# Critical graphs with connected complements

Matěj Stehlík

*Department of Mathematics, Imperial College, London SW7 2BZ, England*

**Abstract**

We show that given any vertex $x$ of a $k$-colour-critical graph $G$ with a connected complement, the graph $G - x$ can be $(k-1)$-coloured so that every colour class contains at least 2 vertices. This extends the well-known theorem of Gallai, that a $k$-colour-critical graph with a connected complement has at least $2k-1$ vertices. Our proof does not use matching theory. It is considerably shorter, conceptually simpler and more general than Gallai's original proof.

## 1 Introduction and Basic Definitions

A graph $G$ is *$k$-colour-critical*, or simply *colour-critical*, if it has chromatic number $k$ but for any vertex $x$ the graph $G - x$ has a $(k-1)$-colouring. A graph $G$ is *decomposable* if its complement $\overline{G}$ is disconnected, otherwise it is *indecomposable*. It is not difficult to check that a decomposable colour-critical graph is the complete join of its indecomposable colour-critical subgraphs; so in a sense the indecomposable colour-critical graphs are the 'building blocks' of colour-critical graphs, and it is natural to study their properties.

The main result of this paper is the following.

**Theorem 1** *If $x$ is any vertex of an indecomposable $k$-colour-critical graph $G$, then $G - x$ has a $(k-1)$-colouring in which every colour class contains at least 2 vertices.*

An immediate corollary is the following beautiful result of Gallai [1].

**Corollary 2 (Gallai)** *Any indecomposable $k$-colour-critical graph has at least $2k-1$ vertices.*

*Email address:* `matej.stehlik@ic.ac.uk` (Matěj Stehlík).

It is worth noting that the proof of Theorem 1 presented in this paper is much simpler and shorter than Gallai's original proof of Corollary 2. Whereas the latter is essentially an application of matching theory to the complements of colour-critical graphs, we develop new techniques that simplify the proof considerably. In fact, our proof uses no matching theory and is entirely self-contained.

Before giving an outline of our proof, a few definitions are needed. Like Gallai, we prefer to work with the complements of colour-critical graphs. A *cover* $T$ of a graph $G$ is a union of disjoint cliques of $G$ containing all vertices of $G$; this corresponds to a colouring of the complement. If $T$ is a cover with $k$ components, it is a *$k$-cover*. The *cover number* $\overline{\chi}(G)$ of a graph $G$ is the minimum $k$ for which there exists a $k$-cover of $G$, and a $\overline{\chi}(G)$-cover of $G$ is a *minimal* cover. A vertex $x \in V(G)$ is *cover-critical* if $\overline{\chi}(G-x) = \overline{\chi}(G) - 1$, and a graph $G$ is *$k$-cover-critical*, or simply *cover-critical*, if $\overline{\chi}(G) = k$ and all vertices of $G$ are cover-critical. Given a cover-critical vertex $x$ of $G$, a cover $T$ of $G$ is *$x$-extreme* if $T - x$ is a minimal cover of $G - x$ with the minimum number of isolated vertices. Note that if $T$ is an $x$-extreme cover of $G$, then $x$ must be an isolated vertex of $T$.

We can now give a brief outline of our proof of Theorem 1. Given two adjacent cover-critical vertices $x_1$ and $x_2$ of a graph $G$, we consider the union $T_1 \cup T_2$ of an $x_1$-extreme cover $T_1$ of $G$ and an $x_2$-extreme cover $T_2$ of $G$. We prove the existence of a component of $T_1 \cup T_2$ that contains the vertices $x_1$ and $x_2$ but does not contain any other isolated vertices of $T_1$ and $T_2$. This crucial fact allows us to prove that, given any vertices $x_1$, $x_2$ of a connected cover-critical graph $G$ and any $x_1$-extreme cover $T_1$ of $G$, there exists an $x_2$-extreme cover $T_2$ of $G$ such that $T_1 - x_1$ has the same isolated vertices as $T_2 - x_2$. Theorem 1 follows easily from this result and the fact that the isolated vertices of a minimal cover of $G$ form a stable set of $G$.

For the benefit of the interested reader, we also give a very brief outline of Gallai's original proof of Corollary 2. Throughout the proof, $G$ is assumed to be a graph with cover number $k$ on at most $2k-1$ vertices, and $T$ is a fixed $k$-cover of $G$ with the minimum number of isolated vertices. A vertex $x$ is *$\alpha$-accessible* if it is connected to an isolated vertex of $T$ by a path whose edges alternately lie in and outside $T$, such that the edge incident with $x$ lies in $T$. Using a number of results from matching theory (and proving important new results), Gallai analysed the components of the subgraph of $G$ induced by the $\alpha$-accessible vertices, and proved that each of these components is a factor-critical graph. Furthermore, he showed that if $G$ is connected and cover-critical, then every vertex of $G$ must be $\alpha$-accessible, so in fact $G$ is factor-critical. Hence $G$ has exactly $2k-1$ vertices, and Corollary 2 follows easily.

## 2 The Proof

It is clear that the cover number $\overline{\chi}(G)$ of a graph $G$ equals the chromatic number $\chi(\overline{G})$ of its complement $\overline{G}$, and that $G$ is $k$-cover-critical if and only if $\overline{G}$ is $k$-colour-critical. We can therefore restate Theorem 1 in terms of covers as follows.

**Theorem 1** *If $x$ is any vertex of a connected $k$-cover-critical graph $G$, then $G$ has a $k$-cover in which $x$ is the only isolated vertex.*

The proof is arranged as a series of lemmas. Before proving the first lemma, we need to make some definitions. The number of components and the set of isolated points of a graph $G$ are denoted by $c(G)$ and $I(G)$, respectively. Given two graphs $G$, $H$ such that $V(G) \supseteq V(H)$, we define $G - H$ to be the induced subgraph of $G$ with vertex set $V(G) \setminus V(H)$. Given a cover $T$ of a graph $G$, a subgraph $H \subseteq G$ is $T$-*closed* if $Q \subseteq H$ or $Q \cap H = \emptyset$, for every component $Q$ of $T$. Note that if $H \subseteq G$ is $T$-closed then $G - H$ is also $T$-closed, with $T \cap H$ and $T - H$ being covers of $H$ and $G - H$, respectively.

We now proceed with the formal proof of Theorem 1. We start with three simple lemmas, the first of which was also used by Gallai [1].

**Lemma 3 (Gallai)** *If $T$ is a minimal cover of a graph $G$ and $H \subseteq G$ is $T$-closed, then $T \cap H$ is a minimal cover of $H$.*

**PROOF.** Since $T \cap H$ is a cover of $H$, we must have $c(T \cap H) \geqslant \overline{\chi}(H)$ by the definition of $\overline{\chi}(H)$. Suppose that $c(T \cap H) > \overline{\chi}(H)$. Then $T \cap H$ is not a minimal cover of $H$, so there exists a cover $T_1$ such that $c(T_1 \cap H) < c(T \cap H)$. But then consider the cover

$$T_2 = (T - H) \cup (T_1 \cap H).$$

We have

$$c(T_2) = c(T - H) + c(T_1 \cap H) < c(T - H) + c(T \cap H) = c(T),$$

contradicting the minimality of $T$. Hence $c(T \cap H) = \overline{\chi}(H)$, as required.

**Lemma 4** *If $T$ is a minimal cover of a graph $G$, then $I(T)$ is a stable set of $G$.*

**PROOF.** Suppose $I(T)$ is not stable, for some minimal cover $T$ of $G$. Say that $x_1$ and $x_2$ are vertices of $I(T)$ that are adjacent in $G$. Define a new cover

$$T_1 = (T - x_1 - x_2) \cup (x_1, x_2),$$

where $(x_1, x_2)$ denotes the single-edge path from $x_1$ to $x_2$. But now

$$c(T_1) = c(T - x_1 - x_2) + c((x_1, x_2)) = c(T) - 1,$$

contradicting the minimality of $T_1$. Hence the result.

**Lemma 5** *Let $x_1$ and $x_2$ be adjacent cover-critical vertices of a graph $G$, and suppose that $T_1$ and $T_2$ are minimal covers of $G$ such that $x_1 \in I(T_1)$ and $x_2 \in I(T_2)$, respectively. Then $x_1$ and $x_2$ lie in the same component of $T_1 \cup T_2$.*

**PROOF.** Let $H$ be the component of $T_1 \cup T_2$ containing $x_1$, and consider the cover

$$T_3 = (T_1 \cap H) \cup (T_2 - H).$$

By Lemma 3 and the fact that $H$ is $T_1$- and $T_2$-closed, it follows that $c(T_1 \cap H) = c(T_2 \cap H)$ and $c(T_1 - H) = c(T_2 - H)$. Therefore

$$c(T_3) = c(T_1 \cap H) + c(T_2 - H) = c(T_1),$$

so $T_3$ is a minimal cover of $G$. If $x_2 \notin V(H)$, then $\{x_1, x_2\} \subseteq I(T_3)$, contradicting Lemma 4 because $x_1$ is adjacent to $x_2$ in $G$. Hence $x_2 \in V(H)$, as required.

The following important lemma essentially generalises assertions (7.2) and (7.3) of Gallai [1].

**Lemma 6** *Let $G$ be a graph with a cover-critical vertex $x$. If $T_1$ is an $x$-extreme cover of $G$ and $T_2$ is any cover of $G$, then any component of $T_1 \cup T_2$ contains at most 1 isolated vertex of $T_1$.*

**PROOF.** Suppose some component $H$ of $T_1 \cup T_2$ contains at least 2 isolated vertices of $T_1$. If $x \in V(H)$, let $x_0 = x$, otherwise let $x_0$ be any vertex of $I(T_1 \cap H)$. Let $P = (x_0, \ldots, x_l)$ be a path of minimum length in $H$ connecting $x_0$ to some other isolated vertex $x_l \in I((T_1 - x_0) \cap H)$.

By the minimality of $P$, the only isolated vertices of $T_1 \cap P$ are $x_0$ and $x_l$. Also by the minimality of $P$, the edges of $P$ alternately lie in $T_1$ and $T_2$. As the edges $x_0x_1$ and $x_{l-1}x_l$ both lie in $T_2$, the length $l$ of $P$ is odd. So $T_1 \cap P$ and $T_2 \cap P$ are covers of $P$ with $c(T_2 \cap P) = c(T_1 \cap P) - 1$. In particular, $T_1 \cap P$ is not a minimal cover of $P$. By Lemma 3, $P$ is not $T_1$-closed, which means that some component of $T_1$ containing an edge of $P$ must have order greater than 2.

Let $x_j$ $(0 < j < l)$ be the closest vertex to $x_l$ on $P$ that lies in a component of $T_1$ of order greater than 2, and let $P_1 = (x_j, \ldots, x_l)$ be the subpath of $P$ with endvertices $x_j$ and $x_l$. By the definition of $x_j$, the edge $x_jx_{j+1}$ lies in $T_2$, so the length $l - j$ of $P_1$ is odd.

Consider the cover

$$T_3 = (T_1 - P_1) \cup (T_2 \cap P_1).$$

As the length of $P_1$ is odd, $T_1 \cap P_1$ and $T_2 \cap P_1$ are covers of $P_1$ with $c(T_2 \cap P_1) = c(T_1 \cap P_1) - 1$. Moreover, as $P_1$ is not $T_1$-closed but $P_1 - x_j$ is, $c(T_1 - P_1) = c(T_1) - c(T_1 \cap P_1) + 1$. Hence

$$c(T_3) = c(T_1 - P_1) + c(T_2 \cap P_1) = c(T_1),$$

so $T_3$ is a minimal cover of $G$. By the definition of $P_1$, $I(T_1) \cap V(P_1) = \{x_l\}$ and $I(T_2) \cap V(P_1) = \emptyset$. Hence

$$I(T_3) = I(T_1) \setminus (I(T_1) \cap V(P_1)) \cup (I(T_2) \cap V(P_1)) = I(T_1 - x_l),$$

contradicting the $x$-extremeness of $T_1$. Therefore every component of $T_1 \cup T_2$ contains at most one isolated vertex of $T_1$.

**Lemma 7** *If $x_1$ and $x_2$ are adjacent cover-critical vertices of a graph $G$ and $T_1$ is an $x_1$-extreme cover of $G$, then there exists an $x_2$-extreme cover $T_2$ of $G$ such that $I(T_1 - x_1) = I(T_2 - x_2)$.*

**PROOF.** Let $T_3$ be any $x_2$-extreme cover of $G$. By Lemma 5, there is a component $H$ of $T_1 \cup T_3$ containing $x_1$ and $x_2$. Define the cover

$$T_2 = (T_1 - H) \cup (T_3 \cap H).$$

By Lemma 3 and the fact that $H$ is $T_1$- and $T_2$-closed, it follows that $T_2$ is a minimal cover of $G$. By the definition of $T_2$, $x_2 \in I(T_2)$. By Lemma 6,

$I(T_1 \cap H) = \{x_1\}$ and $I(T_3 \cap H) = \{x_2\}$, so $I(T_1 - H) = I(T_1 - x_1)$ and $I((T_3 - x_2) \cap H) = \emptyset$. Hence

$$I(T_2 - x_2) = I(T_1 - H) \cup I((T_3 - x_2) \cap H) = I(T_1 - x_1),$$

and $T_2$ is $x_2$-extreme as required.

**Lemma 8** *If $P = (x_0, \ldots, x_l)$ is a path in a graph $G$ and $x_0, \ldots, x_l$ are cover-critical vertices, then for every $x_0$-extreme cover $T_0$ of $G$ there exists an $x_l$-extreme cover $T_l$ of $G$ such that $I(T_0 - x_0) = I(T_l - x_l)$.*

**PROOF.** The case $l = 0$ is trivial, so assume $l > 0$. Using Lemma 7, there exist $x_i$-extreme covers $T_i$ $(i = 1, \ldots, l)$ such that $I(T_{i-1} - x_{i-1}) = I(T_i - x_i)$ for all $1 \leqslant i \leqslant l$. Hence

$$I(T_0 - x_0) = I(T_1 - x_1) = \ldots = I(T_l - x_l),$$

which proves the lemma.

We are now ready to prove Theorem 1.

**Proof of Theorem 1** Suppose $T$ is an $x$-extreme cover of $G$ such that $I(T - x) \neq \emptyset$. Let $x_2 \in I(T - x)$ and let $x_1$ be any vertex adjacent to $x_2$ . The vertex $x_1$ must exist because $G$ is connected, and by Lemma 4 $x_1 \notin I(T)$, so $x_1 \neq x$. By Lemma 8 and the hypothesis that $G$ is connected and cover-critical, there exists an $x_1$-extreme cover $T_1$ of $G$ such that $I(T - x) = I(T_1 - x_1)$. But then $\{x_1, x_2\} \subseteq I(T)$, contradicting Lemma 4. Hence $I(T) = \{x\}$, as required.

Finally, let us remark that Lemmas 4 and 8 can also be used to prove the following slight extension of Theorem 1.

**Theorem 9** *If $x$ is a cover-critical vertex of a graph $G$, then $G$ has a minimal cover $T$ such that $x$ is an isolated vertex of $T$, and $x$ is not connected to any other isolated vertex of $T$ by a path containing only cover-critical vertices.*


## Acknowledgements

The author would like to thank Oliver Pretzel for his valuable advice and comments.

## References


[1] T. Gallai, Kritische Graphen II, *Publ. Math. Inst. Hung. Acad. Sci.* **8** (1963) 373–395.