscr{H} - x$.*

The 2-*section* $\mathscr{H}_{[2]}$ of a hypergraph $\mathscr{H}$ is the graph on $V(\mathscr{H})$ such that two vertices are adjacent if and only if they lie in a common edge of $\mathscr{H}$. As a corollary of Theorem 5 we obtain the following result of Füredi [4], which may be thought of as the dual of Theorem 2.

Theorem 6. *If $\mathscr{H}$ is a connected t-stable hypergraph, then $|V(\mathscr{H})| \leqslant 2t(\mathscr{H}) + 1$. If equality holds then $\mathscr{H}_{[2]}$ is a factor-critical graph.*

It can be shown that $t(G)$ equals the matching number $\nu(G)$ for any graph G. A graph G is ν-stable if $\nu(G - x) = \nu(G)$ for every vertex $x \in V(G)$. Theorem 6 clearly implies the following result of Gallai [6].

Theorem 7. *Every connected ν-stable graph is factor-critical.*

2. Proof of Theorem 1

Before commencing the proof of Theorem 1, let us make the following definitions. A finite family $\mathfrak{S}$ of edge-disjoint stars is a *galaxy*; a subfamily of a galaxy is a *subgalaxy*. The *edge hypergraph* of a galaxy $\mathfrak{S}$ is the hypergraph $\mathscr{E}(\mathfrak{S}) = \bigcup_{\mathscr{S} \in \mathfrak{S}} \mathscr{S}$; the partial hypergraph formed by the edges of 1-stars (stars of size one) of $\mathfrak{S}$ is denoted by $\mathscr{I}(\mathfrak{S})$. If $\mathscr{E}(\mathfrak{S}) = \mathscr{H}$, we say that $\mathfrak{S}$ is a galaxy *of $\mathscr{H}$*. If $\mathscr{F} \subseteq \mathscr{E}(\mathfrak{S})$, the galaxy *induced by* $\mathscr{F}$ is $\mathfrak{S}[\mathscr{F}] = \{\mathscr{S} \cap \mathscr{F} \mid \mathscr{S} \in \mathfrak{S}\}$. A galaxy $\mathfrak{S}$ is *minimal* if there is no galaxy $\mathfrak{S}'$ such that $\mathscr{E}(\mathfrak{S}') = \mathscr{E}(\mathfrak{S})$ and $|\mathfrak{S}'| < |\mathfrak{S}|$; it is easy to see that a galaxy $\mathfrak{S}$ is minimal if and only if $|\mathfrak{S}| = \tau(\mathscr{E}(\mathfrak{S}))$. An edge $E \in \mathscr{H}$ is *τ-critical* if $\tau(\mathscr{H} - \{E\}) < \tau(\mathscr{H})$; clearly, an edge $E \in \mathscr{H}$ is τ-critical if and only if there exists a minimal galaxy $\mathfrak{S}$ of $\mathscr{H}$ such that $\{E\} \in \mathfrak{S}$. A galaxy $\mathfrak{S}$ is *E-extreme* if $\mathfrak{S}$ is minimal, $\{E\} \in \mathfrak{S}$, and $\mathfrak{S} - \{\{E\}\}$ has the minimal number of 1-stars; note that $\mathfrak{S}$ is E-extreme only if E is a τ-critical edge of $\mathscr{E}(\mathfrak{S})$.

As Theorem 1 extends a result of the author [9], the arguments in the proof are similar to those in [9]. The theorem is proved using five lemmas and one corollary. Lemmas 1 and 2 are simple but useful observations about minimal galaxies. Lemmas 3 and 4 give important properties of the union of two galaxies. These properties allow us to modify an E_1-extreme galaxy $\mathfrak{S}_1$ to obtain an E_2-extreme galaxy $\mathfrak{S}_2$ so that $\mathscr{I}(\mathfrak{S}_1) - \{E_1\} = \mathscr{I}(\mathfrak{S}_2) - \{E_2\}$; this is the subject of Lemma 5 and Corollary 2. The proof of Theorem 1 then follows easily.

Lemma 1. *If $\mathfrak{S}$ is a minimal galaxy and $\mathfrak{R} \subseteq \mathfrak{S}$, then $\mathfrak{R}$ is minimal.*

Proof. Suppose there exists a galaxy $\mathfrak{R}'$ such that $\mathscr{E}(\mathfrak{R}') = \mathscr{E}(\mathfrak{R})$ and $|\mathfrak{R}'| < |\mathfrak{R}|$. Then the galaxy $(\mathfrak{S} - \mathfrak{R}) \cup \mathfrak{R}'$ has less stars than $\mathfrak{S}$, contradicting the minimality of $\mathfrak{S}$. Hence $\mathfrak{R}$ is a minimal galaxy. □

Lemma 2. *If $\mathfrak{S}$ is a minimal galaxy, then $\mathscr{I}(\mathfrak{S})$ is a matching.*

Proof. Suppose $\mathscr{I}(\mathfrak{S})$ is not a matching; say that $\{E_1, E_2\} \subseteq \mathscr{I}(\mathfrak{S})$ and $E_1 \cap E_2 \neq \emptyset$. Then the galaxy $(\mathfrak{S} - \{\{E_1\}, \{E_2\}\}) \cup \{\{E_1, E_2\}\}$ has less stars than $\mathfrak{S}$, contradicting the minimality of $\mathfrak{S}$. Hence $\mathscr{I}(\mathfrak{S})$ is a matching. □

Given a family $\mathfrak{S}$ of stars (not necessarily a galaxy), put $\mathscr{S}_1 \sim_{\mathfrak{S}} \mathscr{S}_2$ if there exists a subfamily $\{\mathscr{R}_1, \ldots, \mathscr{R}_k\} \subseteq \mathfrak{S}$ such that $\mathscr{S}_1 = \mathscr{R}_1$, $\mathscr{S}_2 = \mathscr{R}_k$, and $\mathscr{R}_i \cap \mathscr{R}_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-1$. It can be readily checked that $\sim_{\mathfrak{S}}$ is an equivalence relation on $\mathfrak{S}$; we denote the equivalence class of $\mathscr{S}$ by $[\mathscr{S}]_{\sim_{\mathfrak{S}}}$.

Lemma 3. *If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are minimal galaxies of a hypergraph $\mathscr{H}$ such that $E_1 \in \mathscr{I}(\mathfrak{S}_1)$, $E_2 \in \mathscr{I}(\mathfrak{S}_2)$ and $E_1 \cap E_2 \neq \emptyset$, then $\{E_1\} \sim_{\mathfrak{S}_1 \cup \mathfrak{S}_1} \{E_2\}$.*

Proof. Let $\mathfrak{R}_1 = \mathfrak{S}_1 \cap [\{E_1\}]_{\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2}}$ and $\mathfrak{R}_2 = \mathfrak{S}_2 \cap [\{E_1\}]_{\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2}}$. Clearly $\mathfrak{R}_1 \subseteq \mathfrak{S}_1$ and $\mathfrak{R}_2 \subseteq \mathfrak{S}_2$, so $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are minimal by Lemma 1. Furthermore, $|\mathfrak{R}_1| = |\mathfrak{R}_2|$ since $\mathscr{E}(\mathfrak{R}_1) = \mathscr{E}(\mathfrak{R}_2)$. Therefore $|\mathfrak{S}_1 - \mathfrak{R}_1| = |\mathfrak{S}_2 - \mathfrak{R}_2|$, so the galaxy

$$\mathfrak{S}_3 = \mathfrak{R}_1 \cup (\mathfrak{S}_2 - \mathfrak{R}_2)$$

is minimal. If $\{E_1\} \not\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2} \{E_2\}$ then $\{E_1, E_2\} \subseteq \mathscr{I}(\mathfrak{S}_3)$, contradicting Lemma 2. Hence $\{E_1\} \sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2} \{E_2\}$. □

Lemma 4. *If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are galaxies of a hypergraph $\mathscr{H}$ and $\mathfrak{S}_1$ is E-extreme, then every equivalence class of $\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2}$ contains at most one 1-star of $\mathfrak{S}_1$.*

Proof. Suppose some equivalence class $\mathfrak{S}$ of $\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2}$ contains at least two 1-stars of $\mathfrak{S}_1$. Let $(E_1, \ldots, E_k)$ be a path of minimal length in $\mathscr{E}(\mathfrak{S})$ containing two edges of $\mathscr{I}(\mathfrak{S}_1)$, and put $\mathscr{P} = \{E_1, \ldots, E_k\}$.

By the minimality of k it follows that $E_1, E_k \in \mathscr{I}(\mathfrak{S}_1)$ and $|\mathscr{S}| = 2$ for all $\mathscr{S} \in \mathfrak{S}_1[\mathscr{P}] - \{\{E_1\}, \{E_k\}\}$; we may assume $E_1 \neq E$. Moreover, $\{E_i, E_{i+1}\} \in \mathfrak{S}_1[\mathscr{P}]$ if i is even, and $\{E_i, E_{i+1}\} \in \mathfrak{S}_2[\mathscr{P}]$ if i is odd. As $\{E_{k-1}, E_k\} \in \mathfrak{S}_2[\mathscr{P}]$, k is even and $|\mathfrak{S}_1[\mathscr{P}]| = |\mathfrak{S}_2[\mathscr{P}]| + 1$. Hence $\mathfrak{S}_1[\mathscr{P}]$ is not a minimal galaxy, so by Lemma 1, $\mathfrak{S}_1[\mathscr{P}] \not\subseteq \mathfrak{S}_1$. Hence some star of $\mathfrak{S}_1[\mathscr{P}]$ is not a star of $\mathfrak{S}_1$, so there exists a star $\mathscr{S} \in \mathfrak{S}_1$ such that $\mathscr{S} \cap \mathscr{P} \neq \emptyset$ and $\mathscr{S} \not\subseteq \mathscr{P}$; therefore $|\mathscr{S}| \geqslant 3$.

Let $j \in \{2, \ldots, k\}$ be minimal such that $E_j \in \mathscr{S} \in \mathfrak{S}_1$ and $|\mathscr{S}| \geqslant 3$, and let $\mathscr{P}_1 = \{E_1, \ldots, E_j\}$. By the definition of j, $\{E_j, E_{j+1}\} \in \mathfrak{S}_1[\mathscr{P}]$, so j is even. Let

$$\mathfrak{S}_1' = \mathfrak{S}_1[\mathscr{H} - \mathscr{P}_1] \cup \mathfrak{S}_2[\mathscr{P}_1].$$

It can be readily checked that $\mathfrak{S}_1'$ is a minimal galaxy of $\mathscr{H}$ and $\mathscr{I}(\mathfrak{S}_1') = \mathscr{I}(\mathfrak{S}_1) - \{E_1\}$. As $E_1 \neq E$, this contradicts the E-extremeness of $\mathfrak{S}_1$. $\square$

Lemma 5. *If E_1 and E_2 are intersecting τ-critical edges of a hypergraph $\mathscr{H}$ and $\mathfrak{S}_1$ is an E_1-extreme galaxy of $\mathscr{H}$, then there exists an E_2-extreme galaxy $\mathfrak{S}_2$ of $\mathscr{H}$ such that $\mathscr{I}(\mathfrak{S}_1) - \{E_1\} = \mathscr{I}(\mathfrak{S}_2) - \{E_2\}$.*

Proof. Let $\mathfrak{S}_2'$ be any E_2-extreme galaxy of $\mathscr{H}$. By Lemma 3, $E_1 \sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2'} E_2$. Put $\mathfrak{R}_1 = \mathfrak{S}_1 \cap [\{E_1\}]_{\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2'}}$ and $\mathfrak{R}_2 = \mathfrak{S}_2' \cap [\{E_1\}]_{\sim_{\mathfrak{S}_1 \cup \mathfrak{S}_2'}}$. Clearly $\mathfrak{R}_1 \subseteq \mathfrak{S}_1$ and $\mathfrak{R}_2 \subseteq \mathfrak{S}_2$, so $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are minimal by Lemma 1. Furthermore, $|\mathfrak{R}_1| = |\mathfrak{R}_2|$ since $\mathscr{E}(\mathfrak{R}_1) = \mathscr{E}(\mathfrak{R}_2)$. By Lemma 4, $\mathscr{I}(\mathfrak{R}_1) = \{E_1\}$ and $\mathscr{I}(\mathfrak{R}_2) = \{E_2\}$. Hence

$$\mathfrak{S}_2 = (\mathfrak{S}_1 - \mathfrak{R}_1) \cup \mathfrak{R}_2$$

is a minimal galaxy of $\mathscr{H}$ and $\mathscr{I}(\mathfrak{S}_1) - \{E_1\} = \mathscr{I}(\mathfrak{S}_2) - \{E_2\}$. Suppose $\mathfrak{S}_2$ is not E_2-extreme. Then $|\mathscr{I}(\mathfrak{S}_1)| = |\mathscr{I}(\mathfrak{S}_2)| > |\mathscr{I}(\mathfrak{S}_2')|$, so

$$\mathfrak{S}_1' = (\mathfrak{S}_2' - \mathfrak{R}_2) \cup \mathfrak{R}_1$$

is a minimal galaxy of $\mathscr{H}$ with $E_1 \in \mathscr{I}(\mathfrak{S}_1')$ and $|\mathscr{I}(\mathfrak{S}_1')| < |\mathscr{I}(\mathfrak{S}_1)|$, contradicting the E_1-extremeness of $\mathfrak{S}_1$. Hence $\mathfrak{S}_2$ is E_2-extreme. $\square$

The following corollary follows easily by induction.

Corollary 2. *Let $\mathscr{G} \subseteq \mathscr{H}$ be the hypergraph formed by the τ-critical edges of $\mathscr{H}$. If E_1 and E_2 lie in the same component of $\mathscr{G}$ and $\mathfrak{S}_1$ is an E_1-extreme galaxy of $\mathscr{H}$, then there exists an E_2-extreme galaxy $\mathfrak{S}_2$ of $\mathscr{H}$ such that $\mathscr{I}(\mathfrak{S}_1) - \{E_1\} = \mathscr{I}(\mathfrak{S}_2) - \{E_2\}$.*

We are now ready to prove Theorem 1.

Proof of Theorem 1. Suppose $\mathfrak{S}$ is an E-extreme galaxy of $\mathscr{H}$ such that $\mathscr{I}(\mathfrak{S}) - \{E\} \neq \emptyset$. Let $E_2 \in \mathscr{I}(\mathfrak{S}) - \{E\}$, and let E_1 be any edge intersecting E_2 . The edge E_1 must exist because $\mathscr{H}$ is connected, and by Lemma 2 $E_1 \notin \mathscr{I}(\mathfrak{S})$, so $E_1 \neq E$. By Corollary 2 and the hypothesis that $\mathscr{H}$ is connected and τ-critical, there exists an E_1-extreme galaxy $\mathfrak{S}_1$ of $\mathscr{H}$ such that $\mathscr{I}(\mathfrak{S}) - \{E\} = \mathscr{I}(\mathfrak{S}_1) - \{E_1\}$. But then $\{E_1, E_2\} \subseteq \mathscr{I}(\mathfrak{S}_1)$, contradicting Lemma 2. Hence $\mathscr{I}(\mathfrak{S}) = \{E\}$, as required. $\square$

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