A generalisation of matching and colouring

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Abstract

We define an *r-bounded cover* of a graph G to be a subgraph $T \subseteq G$ such that T contains all vertices of G and each component of T is a complete subgraph of G of order at most r. A 2-bounded cover of a graph G corresponds to a matching of G, and an $\omega(G)$-bounded cover of G corresponds to a colouring of the vertices of the complement $\overline{G}$. We generalise a number of results on matching and colouring of graphs to r-bounded covers, including the Gallai-Edmonds Structure Theorem, Tutte's 1-Factor Theorem, and Gallai's theorem on the minimal order of colour-critical graphs with connected complements.

Dedicated to Vladivoj Vojáček on his 80th birthday.

1 Introduction and terminology

Matching and colouring of graphs represent two of the most developed fields of graph theory, but there seems to be very little overlap between them. The purpose of this paper is to show that, in fact, several known results on matching and colouring of graphs can be generalised in terms of what we call *r-bounded covers.*

An r-bounded cover of a graph G is a spanning subgraph $T \subseteq G$ such that each component of T is a complete graph of order at most r. As the case $r = 1$ is trivial, we shall always tacitly assume that $r \geqslant 2$. One can readily verify that a 2-bounded cover of a graph G corresponds to a matching of G, while an $\omega(G)$-bounded cover of G corresponds to a (proper) colouring of the vertices of the complement $\overline{G}$, where $\omega(G)$ denotes the clique number of G. So one may also view an r-bounded cover of a graph as a colouring of the vertices of its complement, with the extra condition that no colour class contains more than r vertices.

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If T is an r-bounded cover of G with the minimal number of components, it is a *minimal r-bounded cover* of G, and the number of components of a minimal r-bounded cover of G is the *r-bounded cover number* $\overline{\chi}_r(G)$. Thus $\overline{\chi}_{\omega(G)}(G)$ equals the chromatic number $\chi(\overline{G})$ of the complement, and it is not hard to check that $\overline{\chi}_2(G) = |G| - \nu(G)$, where $\nu(G)$ denotes the matching number of G.

The paper is divided into four sections. In Section 2 we state, without proof, generalisations of theorems of Dirac [2], Toft [10], and Stehlík [9] on colour-critical graphs. Section 3 is devoted to the proof of the main result of this paper, a generalisation of the Gallai-Edmonds Structure Theorem [3,5,6]. The main theorem has a number of interesting corollaries which are presented in Section 4. In particular, we generalise the following three classical results, due to Tutte [11], Berge [1], and Gallai [5], respectively. (We use $q(G)$ to denote the number of odd components of a graph G.)

Theorem 1 (Tutte's 1-Factor Theorem) *A graph G has a perfect matching if and only if $q(G - X) \leqslant |X|$ for all $X \subseteq V(G)$.*

Theorem 2 (Berge Formula) *For any graph G, $|G| - 2\nu(G) = \max\{q(G - X) - |X| \mid X \subseteq V(G)\}$.*

Theorem 3 *Any k-chromatic vertex-critical graph with a connected complement has at least $2k - 1$ vertices.*

2 Theorems on r-bounded cover-critical graphs

In this section we generalise some results on colour-critical graphs to r-bounded cover-critical graphs. These can all be proved using a straightforward generalisation of the original proofs, so the proofs are omitted. However, the interested reader can find them in [8].

A vertex x of a graph G is *r-bounded cover-critical* if $\overline{\chi}_r(G - x) = \overline{\chi}_r(G) - 1$, and a graph G is *r-bounded cover-critical* if every vertex of G is r-bounded cover-critical. A graph G is *r-bounded cover-maximal* if it is r-bounded cover-critical and $\overline{\chi}_r(G+e) < \overline{\chi}_r(G)$ for every edge $e \notin E(G)$. We denote the number of blocks and components of a graph G by $b(G)$ and $c(G)$, respectively.

The following generalises a theorem of Dirac [2].

Theorem 4 *If a graph G has components $G_1, \ldots, G_{c(G)}$, then*

$$\overline{\chi}_r(G) = \sum_{i=1}^{c(G)} \overline{\chi}_r(G_i).$$

Moreover, G is r-bounded cover-critical (resp. cover-maximal) if and only if each component G_i is r-bounded cover-critical (resp. cover-maximal).

The following two results generalise theorems of Toft [10].

Theorem 5 *A graph G is r-bounded cover-critical if and only if each block of G is r-bounded cover-critical. Moreover, if G is r-bounded cover-critical then*

$$\overline{\chi}_r(G) = c(G) - b(G) + \sum_{i=1}^{b(G)} \overline{\chi}_r(G_i),$$

where $G_1, \ldots, G_{b(G)}$ are the blocks of G.

Theorem 6 *If G is r-bounded cover-maximal, then every non-trivial component of G is 2-connected.*

Finally, we give a generalisation of a theorem of Stehlík [9] on colour-critical graphs.

Theorem 7 *If x is any vertex of a connected r-bounded cover-critical graph G, then G has a minimal r-bounded cover in which x is the only isolated vertex.*

A graph G is *factor-critical* if $G - x$ has a perfect matching for any vertex x of G. Note that with $r = 2$, Theorem 7 implies the following result of Gallai [4] on factor-critical graphs.

Theorem 8 (Gallai's Lemma) *If a graph G is connected and $\nu(G - x) = \nu(G)$ for every vertex x of G, then G is factor-critical.*

3 The Gallai-Edmonds Structure Theorem for r-bounded covers

Given a graph G, a minimal r-bounded cover T of G is *extreme* if it contains the minimal number of isolated vertices. Note that every minimal 2-bounded cover is extreme. The number of isolated vertices of an extreme r-bounded cover of G is called the *r-bounded deficiency* of G, and is denoted by $\text{def}_r(G)$.

Let us partition the vertex set $V(G)$ of any graph G into three subsets $A_r(G)$, $C_r(G)$ and $D_r(G)$ as follows. Let $D_r(G)$ be the set of all vertices of G which are isolated in some extreme r-bounded cover of G. Let $A_r(G)$ be the set of all vertices in $V(G) \setminus D_r(G)$ which are adjacent to at least one vertex in $D_r(G)$. Finally, we let $C_r(G) = V(G) \setminus (A_r(G) \cup D_r(G))$. Note that an r-bounded cover-critical vertex of G is isolated in some minimal r-bounded cover of G, but this cover need not be extreme. So $D_r(G)$ is a subset of the r-bounded cover-critical vertices of G.

Before stating the Gallai-Edmonds Theorem for r-bounded covers, let us make the following definition. A connected r-bounded cover-critical graph G, such that $|G| = 2\overline{\chi}_r(G) - 1$, is called a *basic r-bounded cover-critical graph.* By Theorem 7, every basic r-bounded cover-critical graph is a 2-bounded cover-critical graph.

Theorem 9 *The following assertions hold for any graph G.*

(1) The components of the subgraph induced by $D_r(G)$ are basic r-bounded cover-critical graphs.
(2) The subgraph induced by $C_r(G)$ has a minimal r-bounded cover with no isolated vertices.
(3) Every subset $X \subseteq A_r(G)$ has neighbours in at least $|X| + 1$ components of $G[D_r(G)]$.
(4) If T is any extreme r-bounded cover of G, then for each component G_i of $G[D_r(G)]$, $T \cap G_i$ is an extreme 2-bounded cover of G_i with one isolated vertex and $(|G_i| - 1)/2$ components of order two, $T \cap G[C_r(G)]$ is an extreme r-bounded cover of $G[C_r(G)]$ with no isolated vertices, and each vertex $a_i \in A_r(G)$ lies in a component (a_i, d_i) of T of order two, where each d_i lies in a distinct component G_i of $G[D_r(G)]$.
(5) $|C_r(G)|/r \leqslant \overline{\chi}_r(G) - |D_r(G)|/2 - c(G[D_r(G)])/2 \leqslant |C_r(G)|/2$.
(6) $\mathrm{def}_r(G) = c(G[D_r(G)]) - |A_r(G)|$.

It should be noted that Gallai [5] proved parts of Theorem 9 for the case $r = \omega(G)$, but he was aware that the results also hold for any $r \geqslant 2$. Although we use some of the concepts introduced by Gallai, our proof is quite different from his. It is inspired partly by the proof of the Gallai-Edmonds Structure Theorem given by Lovász and Plummer [7, pp. 94–97].

The proof of Theorem 9 follows from seven lemmas and two corollaries. Lemma 10 contains simple but important observations on minimal and extreme covers. This is followed by the fundamental Lemma 11 on the intersection of an extreme cover with another cover. Theses two lemmas allow us to modify extreme covers while keeping the components with more than two vertices unchanged; this is the subject of Lemma 12 and Corollary 13. In Lemma 14 we consider a special class of graphs G such that $V(G) = D_r(G)$. The remaining Lemmas 15–

17 deal with the subgraph $G - a$, where a is any vertex of $A_r(G)$. Namely, we obtain expressions for $\overline{\chi}_r(G-a)$, $\mathrm{def}_r(G-a)$, $A_r(G-a)$, $C_r(G-a)$ and $D_r(G-a)$ in terms of $\overline{\chi}_r(G-a)$, $\mathrm{def}_r(G)$, $A_r(G)$, $C_r(G)$ and $D_r(G)$, respectively. Corollary 18 simply extends these results to the graph $G - A_r(G)$. The proof of Theorem 9 then follows relatively easily.

Before stating the first lemma, however, some further definitions are required. The set of isolated vertices of a graph G is denoted by $I(G)$. Given an r-bounded cover T of a graph G, a subgraph $H \subseteq G$ is T*-closed* if $Q \cap H = Q$ or $Q \cap H = \emptyset$, for every component Q of T. Note that if $H \subseteq G$ is T-closed then $G - H$ is also T-closed, with $T \cap H$ and $T - H$ being r-bounded covers of H and $G - H$, respectively.

Lemma 10 *Let T be an r-bounded cover of a graph G and let $H \subseteq G$ be a T-closed subgraph.*

(1) If T is minimal, then $T \cap H$ is a minimal r-bounded cover of H and $I(T)$ is a stable set of G.
(2) If T is extreme, then $T \cap H$ is an extreme r-bounded cover of H, $I(T)$ is a stable set of G and all vertices in $N(I(T)) \setminus I(T)$ lie in components of T of order two.

PROOF. We start by proving the first part of the lemma. If $T \cap H$ is not a minimal r-bounded cover of H, there exists an r-bounded cover T_1 of H such that $c(T \cap H) > c(T_1 \cap H)$. But then the r-bounded cover $(T - H) \cup (T_1 \cap H)$ has less components than T, which is impossible. If $I(T)$ is not stable for some r-bounded cover T of G, we can construct a smaller cover by adjoining an edge with endvertices in $I(T)$. This proves the first part of the lemma.

For the second part, suppose $T \cap H$ is a minimal but not an extreme r-bounded cover of H. Then there exists a minimal r-bounded cover T_1 of G such that $|I(T \cap H) > |I(T_1 \cap H)|$. But then the r-bounded cover $(T - H) \cup (T_1 \cap H)$ is minimal and has less isolated vertices than T, which is impossible. Finally, assume T is a minimal r-bounded cover of G, and suppose that some neighbour x_2 of a vertex $x_1 \in I(T)$ lies in a component of T of order greater than two. Define a new r-bounded cover

$$T_2 = (T - x_1 - x_2) \cup (x_1, x_2),$$

where (x_1, x_2) denotes the single-edge path from x_1 to x_2. Then $c(T) = c(T_2)$ and $|I(T)| > |I(T_2)|$, so T is not extreme.

Lemma 11 *Let G be any graph, and let G_1 and G_2 be any subgraphs of G. If T_1 is an extreme r-bounded cover of G_1 and T_2 is any cover of G_2, then every* *component of* $T_1 \cup T_2$ *containing an isolated vertex of* T_1 *contains only one isolated vertex of* T_1*, and contains no components of* T_1 *of order greater than two.*

PROOF. Let H be a component of $T_1 \cup T_2$ containing an isolated vertex x_0 of T_1. Suppose H contains a component of T_1 of order greater than two. Let $P = (x_0, \ldots, x_l)$ be a path of minimal length in H connecting x_0 to a vertex x_l which lies in a component of T_1 of order greater than two. By the minimality of P, the edges of P alternately lie in T_1 and T_2. Also by the minimality of P, the edge $x_{l-1}x_l$ lies in T_2, so the length l of P is odd.

Consider the cover

$$T_3 = (T_1 - P) \cup (T_2 \cap P).$$

The cover T_3 is r-bounded because each component of $T_2 \cap P$ has order at most two. As the length of P is odd, $T_1 \cap P$ and $T_2 \cap P$ are r-bounded covers of P with $c(T_2 \cap P) = c(T_1 \cap P) - 1$. Moreover, as P is not T_1-closed but $P - x_l$ is, $c(T_1 - P) = c(T_1) - c(T_1 \cap P) + 1$. Hence

$$c(T_3) = c(T_1 - P) + c(T_2 \cap P) = c(T_1),$$

so T_3 is a minimal r-bounded cover of G. By the definition of P, $I(T_1) \cap V(P) = \{x_0\}$ and $I(T_2) \cap V(P) = \emptyset$. Hence

$$I(T_3) = (I(T_1) \setminus (I(T_1) \cap V(P))) \cup (I(T_2) \cap V(P)) = I(T_1 - x_0),$$

contradicting the extremeness of T_1. Therefore H contains no components of T_1 of order greater than two.

Suppose the graph $H - x_0$ contains an isolated vertex of T_1, and let $P_1 = (x_0, y_1, \ldots, y_m)$ be a path of minimal length in H connecting x_0 to some other isolated vertex $y_m \in I((T_1 - x_0) \cap H)$. By the minimality of P_1, the only isolated vertices of $T_1 \cap P_1$ are x_0 and y_m. Also by the minimality of P_1, the edges of P_1 alternately lie in T_1 and T_2. As the edges x_0y_1 and $y_{m-1}y_m$ both lie in T_2, the length m of P_1 is odd. So $T_1 \cap P_1$ and $T_2 \cap P_1$ are r-bounded covers of P_1 with $c(T_2 \cap P_1) = c(T_1 \cap P_1) - 1$. In particular, $T_1 \cap P_1$ is not a minimal r-bounded cover of P_1. By Lemma 10, P_1 is not T_1-closed, which means that some component of T_1 containing an edge of P_1 must have order greater than two. But we have already shown this to be impossible. Hence x_0 is the only isolated vertex of H.

Given a graph G, we denote its set of components of order greater than two by $\mathcal{L}(G)$.

Lemma 12 *Let x_1 and x_2 be adjacent vertices in a graph G.*

(1) If x_1 and x_2 are r-bounded cover-critical and T_1 and T_2 are minimal r-bounded covers of G such that $x_1 \in I(T_1)$ and $x_2 \in I(T_2)$, then x_1 and x_2 lie in the same component of $T_1 \cup T_2$.
(2) If x_1 and x_2 are in $G[D_r(G)]$ and T_1 is an extreme r-bounded cover of G with $x_1 \in I(T_1)$, then there exists an extreme r-bounded cover T_2 of G such that $x_2 \in I(T_2)$, $I(T_1 - x_1) = I(T_2 - x_2)$ and $\mathcal{L}(T_1) = \mathcal{L}(T_2)$.

PROOF. To prove the first part, let H be the component of $T_1 \cup T_2$ containing x_1, and consider the r-bounded cover

$$T_3 = (T_1 \cap H) \cup (T_2 - H).$$

By Lemma 10 and the fact that H is T_1- and T_2-closed, T_3 is a minimal r-bounded cover of G. If $x_2 \notin V(H)$, then $\{x_1, x_2\} \subseteq I(T_3)$, contradicting Lemma 10. Hence $x_2 \in V(H)$, as required.

To prove the second part, let T_3 be any extreme r-bounded cover of G with $x_2 \in I(T_3)$. By the first part of the lemma, there is a component H of $T_1 \cup T_3$ containing x_1 and x_2. Define the r-bounded cover

$$T_2 = (T_1 - H) \cup (T_3 \cap H).$$

By Lemma 10 and the fact that H is T_1- and T_3-closed, T_2 is a minimal r-bounded cover of G. By the definition of T_2, $x_2 \in I(T_2)$. By Lemma 11, $I(T_1 \cap H) = \{x_1\}$ and $I(T_3 \cap H) = \{x_2\}$, so $I(T_1 - H) = I(T_1 - x)$ and $I((T_3 - x_2) \cap H) = \emptyset$. Hence

$$I(T_2 - x_2) = I(T_1 - H) \cup I((T_3 - x_2) \cap H) = I(T_1 - x_1),$$

and T_2 is extreme. Moreover, Lemma 11 implies that $\mathcal{L}(T_1 \cap H) = \mathcal{L}(T_3 \cap H) = \emptyset$, so $\mathcal{L}(T_2) = \mathcal{L}(T_1)$, as required.

The following natural extension of the second part of Lemma 12 follows easily by induction.

Corollary 13 *If G is a graph and $P = (x_0, \ldots, x_l)$ is a path in $G[D_r(G)]$, then for every extreme r-bounded cover T_0 of G with $x_0 \in I(T_0)$, there exists an* *extreme r-bounded cover T_l of G such that $x_l \in I(T_l)$, $I(T_0 - x_0) = I(T_l - x_l)$ and $\mathcal{L}(T_0) = \mathcal{L}(T_l)$.*

Lemma 14 *Suppose G is a connected graph and $V(G) = D_r(G)$. If T is an extreme r-bounded cover of G, then T has exactly one isolated vertex, and any other component of T has order two.*

PROOF. Let $x \in I(T)$, and suppose that $T - x$ contains a component of order not equal to two. Let x_2 be a vertex lying in such a component, and let x_1 be a neighbour of x_2. By Corollary 13 there exists an extreme r-bounded cover T_1 of G such that $x_1 \in I(T_1)$, $I(T_1 - x_1) = I(T - x)$ and $\mathcal{L}(T_1) = \mathcal{L}(T)$. But this contradicts Lemma 10, as x_2 lies in a component of T_1 of order not equal to two.

Lemma 15 *If $a \in A_r(G)$ then $\overline{\chi}_r(G - a) = \overline{\chi}_r(G)$. In particular, if T is a minimal r-bounded cover of G, then $T - a$ is a minimal r-bounded cover of $G - a$ and $c(T - a) = c(T)$.*

PROOF. Suppose the claim is false. Let T_1' be an extreme r-bounded cover of $G - a$. Then $T_1 = T_1' \cup \{a\}$ is a minimal r-bounded cover of G with $a \in I(T_1)$. Let x be a vertex in $D_r(G)$ which is adjacent to a, and let T_2 be an extreme r-bounded cover of G with $x \in I(T_2)$. By Lemma 12 there exists a component H of $T_1 \cup T_2$ containing a and x.

Suppose the graph $H - a$ contains an isolated vertex $y \in I(T_1 - a)$. By Lemma 11 all components of $T_1 - a - y$ in H have order two, so H has an even number of vertices. However, by Lemma 11 all components of $T_2 - x$ in H have order two, so H has an odd number of vertices. This contradiction proves that a is the only isolated vertex of T_1 in H, and by Lemma 11 x is the only isolated vertex of T_2 in H.

Now consider the r-bounded cover

$$T_3 = (T_2 - H) \cup (T_1 \cap H).$$

By Lemma 10 and the fact that H is T_1- and T_2-closed, $c(T_1 \cap H) = c(T_2 \cap H)$, so

$$c(T_3) = c(T_2 - H) + c(T_1 \cap H) = c(T_2).$$

Hence T_3 is a minimal r-bounded cover of G. However,

$$I(T_3) = I(T_2 - H) \cup I(T_1 \cap H) = I(T_2 - x) \cup \{a\},$$

so T_3 is an extreme r-bounded cover of G with $a \in I(T_3)$. This is a contradiction, because $a \in A(G)$. Hence $\overline{\chi}_r(G-a) = \overline{\chi}_r(G)$, as required.

Lemma 16 *If $a \in A_r(G)$, then $\mathrm{def}_r(G-a) = \mathrm{def}_r(G)+1$. In particular, if T is an extreme r-bounded cover of G, then $T-a$ is an extreme r-bounded cover of $G-a$ and $|I(T-a)| = |I(T)|+1$.*

PROOF. Let x be a vertex in $D_r(G)$ adjacent to a in G, and let T_1 be an extreme r-bounded cover of G with $x \in I(T_1)$. We will show that $T_1 - a$ is an extreme r-bounded cover of $G-a$ and $|I(T_1-a)| = |I(T_1)|+1 = \mathrm{def}_r(G)+1$.

As x is isolated in T_1 and T_1 is extreme, a must lie in a component of T_1 of order two by Lemma 10. Let y be the other vertex in this component. Consider the r-bounded cover

$$T_2 = (T_1 - a - x) \cup (a, x).$$

We have $c(T_2) = c(T_1)$ and $I(T_2) = (I(T_1) \cup \{y\}) \setminus \{x\}$, so T_2 is an extreme r-bounded cover of G with y as an isolated vertex, so $y \in D_r(G)$.

By Lemma 15 $T_1 - a$ is a minimal r-bounded cover of $G-a$. Suppose that $T_1 - a$ is not an extreme r-bounded cover of $G-a$. Let T_3 be an extreme r-bounded cover of $G-a$, and let H be the component of $T_1 \cup T_3$ containing x. Suppose that $|I((T_1-a) \cap H)| \leqslant |I(T_3 \cap H)|$. Then $|I((T_1-a)-H)| > |I(T_3-H)|$, and define the r-bounded cover

$$T_4 = (T_3 - H) \cup ((T_1 - a - x) \cap H) \cup (a, x).$$

By Lemma 10 and the fact that $H-a$ is (T_1-a)- and T_3-closed, $c((T_1-a)-H) = c(T_3-H)$, so

$$c(T_4) = c(T_3 - H) + c((T_1 - a - x) \cap H) + c((a, x)) = c(T_1 - a) = c(T_1),$$

where the last equality follows from Lemma 15. Hence T_4 is a minimal r-bounded cover of G. However,

$$\begin{aligned} |I(T_4)| &= |I(T_3 - H)| + |I((T_1 - a - x) \cap H)| + |I((a, x))| \\ &< |I(T_1 - a)| - 1 \leqslant |I(T_1)|, \end{aligned}$$

contradicting the extremeness of T_1.

Hence $|I((T_1 - a) \cap H)| > |I(T_3 \cap H)|$, so by Lemma 11 $I((T_1 - a) \cap H) = \{x\}$ and $I(T_3 \cap H) = \emptyset$. If $y \notin V(H)$ then $a \notin V(H)$, and consider the r-bounded cover

$$T_5 = (T_1 - H) \cup (T_3 \cap H).$$

By Lemmas 10 and 15, and the fact that $H - a$ is $(T_1 - a)$- and T_3-closed, $c(T_1 \cap H) = c((T_1 - a) \cap H) = c(T_3 \cap H)$. Hence

$$c(T_5) = c(T_1 - H) + c(T_3 \cap H) = c(T_1),$$

so T_5 is a minimal cover of G. However,

$$I(T_5) = I(T_1 - H) \cup I(T_3 \cap H) = I(T_1 - x),$$

contradicting the extremeness of T_1.

Hence $y \in V(H)$ and $a \in V(H)$. Let $P = (x_0, \ldots, x_l)$ be a path of minimal length in H such that $x_0 = x$ and $x_l = a$. By the minimality of P, the edges of P lie alternately in T_1 and T_3. As the edge x_0x_1 lies in T_3 and the edge $x_{l-1}x_l$ in T_1, the length l of P is even. By Lemma 11 all components of $T_1 \cap H$ have order at most two, so P is T_1-closed. Consider the r-bounded cover

$$T_6 = (T_1 - P) \cup ((T_3 \cup a) \cap P).$$

As the length of P is even, $c(T_1 \cap P) = c((T_3 \cup a) \cap P)$, so

$$c(T_6) = c(T_1 - P) + c((T_3 \cup a) \cap P) = c(T_1).$$

Hence T_6 is a minimal r-bounded cover of G. However,

$$I(T_6) = I(T_1 - P) \cup I((T_3 \cup a) \cap P) = I(T_1 - x) \cup \{a\},$$

which is impossible because $a \in A_r(G)$.

We have thus proved that $T_1 - a$ is an extreme r-bounded cover of $G - a$. As $I(T_1 - a) = I(T_1) \cup \{y\}$, $\mathrm{def}_r(G - a) = \mathrm{def}_r(G) + 1$ and the lemma is proven.

Lemma 17 *Let G be any graph. If $a \in A_r(G)$ then $A_r(G - a) = A_r(G) \setminus \{a\}$, $C_r(G - a) = C_r(G)$ and $D_r(G - a) = D_r(G)$.*

PROOF. It suffices to show that $D_r(G-a) = D_r(G)$. We first show that $D_r(G) \subseteq D_r(G-a)$. Let $x \in D_r(G)$ and let T be an extreme r-bounded cover of G with $x \in I(T)$. By Lemma 15 $T-a$ is a minimal r-bounded cover of $G-a$ with $x \in I(T-a)$. As $|I(T-a)| \leqslant |I(T)|+1$, Lemma 16 implies that $T-a$ is an extreme r-bounded cover of $G-a$. Hence $x \in D_r(G-a)$, so $D_r(G) \subseteq D_r(G-a)$ as required.

To show $D_r(G-a) \subseteq D_r(G)$, let $x \in D_r(G-a)$, and let T_1 be an extreme r-bounded cover of $G-a$ with $x \in I(T_1)$. Furthermore, let $y \in D_r(G)$ be adjacent to a in G, and let T_2 be an extreme r-bounded cover of G with $y \in I(T_2)$. If $x \in I(T_2)$ then $x \in D_r(G)$ as required. So assume $x \notin I(T_2)$. Let H be the component of $T_1 \cup T_2$ containing a. Suppose $|I(T_1 \cap H)| \leqslant |I(T_2 \cap H)|$. Then $|I(T_1 - H)| > |I(T_2 - H)|$, and define the r-bounded cover

$$T_3 = (T_2 - H) \cup (T_1 \cap H).$$

By Lemma 10 and the fact that $H-a$ is T_1- and (T_2-a)-closed, $c(T_1 \cap H) = c((T_2-a) \cap H)$. Hence

$$c(T_3) = c(T_2 - H) + c(T_1 \cap H) = c(T_2 - a),$$

so T_3 is a minimal r-bounded cover of $G-a$ by Lemma 15. However,

$$|I(T_3)| = |I(T_2 - H)| + |I(T_1 \cap H)| < |I(T_1)|,$$

contradicting the extremeness of T_1.

Hence $|I(T_1 \cap H)| > |I(T_2 \cap H)|$, so by Lemma 11 H contains one isolated vertex of T_1 and no isolated vertex of T_2. By Lemma 16, $|I(T_1)| = |I(T_2)|+1$. Suppose $x \notin V(H)$, and define the r-bounded cover

$$T_4 = (T_1 - H) \cup (T_2 \cap H).$$

By Lemma 10 and the fact that $G-H$ is T_1- and T_2-closed, $c(T_1 - H) = c(T_2 - H)$. Hence

$$c(T_4) = c(T_1 - H) + c(T_2 \cap H) = c(T_2),$$

so T_4 is a minimal r-bounded cover of G. By Lemma 16 and the fact that H contains one isolated vertex of T_1 and no isolated vertex of T_2,

$$|I(T_4)| = |I(T_1 - H)| + |I(T_2 \cap H)| = |I(T_2)|.$$

Hence T_4 is an extreme r-bounded cover of G with $x \in I(T_4)$, so $x \in D_r(G)$.

Now suppose $x \in V(H)$. As H contains no isolated vertex of T_2, $y \notin V(H)$. Define the r-bounded cover

$$T_5 = (T_2 - a - y) \cup (a, y).$$

We have $c(T_5) = c(T_2)$, so T_5 is a minimal r-bounded cover of G. Moreover, $|I(T_5)| \leqslant |I(T_2)|$, so T_5 is an extreme r-bounded cover of G with $x \in I(T_5)$, so $x \in D_r(G)$. This completes the proof.

The following result can easily be deduced from Lemmas 15–17 using induction. The proof is left to the reader.

Corollary 18 *The following assertions hold for any graph G.*

(1) $A_r(G - A_r(G)) = \emptyset$.
(2) $C_r(G - A_r(G)) = C_r(G)$.
(3) $D_r(G - A_r(G)) = D_r(G)$.
(4) $\overline{\chi}_r(G - A_r(G)) = \overline{\chi}_r(G)$.
(5) $\mathrm{def}_r(G - A_r(G)) = \mathrm{def}_r(G) + |A_r(G)|$.
(6) If T is an extreme r-bounded cover of G, then $T - A_r(G)$ is an extreme r-bounded cover of $G - A_r(G)$.

We are finally ready to prove Theorem 9.

Proof of Theorem 9 We first prove Theorem 9 (4). By Corollary 18 (2) and (3), each component H of $G[C_r(G)]$ and $G[D_r(G)]$ is a component of $G - A_r(G)$, so H must be $(T - A_r(G))$-closed. Hence by Lemma 10, $T \cap H$ is an extreme r-bounded cover of H. By Corollary 18 (2), T contains no isolated vertices in $G[C_r(G)]$. If H is a component of $G[D_r(G)]$, Lemma 14 implies that $T \cap H$ contains one isolated vertex and $(|H| - 1)/2$ components of order two.

Since no vertex of $A_r(G)$ is isolated in T and $|I(T - A_r(G))| = |I(T)| + |A_r(G)|$ by Corollary 18 (5), each $a_i \in A_r(G)$ lies in a component (a_i, d_i) of T of order two, and each d_i is isolated in $T - A_r(G)$, so $d_i \in D_r(G - A_r(G) = D_r(G)$. As each component of $G[D_r(G)]$ contains precisely one isolated vertex of T, the d_i must lie in distinct components of $G[D_r(G)]$. So we have proved Theorem 9 (4).

To prove Theorem 9 (1) and (2), note that given any component H of $D_r(G)$ and any vertex $x \in V(H)$, there exists an extreme r-bounded cover T_x of G such that x is an isolated vertex of T_x. By Theorem 9 (4) x is the only isolated vertex of $T_x \cap H$ and all other components of $T_x \cap H$ have order two. As H and

x are arbitrary, this shows that every component of $G[D_r(G)]$ is factor-critical. Theorem 9 (2) follows immediately from Theorem 9 (4).

To prove Theorem 9 (3), let G_h be any component of $G[D_r(G)]$ adjacent to at least one vertex of X. As $X \subseteq A_r(G)$, every vertex in X is adjacent to at least one vertex in $D_r(G)$, so such a G_h exists. Let x be a vertex in $V(G_h)$ and let T be an extreme r-bounded cover of G with $x \in I(T)$. By Theorem 9 (4), each $a_i \in X$ lies in a component (a_i, d_i) of T, where the d_i are vertices in distinct components G_i of $G[D_r(G)]$. So together with G_h there are at least $|X| + 1$ components G_i adjacent to X, as required.

Theorem 9 (5) follows from Corollary 18 (4), and Theorem 9 (1) and (2). We have

$$\begin{aligned}\overline{\chi}_r(G) = \overline{\chi}_r(G - A_r(G)) &= \sum_{i=1}^{c(G[D_r(G)])} \frac{(|G_i| + 1)}{2} + \overline{\chi}_r(C_r(G)) \\ &= \frac{|D_r(G)| + c(G[D_r(G)])}{2} + \overline{\chi}_r(C_r(G)),\end{aligned}$$

and

$$\frac{|C_r(G)|}{r} \leqslant \overline{\chi}_r(C_r(G)) \leqslant \frac{|C_r(G)|}{2},$$

because given any extreme r-bounded cover T of G, all components of $T \cap G[C_r(G)]$ have order at least two and at most r.

Finally, to prove Theorem 9 (6), let T be an extreme r-bounded cover of G. By Theorem 9 (4), each vertex $a_j \in A_r(G)$ lies in a component (a_j, d_j) of T, and for each component G_i of $G[D_r(G)]$, $T \cap G_i$ contains one isolated vertex. Hence T contains $c(G[D_r(G)]) - |A_r(G)|$ isolated vertices, as required.

4 Applications of Theorem 9

We can use Theorem 9 to prove the following generalisation of Theorems 3 and 8. However, note that it is also a straightforward corollary of Theorem 7.

Theorem 19 *Any connected r-bounded cover-critical graph has at least $2\overline{\chi}_r(G)-1$ vertices.*

PROOF. Let G be an r-bounded cover-critical graph on less than $2\overline{\chi}_r(G)-1$ vertices. Every extreme r-bounded cover of G contains at least two isolated vertices, so $\mathrm{def}_r(G) \geqslant 2$. By Corollary 18 (4), and the fact that G is r-bounded cover-critical, $A_r(G) = \emptyset$. Hence using Theorem 9 (6), $c(G[D_r(G)]) \geqslant 2$. As $A_r(G) = \emptyset$ and no vertex of $D_r(G)$ is adjacent to a vertex of $C_r(G)$, the components of $G[D_r(G)]$ are components of G. Hence $c(G) \geqslant 2$, so G is not connected.

Recall that an r-bounded cover-critical graph G is basic if $|G| = 2\overline{\chi}_r(G) - 1$. Theorem 19 immediately implies the following result, which we state without proof.

Corollary 20 *Any r-bounded cover-critical graph G with at most $2\overline{\chi}_r(G) - t$ vertices, where $1 \leqslant t \leqslant \overline{\chi}_r(G)$, contains at least t basic r-bounded cover-critical components.*

The next result may seem somewhat surprising. It states that any (not necessarily minimal) r-bounded cover of G contains at least as many isolated vertices as an extreme r-bounded cover. Thus $\mathrm{def}_r(G)$ could also be defined as the minimal number of isolated vertices of an r-bounded cover, taken over all r-bounded covers of G.

Theorem 21 *If T is any r-bounded cover of G, then $|I(T)| \geqslant \mathrm{def}_r(G)$.*

PROOF. Let T be any r-bounded cover of G. By Theorem 9 (6) $G[D_r(G)]$ contains $|A_r(G)| + \mathrm{def}_r(G)$ components G_i. By Theorem 9 (1) each G_i contains at least one isolated vertex d_i of $T \cap G_i$. If d_i is not an isolated vertex of T then it lies in a component Q_i of T containing a vertex $a_i \in A_r(G)$. As the components Q_i are disjoint, there are at most $|A_r(G)|$ of them. Hence at least $\mathrm{def}_r(G)$ vertices d_i are isolated in T.

Recall that $q(G)$ denotes the number of odd components of a graph G. Given a graph G, let $q_r(G)$ denote the number of odd components G_i of G which satisfy $|G_i| \leqslant 2\overline{\chi}_r(G_i) - 1$. Any component G_i with an odd number of vertices satisfies $|G_i| \leqslant 2\overline{\chi_2}_r(G_i) - 1$, so $q_2(G) = q(G)$. Theorem 9 can also be used to deduce the following generalisation of Theorem 2.

Theorem 22 *For any graph G, $\mathrm{def}_r(G) = \max\{q_r(G - X) - |X| \mid X \subseteq V(G)\}$.*

PROOF. Let T be an extreme r-bounded cover of G, so $\mathrm{def}_r(G) = |I(T)|$. We first show that $\mathrm{def}_r(G) \geqslant \max\{q_r(G - X) - |X| \mid X \subseteq V(G)\}$. Let X be any subset of $V(G)$, and put $t = q_r(G - X)$. Let $G_1, \ldots, G_t$ be the odd components of $G - X$ satisfying $|G_i| \leqslant 2\overline{\chi}_r(G_i) - 1$. Then for each $i \in \{1, \ldots, t\}$, $T \cap G_i$

contains at least one isolated vertex x_i. If x_i is not an isolated vertex of T, then x_i must lie in a component Q_i of T which contains a vertex in X. As the components Q_i are disjoint, there are at most $|X|$ of them. Hence

$$\mathrm{def}_r(G) = |I(T)| \geqslant t - |X| = q_r(G - X) - |X|.$$

As X was arbitrary, this shows that $\mathrm{def}_r(G) \geqslant \max\{q_r(G - X) - |X| \mid X \subseteq V(G)\}$.

To show $\mathrm{def}_r(G) \leqslant \max\{q_r(G - X) - |X| \mid X \subseteq V(G)\}$, note that by Theorem 9 (1) and (2), every component G_i of $G[D_r(G)]$ has order $2\overline{\chi}_r(G) - 1$, while every component H of $G[C_r(G)]$ has order at least $2\overline{\chi}_r(G)$. Hence $q_r(G - A_r(G)) = c(G[D_r(G)])$, and by Theorem 9 (6), $\mathrm{def}_r(G) = c(G[D_r(G)]) - |A_r(G)|$. Hence

$$\mathrm{def}_r(G) = q_r(G - A_r(G)) - |A_r(G)|,$$

so $\mathrm{def}_r(G) \leqslant \max\{q_r(G - X) - |X| \mid X \subseteq V(G)\}$, as required.

An immediate corollary of Theorem 22 is the following generalisation of Theorem 1. Note that by Theorem 21 the word ‘extreme’ may be omitted.

Theorem 23 *A graph G has an extreme r-bounded cover with no isolated vertices if and only if $q_r(G - X) \leqslant |X|$ for all $X \subseteq V(G)$.*

PROOF. If G has an extreme cover with no isolated vertices, then $\mathrm{def}_r(G) = 0$, so by Theorem 22

$$0 = \mathrm{def}_r(G) = \max\{q_r(G - X) - |X| \mid X \subseteq V(G)\},$$

so $q_r(G - X) \leqslant |X|$ for all $X \subseteq V(G)$.

Conversely, suppose $q_r(G - X) \leqslant |X|$ for all $X \subseteq V(G)$. Then by Theorem 22 $\mathrm{def}_r(G) = 0$, so there exists an extreme r-bounded cover of G with no isolated vertices.

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