Characterizing Noninteger Polyhedra with 0–1 Constraints

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Abstract. We characterize when the intersection of a set-packing and a set-covering polyhedron or of their corresponding minors has a noninteger vertex. Our result is a common generalization of Lovász's characterization of 'imperfect' and Lehman's characterization of 'nonideal' systems of inequalities, furthermore, it includes new cases in which both types of inequalities occur and interact in an essential way. The proof specializes to a conceptually simple and short common proof for the classical cases, moreover, a typical corollary extracting a new case is the following: *if the intersection of a perfect and an ideal polyhedron has a noninteger vertex, then they have minors whose intersection's coefficient matrix is the incidentce matrix of an odd circuit graph.*

1 Introduction

Let $A^{\leq}$ and $A^{\geq}$ be *0–1-matrices* (meaning that each entry is 0 or 1) with n columns. We will study the integrality of the intersection $P(A^{\leq}, A^{\geq}) := P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})$ of the *set-packing polytope* $P^{\leq}(A^{\leq}) = \{x \in \mathbb{R}^n : A^{\leq}\, x \leq 1,\ x \geq 0\}$ and the *set-covering polyhedron* $P^{\geq}(A^{\geq}) = \{x \in \mathbb{R}^n : A^{\geq}\, x \geq 1,\ x \geq 0\}$. We will speak about $(A^{\leq}, A^{\geq})$ as a system of inequalities, or simply *system.*

Obviously, one can suppose that both the rows of $A^{\leq}$ and those of $A^{\geq}$ are incidence ('characteristic') vectors of a *clutter*, that is of a family of sets none of which contains the other. The sets in the clutters and their 0–1 incidence vectors will be confused, and with the same abuse of terminology, clutters and their matrix representations (where the rows are the members of the clutter) will not be distinguished. If $A^{\leq}$ and $A^{\leq}$ do not have equal rows, that is (explicit) *equalities*, we will say that $(A^{\leq}, A^{\geq})$ is *simple.*

The constraints defining $P^{\leq}(A^{\leq})$ will be called of *packing* type, and those defining $P^{\geq}(A^{\geq})$ of *covering* type. A vertex of $P(A^{\leq}, A^{\geq})$ can also be classified to be of *packing* type, of *covering* type, or of *mixed* type, depending on whether all nonequality constraints containing the vertex are of packing type, of covering type, or both types occur.

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The blocker of a clutter $A^{\geq}$ (or the antiblocker of $A^{\leq}$) is the set of inclusionwise minimal (resp. maximal) integer vectors in $P^{\geq}(A^{\geq})$ (resp. $P^{\leq}(A^{\leq})$). These are 0–1-vectors and define another clutter.

A *polyhedron* in this paper is the set of all (real) solutions of a system of linear inequalities with integer coefficients. A *polytope* is a bounded polyhedron. For basic definitions and statements about polyhedra we refer to Schrijver [11], and we only repeat now shortly the definition of the terms we are using directly. A *face* of a polyhedron is a set we get if we replace certain defining inequalities with the equality so that the resulting polyhedron is nonempty. A polyhedron is *integer* if each of its faces contains an integer point, otherwise it is *noninteger.*

If $X \subseteq \mathbb{R}^n$, we will denote by $r(X)$ the (linear) *rank* of X, and by $\dim(X)$ the *dimension* of X, meaning the rank of the differences of pairs of vectors in X, that is, $\dim(X) := r\big(\{x - y : x, y \in X\}\big)$.

If P is a polyhedron, then its faces of dimension $\dim(P) - 1$ are called *facets*, and its faces of dimension 0, *vertices.* All (inclusionwise) minimal faces of P have the same dimension. We say that P *has vertices*, if this dimension is 0.

It is easy to see that $P(A^{\leq}, A^{\geq})$ has vertices for all $A^{\leq}$, $A^{\geq}$. (If a minimal face is not of dimension 0, it contains an entire line, contradicting some non-negativity constraint.) So $P(A^{\leq}, A^{\geq})$ is integer if and only if it has integer vertices.

A vertex of a full dimensional polyhedron is *simplicial*, if it is contained in exactly n facets. A simplicial vertex has n neighbouring vertices. Neighbours share $n - 1$ facets.

If $A^{\leq}$ is empty, a combinatorial coNP characterization of the integrality of $P(A^{\leq}, A^{\geq})$ is well-known (Lovász [8], Padberg [9]). If $A^{\geq}$ is empty, a recent result of Lehman solves the problem (Lehman [6], Seymour [12]). A common generalization of these could be a too modest goal: if for every $i \in \{1, \ldots, n\}$ either the i-th column of $A^{\leq}$ or that of $A^{\geq}$ is 0, then the nonintegrality of $P(A^{\leq}, A^{\geq})$ can be separated to the two 'classical' special cases. Such systems $(A^{\leq}, A^{\geq})$ contain both special cases, but nothing more. There are less trivial examples where $P^{\leq}(A^{\leq})$ and $P^{\geq}(A^{\geq})$ do not really interact in the sense that all fractional vertices of $P^{\leq}(A^{\leq}, A^{\geq})$ are vertices of $P^{\leq}(A^{\leq})$ or of $P^{\geq}(A^{\geq})$.

In this work we characterize when the intersection of a set-packing and a set-covering polyhedron or that of any of their corresponding minors is noninteger. The results contain the characterizations of perfect and ideal polyhedra and new cases involving mixed vertices. The special cases are not used and are not treated separately by the proof: a common proof is provided for them instead.

Graphs $G = (V, E)$ are always undirected, $V = V(G)$ is the vertex-set, $E = E(G)$ the edge-set; $\underline{1}$ is the all 1 vector of appropriate dimension.

If $x \in \mathbb{R}^n$, the *projection* of x parallel to the i-th coordinate is the vector $x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Let us fix the notation $V := \{1, \ldots, n\}$. If $X \subseteq \mathbb{R}^n$, the projection parallel to the i-th coordinate of the set X is $X^i := \{x^i : x \in X\}$; if $I \subseteq V$, X^I is the result of successively projecting parallel to $i \in I$ (the order does not matter).

Let $P := P^{\leq}(A^{\leq})$ or $P := P^{\geq}(A^{\geq})$, and $I, J \subseteq V$, $I \cap J = \emptyset$. A *minor* of P is a polyhedron $P \setminus I/J := (P \cap \{x : x_i = 0 \text{ if } i \in I\})^{I \cup J}$. The set I is said to be *deleted*, whereas J is *contracted*. For set-packing polyhedra the contraction of J is the same as the deletion of J.

It is easy to see that $P^{\leq} \setminus I/J = P^{\leq}(A'^{\leq})$, and $P^{\geq} \setminus I/J = P^{\geq}(A'^{\geq})$, where $A'^{\leq}$, $A'^{\geq}$ arise from $A^{\leq}$, resp. $A^{\geq}$ in a simple way: delete the columns indexed by I, and then delete those rows that are no more maximal, resp. minimal; for $A^{\leq}$ do the same with J; for $A^{\geq}$ delete the columns indexed by J and also delete all the rows having a 1 in at least one of these columns. Hence *minors of set-packing or set-covering polyhedra are of the same type*.

We do not use the terms 'contraction' or 'deletion' for matrices (or clutters), because that would be confusing here for several reasons, one of which being that these operations do not only depend on the matrix (or clutter) itself. But we define the *minors of the ordered pair* $(A^{\leq}, A^{\geq})$: $(A^{\leq}, A^{\geq}) \setminus I/J := (A'^{\leq}, A'^{\geq})$, where I, J, $A'^{\leq}$, $A'^{\geq}$ are as defined above. The polyhedra $P^{\leq}(A'^{\leq})$ and $P^{\geq}(A'^{\geq})$ will be called *corresponding minors* of the two polyhedra $P^{\leq}(A'^{\leq})$ and $P^{\geq}(A'^{\geq})$. Parallelly, for a clutter (matrix) $\mathcal{A}$ and $v \in V$ we define the clutter $\mathcal{A} - v := \{A \in \mathcal{A} : v \notin A\}$ on $V \setminus \{v\}$.

If $P = P^{\leq}(A^{\leq})$ is integer, then P and $A^{\leq}$ are called *perfect*, whereas if $P = P^{\geq}(A^{\geq})$ is integer, P and $A^{\geq}$ are called *ideal*. All minors of perfect and ideal matrices are also ideal or perfect respectively. If a matrix is not perfect (not ideal) but all its proper minors are, then it is called *minimal imperfect*, or *minimal nonideal* respectively. It is easy to see that the family $\mathcal{H}_n^{n-1}$ of the $n-1$-tuples of an n-set is minimal imperfect, and it is also an easy and well-known exercise to show that matrices not containing such a minor (or equivalently having the 'dual Helly property') can be represented as the (inclusionwise) maximal cliques of a graph. We will call $\mathcal{H}_n^{n-1}$ $(n = 3, 4, \ldots,)$ *minimal nongraph* clutters. The *degenerate projective plane* clutters $\mathcal{F}_n = \big\{\{1, \ldots, n-1\}, \{1, n\}, \{2, n\}, \ldots, \{n-1, n\}\big\}$, $(n = 3, 4, \ldots)$ are minimal nonideal.

It is easy to show that the blocker of the blocker is the original clutter. The antiblocker of the antiblocker of $\mathcal{H}_n^{n-1}$ is not itself, and this is the only exception: it is another well-known exercise to show that the antiblocker of the antiblocker of a clutter that has no $\mathcal{H}_n^{n-1}$ minor (dual Helly property), is itself.

A graph G is called *perfect* or *minimal imperfect* if its clique-matrix is so. It is said to be *partitionable*, if it has $n = \alpha\omega + 1$ vertices $(\alpha, \omega \in \mathbb{N})$, and for all $v \in V(G)$, $G - v$ can be partitioned both into α cliques and into ω stable-sets. Lovász [8] proved that minimal imperfect graphs are partitionable and Padberg [9] proved further properties of partitionable graphs.

Analogous properties have been proved for nondegenerate minimal nonideal clutters by Lehman [6], from which we extract: a pair of clutters $(\mathcal{A}, \mathcal{B})$, where $\mathcal{B}$ is the blocker or the antiblocker of $\mathcal{A}$ will be called *partitionable*, if they are defined on $V := \{1, \ldots, n\}$, $n = rs - \mu + 1$, $(r, s \in \mathbb{N}, \mu \in \mathbb{Z}, 0 \leq \mu \leq \min\{r, s\}), \mu \neq 1$, and for all $v \in V$ there exist sets $A_1, \ldots, A_s \in \mathcal{A}$ and sets $B_1, \ldots, B_r \in \mathcal{B}$ such that $v \in A_1, \ldots, A_\mu, B_1, \ldots, B_\mu$ and both $\{A_1 \setminus v, \ldots, A_\mu \setminus v, A_{\mu+1}, \ldots, A_s\}$ and $\{B_1 \setminus v, \ldots, B_\mu \setminus v, B_{\mu+1}, \ldots, B_r\}$ are partitions of $V \setminus \{v\}$.

Remark 1. The clique matrix of a partitionable graph is a partitionable clutter with $\omega = r, \alpha = s, \mu = 0$.

Supposing that $(\mathcal{A}, \mathcal{B})$ is partitionable, it is easy to see that they are antiblockers of each other if and only if $\mu = 0$, and they are blockers of each other if and only if $\mu \geq 2$. Indeed, if $(\mathcal{A}, \mathcal{B})$ are partitionable it can be shown (Padberg [9], [10]) that $\mathcal{A}$ and $\mathcal{B}$ have exactly n members and these can be indexed $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ so that $|A_i \cap B_j|$ is 1 if $i \neq j$ and is μ if $i = j$. These properties will be proved directly for general 'minimal noninteger systems'.

We will call $\mathcal{A}$ partitionable, if $(\mathcal{A}, \mathcal{B})$ is partitionable, where $\mathcal{B}$ is the blocker or the antiblocker of $\mathcal{A}$ – we will always make clear which of the two is meant.

Let $\mathcal{A}$ be partitionable. Clearly, $1/r\underline{1} \in P^{\leq}(\mathcal{A})$ if $\mu = 0$, and $1/r\underline{1} \in P^{\geq}(\mathcal{A})$ if $\mu \geq 2$ (it is actually the unique full support noninteger vertex of $P^{\leq}$ or $P^{\geq}$, for minimal nonideal or minimal imperfect polyhedra, it is the unique fractional vertex). Let us call this the *regular* vertex of $P^{\leq}(\mathcal{A})$, or of $P^{\geq}(\mathcal{A})$. The regular vertex of $\mathcal{F}_n$ and that of $\mathcal{H}_n^{n-1}$ is defined as their unique fractional vertex.

The idea of this work originates in the frustrating similarities between minimal imperfect and minimal nonideal matrices and the proofs of the results. This similarity becomes fascinating when comparing Seymour's proof [12] of Lehman's, and Gasparyan's direct proof [3] of Lovász's and Padberg's theorems.

Despite these similarities, the generalization has to deal with several new phenomena, for instance $P(A^{\leq}, A^{\geq})$ can be empty, and its dimension can also vary. (Antiblocking and blocking polyhedra are trivially full dimensional !) We will meet many other difficulties that oblige us to generalize the notions and arguments of the special cases – without making the solution much more complicated. The proof synthesizes polyhedral and combinatorial arguments, moreover a lemma involving the divisibility relations between the parameters will play a crucial role when mixed fractional vertices occur.

We show now an example with mixed vertices. Surprisingly, this will be the only essential ('minimal noninteger') new example where the two types of inequalities interact in a nontrivial way. In a sense, a kind of 'Strong Perfect Graph Conjecture' is true for mixed polyhedra.

If $\mathcal{A}^{\leq} \cup \mathcal{A}^{\geq} = E(C_{2k+1}) \subseteq 2^{V(G)}$ $(k \in \mathbb{N})$, and neither $A^{\leq}$ nor $A^{\geq}$ is empty, then $P(A^{\leq}, A^{\geq})$ will be called a *mixed odd circuit polyhedron*, and $(A^{\leq}, A^{\geq})$ will be called a *mixed odd circuit*. The unique fractional vertex of a mixed odd circuit polyhedron is $1/2\underline{1}$.

Let now $(A^{\leq}, A^{\geq})$ be a simple odd circuit. Let us define B_i to be the (unique) subset of vertices of the graph C_{2k+1} having exactly one common vertex with every edge of C_{2k+1} except with $(i, i+1)$; the number of common vertices of B_i with the edge $(i, i+1)$ is required to be zero or two depending on whether its incidence vector is in $A^{\leq}$ or $A^{\geq}$ respectively $(i = 1, \ldots, 2k+1,\ i+1$ is understood mod $n = 2k+1)$. The neighbors of the vertex $1/2\underline{1}$ on $P(A^{\leq}, A^{\geq})$ are the characteristic vectors of the B_i, $(i = 1, \ldots, n = 2k+1)$. Follow these and other remarks on C_7:

Example 1. (an odd circuit polyhedron) Let us define $P(A^{\leq}, A^{\geq}) \subseteq \mathbb{R}^7$ with:
$x_i + x_{i+1} \leq 1\,(i = 1, 2, 3, 4), \quad x_i + x_{i+1} \geq 1\,(i = 5, 6, 7;$ for $i = 7$, $i + 1 := 1)$.

This polyhedron *remains noninteger* after projecting 1: indeed, the inequality $x_7 - x_2 \geq 0$ is a sixth facet-inducing inequality (containing the vertex $1/2\,\underline{1}$) besides the five remaining edge-inequalities. These six inequalities are linearly independent ! (The projection of a vertex is still a vertex if and only if the projection is parallel to a coordinate which is nonzero both in some set-packing and some set-covering facet containing the vertex.) But the new inequality is not 0–1 ! However, a study of nonintegrality should certainly include this example.

The vertices of $P(A^{\leq}, A^{\geq})$ are, besides $(1/2)\underline{1}$, the sets B_i, $(i = 1, \ldots, 7)$. These are the shifts of $B_2 := \{1, 4, 6\}$ by ± 1 and $0, 2$, and of $B_6 := \{2, 4, 6, 7\}$ by $0, \pm 1$. Note that the vector $(0, 1, 1, 1, 0, -1, -1)$ is orthogonal to all the B_i $(i = 1, \ldots, 7)$, whence the 7×7 matrix $\mathcal{B}$ whose rows are these, *is singular*!

In general, if $(A^{\leq}, A^{\geq})$ is a simple mixed odd circuit, and $A^{\leq}$ has one more row than $A^{\geq}$, then $\underline{1}^T A^{\leq} - \underline{1}^T A^{\geq}$ (defines a Chvátal-Gomory cut and) is orthogonal to all the B_i-s $(i = 1, \ldots, 2k + 1)$, so *they are linearly dependent !*

Linear independence of the neighbors of fractional vertices play a fundamental role in the special case of Padberg [9],[10], Lehman[6], and also in Gasparyan [3],[4]. Mixed odd circuits show that *we have to work here without this condition.* As a consequence we will not be able to stay within matrix terms, but will have to mix combinatorial and polyhedral arguments: Lemma 8 is mostly a self-contained lemma on matrices, where the polyhedral context, through Lemma 7 brings in a stronger combinatorial structure: '$r = 2$'. The matricial part of Lemma 8 reoccurs in papers [4] and [5], studying the arising matrix equations. The latter avoids the 'nonsingularity assumption' replacing Lemma 7 by combinatorial (algebraic) considerations.

This paper is organized as follows: Section 2 states the main result, its corollaries, and reformulations. The proof of the main result is provided in sections 3 and 4. Section 5 is devoted to some more examples and other comments.

2 Results

When this does not cause missunderstanding, we will occasionnally use the shorter notations $P^{\leq} := P^{\leq}(A^{\leq})$, $P^{\geq} := P^{\geq}(A^{\geq})$, $P := P(A^{\leq}, A^{\geq}) = P^{\leq} \cap P^{\geq}$. Recall that the polyhedra $P^{\leq}(A'^{\leq}) := P^{\leq} \setminus I/J$ and $P^{\geq}(A'^{\geq}) := P^{\geq} \setminus I/J$, $(I, J \subseteq V := \{1, \ldots, n\},\ I \cap J = \emptyset)$ are called corresponding minors, and $(A'^{\leq}, A'^{\geq}) =: (A^{\leq}, A^{\geq}) \setminus I/J$ is a minor of $(A^{\leq}, A^{\geq})$. (Note that two minors are corresponding if and only if the two $I \cup J$ are the same, since for set-packing polyhedra deletion is the same as contraction.) Furthermore, if for all such I, J the polyhedron $(P^{\leq}(A^{\leq}) \setminus I/J) \cap (P^{\geq}(A^{\geq}) \setminus I/J)$ is integer, then the system $(A^{\leq}, A^{\geq})$ will be called *fully integer.*

Theorem 1. *Let $A^{\leq}$ and $A^{\geq}$ be 0–1-matrices with n columns. Then $(A^{\leq}, A^{\geq})$ is not fully integer if and only if it has a minor $(A'^{\leq}, A'^{\geq})$ for which at least one of the following three statements holds:*

– *$A'^{\leq}$ is a minimal nongraph clutter, or it is partitionable with $\mu = 0$, moreover in either case the regular vertex of $P^{\leq}(A'^{\leq})$ is in $P^{\geq}(A'^{\geq})$, and it is the unique packing type fractional vertex of $P^{\leq}(A'^{\leq}) \cap P^{\geq}(A'^{\geq})$.*
– *$A'^{\geq}$ is a degenerate projective plane, or it is partitionable with $\mu \geq 2$, moreover in either case the regular vertex of $P^{\geq}(A'^{\geq})$ is in $P^{\leq}(A'^{\leq})$, and it is the unique covering type fractional vertex of $P^{\leq}(A'^{\leq}) \cap P^{\geq}(A'^{\geq})$.*
– *$(A'^{\leq}, A'^{\geq})$ is a mixed odd circuit.*

Lovász's NP-characterization of imperfect graphs [8] (with the additional properties proved by Padberg[10]), follow:

Corollary 1. *Let $A^{\leq}$ be a 0–1-matrix with n columns. Then $A^{\leq}$ is imperfect if and only if it has either a minimal nongraph or a partitionable minor $A'^{\leq}$, moreover $P(A'^{\leq})$ has a unique fractional vertex.*

Specializing Theorem 1 to set-covering polyhedra one gets Lehman's celebrated result [6], see also Seymour [12]:

Corollary 2. *Let $A^{\geq}$ be a 0–1-matrix with n columns. Then $A^{\geq}$ is nonideal if and only if it has either a degenerate projective plane or a partitionable minor $A'^{\geq}$, moreover $P(A'^{\geq})$ has a unique fractional vertex.*

The following two consequences are stated in a form helpful for coNP characterization theorems (see Section 5):

Corollary 3. *Let $A^{\leq}$ and $A^{\geq}$ be 0–1-matrices with n columns. Then $(A^{\leq}, A^{\geq})$ is not fully integer if and only if at least one of the following statements holds:*

– *$A^{\leq}$ has a minimal nongraph or a partitionable, furthermore minimal imperfect minor with its regular vertex in the corresponding minor of $P^{\geq}(A^{\geq})$,*
– *$A^{\geq}$ has a degenerate projective plane or a partitionable minor with its regular vertex in the corresponding minor $P^{\leq}(A'^{\leq})$ of $P^{\leq}(A^{\leq})$, where $A'^{\leq}$ is perfect.*
– *$(A^{\leq}, A^{\geq})$ has a mixed odd circuit minor.*

If we concentrate on the structural properties of the matrices $A^{\leq}$ and $A^{\geq}$ implied by the existence of a fractional vertex we get the following.This statement is not reversible: if $A^{\leq}$ consists of the maximal stable-sets of an odd antihole, and $A^{\geq}$ of one maximal but not maximum stable-set, then $(A^{\leq}, A^{\geq})$ is fully integer, although $A^{\leq}$ is minimal imperfect !

Corollary 4. *Let $A^{\leq}$ and $A^{\geq}$ be 0–1-matrices with n columns and assume that $P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})$ is a noninteger polyhedron. Then*

– *either $A^{\leq}$ has a minimal imperfect minor,*
– *or $A^{\geq}$ has a degenerate projective plane, or a partitionable minor,*
– *or $(A^{\leq}, A^{\geq})$ has a mixed odd circuit minor.*

Note the asymmetry between 'minimal imperfect' in the first, and 'partitionable' in the second case (for an explanation see 5.2).

The results certainly provide a coNP characterization in the following case:

Corollary 5. *Let $A^{\leq}$ be a perfect, and $A^{\geq}$ an ideal 0–1-matrix with the same number of columns. Then $(A^{\leq}, A^{\geq})$ is fully integer if and only if it has no mixed odd circuit minor.*

These results provide a certificate for the intersection of a set-covering polyhedron and a set-packing polytope or of their corresponding minors to be noninteger. This certificate can be checked in polynomial time in the most interesting cases (see Section 5). We will however prove Theorem 1 in the following, slightly sharper form which leaves the possibility to other applications open − and corresponds better to our proof method:

We call $(A^{\leq}, A^{\geq})$ *combinatorially minimal noninteger*, if $P := P(A^{\leq}, A^{\geq})$ is noninteger, but $(P^{\leq} \setminus i) \cap (P^{\geq} \setminus i)$ and $(P^{\leq}/i) \cap (P^{\geq}/i)$ are fully integer for all $i = 1, \ldots, n$. Clearly, mixed odd circuits have this property.

Note the difference with the following definition which takes us out of 0–1 constraints: P is *polyhedrally minimal noninteger*, if it is noninteger, but $P \cap \{x \in \mathbb{R}^n : x_i = 0\}$ and P^i are integer for all $i \in V$.

Both the combinatorial and the polyhedral definitions require that the intersection of P with each hyperplane $x_i = 0$ $(i \in V)$ is integer.

The two definitions are different only in what they require from projections, and this is what we are going to generalize now. When we are contracting an element, combinatorially minimal noninteger systems require the integrality of $P^{\leq}(A^{\leq})^i \cap P^{\geq}(A^{\geq})^i$ instead of the integrality of $\left[P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})\right]^i$ in the polyhedral definition, and this is the only difference between the two. It is easy to see that $P^{\leq}(A^{\leq})^i \cap P^{\geq}(A^{\geq})^i \supseteq \left[P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})\right]^i$, and we saw (see Example 1) that the equality does not hold in general, so the integrality of $P^{\leq}(A^{\leq})^i \cap P^{\geq}(A^{\geq})^i$ and that of $\left[P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})\right]^i$ are seemingly independent of each other. The combinatorial definition looks actually rather restrictive, since it also requires that fixing a variable to 1 in $P^{\geq}(A^{\geq})$, and fixing the same variable to 0 in $P^{\leq}(A^{\leq})$ the intersection of the two polyhedra we get is integer.

Note however, that surprisingly, the results confirm the opposite: the combinatorial definition is *less* restrictive, since besides partitionable, minimal non-graph and degenerate projective clutters, it also includes mixed odd circuit polyhedra, which are not polyhedrally minimal noninteger !

Our proofs will actually not use more about the projections than the following simple *sandwich property* of P which is clearly implied by both combinatorial and polyhedral minimal nonintegrality (Q_i can be chosen to be the polyhedron on the left hand side or the one on the right hand side respectively):

for all $i = 1, \ldots, n$, there exists an integer polyhedron Q_i such that

$$\left[P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})\right]^i \subseteq Q_i \subseteq P^{\leq}(A^{\leq})^i \cap P^{\geq}(A^{\geq})^i.$$

Let us call the system $(A^{\leq}, A^{\geq})$ *minimal noninteger*, if

- P is noninteger, and
- $P \cap \{x \in \mathbb{R}^n : x_i = 0\} (= P^{\leq} \cap \{x \in \mathbb{R}^n : x_i = 0\} \cap P^{\geq} \cap \{x \in \mathbb{R}^n : x_i = 0\})$ is an integer polyhedron for all $i \in V$, and
- P has the sandwich property.

Theorem 2. *If* $(A^{\leq}, A^{\geq})$ *is minimal noninteger, simple, and* $w \in P$ *is a fractional vertex, then* P *is full dimensional,* w *is simplicial, and at least one of the following statements hold:*

- w *is of packing type, and then* $A^{\leq}$ *is either a minimal nongraph clutter, or the clique-matrix of a partitionable graph,*
- w *is of covering type, and then* $A^{\geq}$ *is either a degenerate projective plane or a partitionable clutter,* $\mu \geq 2$*,*
- w *is a mixed vertex, and then* $(A^{\leq}, A^{\geq})$ *is a mixed odd circuit.*

Moreover, P *has at most one fractional vertex of covering type, at most one of packing type, and if it has a vertex of mixed type, then that is the unique fractional vertex of* P*.*

Note that Theorem 2 sharpens Theorem 1 in two directions: first, the constraint of Theorem 2 does not speak about all minors, but only about the deletion and contraction of elements; second, the integrality after the contraction of elements is replaced by the sandwich property.

The corollaries about combinatorial and polyhedral minimal nonintegrality satisfy the condition of Theorem 2 for two distinct reasons. In the combinatorial case *simplicity does not necessarily hold*, but deleting the certain equalities from $A^{\geq}$, the system remains combinatorially minimal noninteger (see 5.2).

Corollary 6. *If* $(A^{\leq}, A^{\geq})$ *is combinatorially minimal noninteger, then at least one of the following statements holds:*

- $A^{\leq}$ *is a minimal nongraph or a partitionable clutter with* $\mu = 0$*, furthermore it is minimal imperfect, and the regular vertex of* $P^{\leq}(A^{\leq})$ *is the unique packing type fractional vertex of* $P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})$*.*
- $A^{\geq}$ *is a degenerate projective plane, or a partitionable clutter with* $\mu \geq 2$*, while* $A^{\leq}$ *is perfect, and the regular vertex of* $P^{\geq}(A^{\geq})$ *is in* $P^{\leq}(A^{\leq})$*.*
- $(A^{\leq}, A^{\geq})$ *is a mixed odd circuit, and* $1/2\underline{1}$ *is its unique fractional vertex.*

This easily implies Theorem 1 and its corollaries using the following remark. (it is particularly close to Corollary 3), while the next corollary does not have similar consequences. This relies on the following:

- If P is noninteger, $(A^{\leq}, A^{\geq})$ *does contain* a combinatorially minimal noninteger minor. (Proof: In both $P^{\leq}$ and $P^{\geq}$ delete and contract elements so that the intersection is still noninteger. Since the result *has still 0–1 constraints this can be applied successively* until arriving at a combinatorially minimal noninteger system.)
- If P is noninteger, one *does not necessarily arrive* at a polyhedrally minimal noninteger polyhedron with deletions and restrictions of variables. (Counterexample: Example 1.)

Corollary 7. *If* $P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})$ *is polyhedrally minimal noninteger, then at least one of the following statements holds:*

- *either* $A^{\leq}$ *is a minimal nongraph or a partitionable clutter with* $\mu = 0$*, and the regular vertex of* $P^{\leq}(A^{\leq})$ *is the unique packing type fractional vertex of* $P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})$*.*
- *or* $A^{\geq}$ *is a degenerate projective plane, or a partitionable clutter with* $\mu \geq 2$*, and the regular vertex of* $P^{\geq}(A^{\geq})$ *is in* $P^{\leq}(A^{\leq})$*.*

Proof. Express w^i as a convex combination of vertices of P^i. Replacing the vectors in this combination by their lift, we get a vector which differs from w exactly in the i-th coordinate ($i = 1, \dots, n$) – if it did not differ, w would be the convex combination of integer vertices of P. So the i-th unit vector is in the linear space generated by P for all $i = 1, \dots, n$, proving that P is full dimensional, in particular, simple. So Theorem 2 can be applied, and its third alternative cannot hold (see Example 1). □

Gasparyan [4] has deduced this statement by proving that in the polyhedral minimal case the matrices involved in the matrix equations are nonsingular (see comments concerning nonsingularity in Example 1).

The main frame of the present paper tries to mix (the polar of) Lehman's polyhedral and Padberg's matricial approaches so as to arrive at the simplest possible proof. Lemmas 1–4 and Lemma 7 are more polyhedral, Lemma 5, Lemma 6 and Lemma 8 are matricial and combinatorial. When specializing these to ideal clutters, their most difficult parts fall out and quite short variants of proofs of Lehman's or Padberg's theorem are at hand.

3 From Polyhedra to Combinatorics

The notation $\mathcal{A}$, $\mathcal{B}$ will be used for families of sets. (We will also use the notation $\mathcal{A}$ for the matrices whose rows are the members of $\mathcal{A}$.) The *degree* $d_{\mathcal{A}}(v)$ of v in $\mathcal{A}$ is the number of $A \in \mathcal{A}$ containing v.

Given $w \in P$, let $\mathcal{A}_w$ be the set of those rows A of $A^{\geq}$ or of $A^{\leq}$ for which $w(A) = 1$. (We do not give multiplicites to the members of $\mathcal{A}_w$, regardless of whether some of its elements are contained in both $A^{\geq}$ and $A^{\leq}$!) We also define these if the polyhedron also has non-0–1-constraints. Then $A^{\geq}$ and $A^{\leq}$ denote the set-covering and set-packing inequalities in the defining system.

If P is integer, we define $\mathcal{B}_w$ as the family of those 0–1 vectors (vertices of P) which are on the minimal face of P containing w. (Equivalently, $\mathcal{B}_w$ is the set of vertices having a nonzero coefficient in some convex combination expressing w.) If $A \in \mathcal{A}_w$, and $B \in \mathcal{B}_w$, then $|A \cap B| = 1$. Clearly, $r(\mathcal{A}_w) + r(\mathcal{B}_w) = \dim P + 1$. If it is necessary in order to avoid misunderstanding, we will write $\mathcal{A}_w(P)$, $\mathcal{B}_w(P)$.

The following lemma is based on the polar (in the sense of interchanging vertices and facets) of a statement implicit in arguments of Lehman's and Seymour's work (see Seymour [12]).

Lemma 1. *If* Q *is a polyhedron with 0–1 vertices (and not necessarily 0–1-constraints) and* $w \in Q$*,* $w > 0$*, then* $\bigcup_{B \in \mathcal{B}_w} B = V$*, and*

$$r(\mathcal{B}_w) \geq \max\{|A| : A \in \mathcal{A}_w\}, \quad r(\mathcal{A}_w) \leq n - \max\{|A| : A \in \mathcal{A}_w\} + 1.$$

Proof. Indeed, since Q is integer, w is the convex combination of 0–1 vertices in $\mathcal{B}_w$, whence $\bigcup_{B\in\mathcal{B}_w} B = V$. In particular, for $A \in \mathcal{A}_w$ and all $a \in A$ there exists $B_a \in \mathcal{B}_w$, such that $a \in B_a$.

Since $A \in \mathcal{A}_w$, and $B_a \in \mathcal{B}_w$, we have $|A \cap B_a| = 1$, and consequently $A \cap B_a = \{a\}$. Thus $\{B_a : a \in A\}$ consists of $|A|$ linearly independent sets of $\mathcal{B}_w$, whence $r(\mathcal{B}_w) \geq |A|$. □

Remark 2. Compare Lemma 1 with Fonlupt, Sebő [2]: a graph is perfect if and only if the linear rank of the maximum cliques (as vertex-sets) in every induced subgraph is at most $n - \omega + 1$ where ω is the size of the maximum clique in the subgraph; the equality holds if and only if the subgraph is uniquely colorable.

We note and use in the sequel without reference that if P is minimal noninteger, then $w > 0$ for all fractional vertices w of P ($w_i = 0$ would imply that $(P^{\leq} \setminus i) \cap (P^{\geq} \setminus i)$ is also noninteger).

In sections 3 and 4 I will denote the identity matrix, J the all 1 matrix of appropriate dimensions; A is called *r-regular*, if $\underline{1}\mathcal{A} = r\underline{1}$, and *$r$-uniform* if $\mathcal{A}\underline{1} = r\underline{1}$; $\mathcal{A}^c := \{V \setminus A : A \in \mathcal{A}\}$. $\mathcal{A}$ is said to be *connected* if V cannot be partitioned into two nonempty classes so that every $A \in \mathcal{A}$ is a subset of one of the two classes. There is a unique way of partitioning $\mathcal{A}$ and V into the *connected components* of $\mathcal{A}$.

Lemma 2. *If $(A^{\leq}, A^{\geq})$ is minimal noninteger, w is a fractional vertex of $P := P(A^{\leq}, A^{\geq})$, and $\mathcal{A} \subseteq \mathcal{A}_w$ is a set of n linearly independent members of $\mathcal{A}_w$, then every connected component K of $\mathcal{A}^c$ is $n - r_K$-regular and $n - r_K$-uniform ($r_K \in \mathbb{N}$), and $r(\mathcal{A} - v) = n - d_{\mathcal{A}}(v)$.*

Proof. Recall that $w > 0$. If P is minimal noninteger, then for arbitrary $i \in V$ the sandwich property provides us $Q_i \subseteq \mathbb{R}^{V\setminus\{v\}}$, $w^i \in \left[P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})\right]^i \subseteq Q_i \subseteq P^{\leq}(A^{\leq})^i \cap P^{\geq}(A^{\geq})^i$, that is, $w^i \in Q_i$ and $w^i > 0$. Applying the inequality in Lemma 1 to Q_i and w^i, and using the *trivial but crucial fact that $\mathcal{A}_{w^i}(Q_i) \supseteq \mathcal{A} - i$*, we get the inequality $r(\mathcal{A} - i) \leq n - \max\{|A| : A \in \mathcal{A} - i\}$.

On the other hand, $r(\mathcal{A}) = n$ by assumption. One can now finish in a few lines like Conway proves de Bruijn and Erdős's theorem [7], which is actually the same as Seymour [12, Lemma 3.2]:

Let $\mathcal{H} := \mathcal{A}^c$ for the simplicity of the notation. What we have proved so far translates as $d_{\mathcal{H}}(v) \leq |H|$ for all $v \in H \in \mathcal{H}$. But then,

$$n = \sum_{H\in\mathcal{H}} 1 = \sum_{H\in\mathcal{H}} \sum_{v\in H} 1/|H| = \sum_{v\in V} \sum_{H\in\mathcal{H}, v\in H} 1/|H| = \sum_{v\in V} d_{\mathcal{H}}(v)/|H| \leq \sum_{v\in V} 1,$$

and the equality follows. □

Remark 3. The situation of the above proof will be still repeated several times: when applying Lemma 1, the 0–1 vectors that have an important auxiliary role for bounding the rank of some sets are in $\mathcal{B}_w(Q_i)$, *and are not necessarily vertices* *of* P. The reader can check on mixed odd circuits that the neighbors $\mathcal{B} = \{B_1, \dots, B_n\}$ of $1/2\underline{1}$ are not suitable for the same task (unlike in the special cases): the combinatorial ways that use $\mathcal{B}$ had to be replaced by this more general polyhedral argument. Watch for the same technique in Lemma 7 !

The next lemma synthesizes two similar proofs occurring in the special cases:

Lemma 3. *If* $(A^{\leq}, A^{\geq})$ *is minimal noninteger,* w *and* w' *are fractional vertices of* P*, then defining* $\mathcal{A}$ *and* $\mathcal{A}'$ *to be a set of* n *linearly independent vectors from* $\mathcal{A}_w$*,* $\mathcal{A}_{w'}$ *respectively,* $\mathcal{A}$ *and* $\mathcal{A}'$ *cannot have exactly* $n-1$ *common elements.*

Proof. (Sketch) Apply Lemma 6 to both w and w'. With the exception of some degenerate cases easy to handle, any member of an r-regular clutter can be uniquely reconstructed from the others. □

Lemma 4. *If* $(A^{\leq}, A^{\geq})$ *is minimal noninteger and simple, and* w *is a fractional vertex of* P*, then* $P := P(A^{\leq}, A^{\geq})$ *is full dimensional,* w *is simplicial, and the vertices neighbouring* w *on* P *are integer.*

The proof can be summarized with the sentence: a minimal noninteger system cannot contain implicit equalities (only 'explicit' equalities).

Proof. Let us first prove that P is full dimensional. By Lemma 3 $\mathcal{A}_w$ is linearly independent (recall that every member was included only once). Suppose $0 \in \mathbb{R}^n$ can be written as a nontrivial nonnegative linear combination of valid inequalities. Clearly, all of these are implicit equalities (see [11]) of P. In particular their coefficient vectors are in $\mathcal{A}_w$. In this nontrivial nonnegative combination there is no nonnegativity constraint $x_i \geq 0$, because otherwise $P \subseteq \{x : x_i = 0\}$, contradicting $w > 0$. So everything participating in it is in $\mathcal{A}_w$ contradicting its linear independence.

Since $\mathcal{A}_w$ is linearly independent, w is simplicial. If a neighbour w' of w is noninteger, we arrive at a contradiction with Lemma 3. □

We will say that a polyhedron P is *minimal noninteger* if $P = P(A^{\leq}, A^{\geq})$ for some simple, minimal noninteger system. (Since P is full dimensional by Lemma 4, it determines $(A^{\leq}, A^{\geq})$ uniquely.)

Given a minimal noninteger polyhedron P and a fractional vertex w of P, fix $\mathcal{A} := \mathcal{A}(w) := \mathcal{A}_w$ and let $\mathcal{B} := \mathcal{B}(w)$ denote the set of vertices neighboring w in P.

Note that $\cup_{i=1}^n \mathcal{B}_{w^i}(P^i) = \mathcal{B}(w)$ *holds in the polyhedrally minimal noninteger case, but does not necessarily hold otherwise,* and therefore we need essential generalizations. Do not confuse $\mathcal{B}_w$ (which is just $\{w\}$) with $\mathcal{B}(w)$.

We will say that a vertex $B \in \mathcal{B}$ and a facet $A \in \mathcal{A}$ not containing it are *associates*. By Lemma 4 w is simplicial, whence this relation perfectly matches $\mathcal{A}$ and $\mathcal{B}$. We will suppose that the associate of the i-th row A_i of A is the associate B_i of A_i; $\mu_i := |A_i \cap B_i|$. Clearly, $\mu_i \neq 1$ $(i = 1, \dots, n)$. Denoting by $\mathrm{diag}(d_1, \dots, d_n)$ the $n \times n$ diagonal matrix whose diagonal entries are $d_1, d_2, \dots, d_n$, we have proved:

Lemma 5. $\mathcal{A}\mathcal{B}^T = J + \mathrm{diag}(\mu_1 - 1, \ldots, \mu_n - 1)$, *where* $\mu_i \neq 1$, $(i = 1, \ldots, n)$.

If μ_i does not depend on i, we will simply denote it by μ. (This notation is not a coincidence: in this case $(\mathcal{A}, \mathcal{B})$ turns out to be partitionable where μ is the identically denoted parameter.) By Lemma 5, $\mu \neq 1$.

The main content of Lemma 3, 5, some aspects of Lemma 4 and most of Lemma 6 are already implicitly present already in Padberg[9].

4 Associates and the Divisibility Lemma

The following lemma extracts and adapts to our needs well-known statements from Lehman's, Seymour's and Padberg's works, and reorganizes these into one statement. It can also be deduced by combining results of Gasparyan [4], which investigate combinatorial properties implied by matrix equations. For instance the connectivity property of Lemma 6 below is stated in [4] in a general self-contained combinatorial setting.

Lemma 6. *If P is minimal noninteger, and $w \in P$ is a fractional vertex of P, then $\mathcal{A} = \mathcal{A}(w)$ is nonsingular and connected, moreover,*

- *if the clutter $\mathcal{A}^c$ is connected, then $\underline{1}\mathcal{A} = \mathcal{A}\underline{1} = r\underline{1}$, $r \geq 2$.*
- *if the clutter $\mathcal{A}^c$ has two components, then $\mathcal{A}$ is a degenerate projective plane.*
- *if the clutter $\mathcal{A}^c$ has at least three components, then $\mathcal{A} = \mathcal{H}_n^{n-1}$.*

Proof. (Sketch) If $\mathcal{A}^c$ has at least two components, then any two sets whose complements are in different components cover V. This, and the matrix equation of Lemma 5 determine a degenerate combinatorial structure. (For instance one can immediately see that the associate of a third set has cardinality at most two, and it follows that all but at most one members of $\mathcal{B}$ have at most two elements.)

If $\mathcal{A}^c$ has one component,then the uniformity and regularity of $\mathcal{A}^c$ claimed by Lemma 2 implies that of $\mathcal{A}$. □

Recall that the nonsingularity of $\mathcal{B}$ cannot be added to Lemma 6 !

It is well-known that both for minimal imperfect and minimal nonideal matrices the associates of intersecting sets are (almost) disjoint. In our case they can also *contain one another*, and the proof does not fit into the combinatorial properties we have established (namely Lemma 5). We have to go back to our polyhedral context (established in the proof of Lemma 2, see also Remark 3):

Let us say that $\mathcal{A}$ with $\mathcal{A}\mathcal{B}^T = J + \mathrm{diag}(\mu_1 - 1, \ldots, \mu_n - 1)$, where $\mu_i \neq 1$ $(i = 1, \ldots, n)$ is *nice*, if for $A_1, A_2 \in \mathcal{A}$, $v \in A_1 \cap A_2$ the associates $B_1, B_2 \in \mathcal{B}$ of A_1 and A_2 respectively, either satisfy $B_1 \cap B_2 \setminus \{v\} = \emptyset$ or $B_1 \setminus \{v\} = B_2 \setminus \{v\}$. (In the latter case, since B_1 and B_2 cannot be equal, one of the two contains v.)

Lemma 7. *Let P be minimal noninteger, and $\mathcal{A} = \mathcal{A}(w)$, $\mathcal{B} = \mathcal{B}(w)$ for some noninteger vertex $w \in P$. Then $\mathcal{A}$ is nice.*

Check the statement for the mixed C_7 of Example 1 ! (It can also be instructive to follow the proof on this example.)

Proof. Let $v \in A_1,\ A_2 \in \mathcal{A}$, and let $B_1, B_2 \in \mathcal{B}$ be their associates. Moreover assume $u \in B_1 \cap B_2 \setminus \{v\}$. Let $A_0 \in \mathcal{A}$, $u \in A$, $v \notin A_0$. (There exists such an $A_0 \in \mathcal{A}$ since for instance Lemma 6 implies that a column of $\mathcal{A}$ cannot dominate another.) Since P is minimal noninteger, there exists an integer polyhedron Q_v such that $\left[P^{\leq}(A^{\leq}) \cap P^{\geq}(A^{\geq})\right]^v \subseteq Q_v \subseteq P^{\leq}(A^{\leq})^v \cap P^{\geq}(A^{\geq})^v$. Now because of $\mathcal{A}_{w^v}(Q_v) \supseteq \mathcal{A} - i$, the scalar product of the vertices of $\mathcal{B}_{w^v}(Q_v)$ with all vectors in $\mathcal{A} - i$ is 1, and the proof method of Lemma 1 can be applied:

For every $a \in A_0 \setminus u$ fix some $B_a \in \mathcal{B}_{w^v}(Q_v)$ so that $a \in B_a$. Now $\{B_a : a \in A_0 \setminus u\} \cup \{B_1 \setminus v, B_2 \setminus v\}$ are $r+1$ vectors in $\mathbb{R}^{V \setminus v}$ all of which have exactly one common element with each $A \in \mathcal{A} - v$. On the other hand, by Lemma 2 $r(\mathcal{A} - v) = n - r = (n-1) - (r-1)$, so there can be at most r linearly independent sets with this property. Hence there exists a nontrivial linear combination $\lambda_1(B_1 \setminus v) + \lambda_2(B_2 \setminus v) + \sum_{a \in A \setminus u} \lambda_a B_a = 0$. Since for $a \in A_0 \setminus u$ the unique vector in this linear combination which contains a is B_a, one gets that $\lambda_a = 0$ for all $a \in A \setminus u$. It follows that $B_1 \setminus v = B_2 \setminus v$, and $\lambda_1 = \lambda_2$. □

Although the following statement is the heart of our proof, it is independent of the other results. The very root of the statement is the simple observation that $n + d_j$ is a multiple of r. Note that in order to deduce $r = 2$ we need more than just the matrix equation !

Lemma 8. *Assume that $\mathcal{A}$, $\mathcal{B}$ are 0–1 matrices, $\underline{1}\mathcal{A} = \mathcal{A}\underline{1} = r\underline{1}$, and $\mathcal{A}\mathcal{B}^T = J + \mathrm{diag}(\mu_1 - 1, \ldots, \mu_n - 1)$, $\mu_i \neq 1$. Then*

- *either $\mu_1 = \ldots = \mu_n =: \mu$, and then $\mathcal{A}\mathcal{B}^T = \mathcal{B}^T\mathcal{A} = J + (\mu - 1)I$, $\mathcal{B}J = J\mathcal{B} = sJ$, $(s = (n + \mu - 1)/r)$,*
- *or $\{\mu_1, \ldots, \mu_n\} = \{0, r\}$, and if $\mathcal{A}$ is connected and nice, then $r = 2$.*

Proof. If $\mu_1 = \mu_2 = \ldots = \mu_n =: \mu$, then we finish easily, like [3]: since $\mu \neq 0$, $\mathcal{A}$ is invertible; since $\mathcal{A}$ commutes with I, and by assumption with J too, so does its inverse; now expressing $\mathcal{B}^T$ from $\mathcal{A}\mathcal{B}^T = J + (\mu - 1)I$ we get that it is the product of two matrices which commute with both A and J. So $\mathcal{B}^T$ also commutes with these matrices, proving the statement concerning this case. (The matrices X and Y are said to commute, if $XY = YX$.)

So suppose that there exist $i, j \in V$ such that $\mu_i \neq \mu_j$.

Claim (1). $r|B_j| = n + \mu_j - 1$, and $0 \leq \mu_j \leq r$, $(j = 1, \ldots, n)$.

Indeed, $r\underline{1}\mathcal{B}^T = (\underline{1}\mathcal{A})\mathcal{B}^T = \underline{1}(\mathcal{A}\mathcal{B}^T) = \underline{1}\big(J + \mathrm{diag}(\mu_1 - 1, \ldots, \mu_n - 1)\big) = (n + \mu_1 - 1, \ldots, n + \mu_n - 1)$.

The inequality is obvious: $0 \leq \mu_j = |A_j \cap B_j| \leq |A_j| = r, (j = 1, \ldots, n)$.

Claim (2). If there exist $i, j \in V$, $\mu_i \neq \mu_j$, then $\mu_j \in \{0, r\}$ for all $j \in V$.

Indeed, according to Claim (1) we have $n + \mu_j - 1 \equiv 0 \bmod r$, where μ_j is in an interval of $r + 1$ consecutive integers representing every residue class mod r exactly once, except 0, which is represented twice, by 0 and r. Hence if $\{\mu_1, \dots, \mu_n\}$ contains two different values, then these values can only be 0 and r as claimed.

Claim (3). If $v \in A_1 \cap A_2$, $\mu_1 = 0$, $\mu_2 = r$, then $B_1 = B_2 \setminus \{v\}$.

Indeed, let $u \in A_2 \cap B_1$. (Because of the matrix equation in the constraint, we also know $|A_2 \cap B_1| = 1$.) We have $|A_1 \cap B_1| = \mu_1 = 0$, and since $|A_2 \cap B_2| = \mu_2 = r = |A_2|$, we also have $A_2 \subseteq B_2$.

Since $v \in A_1$ and $A_1 \cap B_1 = \emptyset : u \neq v$. Because of $A_2 \subseteq B_2$ we have $u \in (B_1 \cap B_2) \setminus \{v\}$. So we must have $B_1 \setminus \{v\} = B_2 \setminus \{v\}$ by the condition, and since $v \notin B_1$, $v \in B_2 : B_1 = B_2 \setminus \{v\}$. The claim is proved.

Now we finish the proof. Since there exist $i, j \in V$ such that $\mu_i \neq \mu_j$, by Claim (2) $\mu_j \in \{0, r\}$ for all $j \in V$. Since $\mathcal{A}$ is a connected clutter, there exists $v \in V$ so that $v \in A_i \cap A_j$ and $\mu_i = 0$, $\mu_j = r$. After possible renumbering, we can assume $i = 1$, $j = 2$.

So let $A_1, A_2 \in \mathcal{A} = \mathcal{A}(w)$, $v \in A_1 \cap A_2$, $\mu_1 = 0$, $\mu_2 = r$ and denote the associates of A_1, A_2 by B_1, B_2 respectively.

By Claim (3), $1 = |A_2 \cap B_1| = |A_2 \cap B_2| - 1 = r - 1$, so $r = 2$. □

Proof of Theorem 2. (Sketch) Let $(A^{\leq}, A^{\geq})$ be minimal noninteger. Furthermore, let $w \in P$ a fractional vertex of P. Let $\mathcal{A} := \mathcal{A}(w)$ and $\mathcal{B} := \mathcal{B}(w)$. Then we have the matrix equation of Lemma 5.

Case 1. $\mathcal{A}^c$ is connected: according to Lemma 6 and Lemma 7 the conditions of Lemma 8 are satisfied, and using Lemma 8 it is straightforward to finish.

Case 2. $\mathcal{A}^c$ has two components: by Lemma 6 $\mathcal{A}$ is a degenerate projective plane. It can be checked then that either $\mathcal{A} = A^{\geq}$ or $\mathcal{A}$ is not minimal noninteger. We prove this with the following technique (and use similar arguments repeatedly in the sequel): we prove first that there exist an $i \in V$ so that $(P^{\geq}/i) \cap (P^{\leq}/i)$ is noninteger. It turns out then that the maximum p of the sum of the coordiates of a vector on $(P^{\geq}/i) \cap (P^{\leq}/i)$ and the maximum q of the same objective function on P^i are close to each other: $[p, q]$ does not contain any integer (we omit the details). So for all Q_i such that $P^i \subseteq Q_i \subseteq (P^{\geq}/i) \cap (P^{\leq}/i)$ the maximum of the sum of coordinates on Q_i must lie in the interval $[p, q]$. Thus Q_i cannot be chosen to be integer, whence P does not have the sandwich property.

Case 3. $\mathcal{A}^c$ has at least three components: by Lemma 6 $\mathcal{A}$ is the set of $n-1$-tuples of an n-set. If $\mathcal{A} = A^{\leq}$, then we are done (again the first statement holds in the theorem). In all the other cases P turns out not to be minimal noninteger (with the above-described technique). □

5 Comments

5.1 Further Examples

A system $(A^{\leq}, A^{\geq})$ for which a $P(A^{\leq}, A^{\geq}) \subseteq \mathbb{R}^5$ is integer, but $(A^{\leq}, A^{\geq})$ is not fully integer: the rows of $A^{\leq}$ are $(1,1,0,0,0)$, $(0,1,1,0,0)$, $(1,0,1,0,0)$ and $(0,0,1,1,1)$; $A^{\geq}$ consists of only one row, $(0,0,0,1,1)$.

We mention that a class of minimal noninteger simple systems $(A^{\leq}, A^{\geq})$ with the property that $(A^{\leq}, A^{\geq}) \setminus i$ $(i \in V)$ defines an integer, but not always fully integer polyhedron, can be defined with the help of 'circular' minimal imperfect and minimal nonideal systems (see Cornuéjols and Novick [1]): define $A^{\leq} := C_n^r$, $A^{\geq} := C_n^s$, where $r \leq s$ and $A^{\leq}$ is minimal imperfect, $A^{\geq}$ is minimal nonideal.

Such examples do not have mixed vertices, so they also show that the first two cases of our results can both occur in the same polyhedron.

5.2 A Polynomial Certificate

We sketch why Corollary 6 follows from Theorem 2. Note that Corollary 6 immediately implies Corollary 3.

In a combinatorially minimal noninteger system $(A^{\leq}, A^{\geq})$, $A^{\leq}$ is in fact minimal imperfect or perfect. This is a simple consequence of the following:

Claim. If $P^{\leq} := P^{\leq}(A^{\leq})$ or $P^{\geq} := P^{\geq}(A^{\geq})$ is partitionable with a regular vertex $w \in P := P^{\leq} \cap P^{\geq}$, and $P^{\leq}/I$ $(I \subseteq V)$ is partitionable with regular vertex w', then $w' \in P^{\geq}/I$.

Indeed, suppose that w is the regular vertex of a polyhedron whose defining clutter has parameters (r, s), and let the parameters of w' be (r', s'). So $w := 1/r\underline{1}$ and $w' := 1/r'\underline{1}$.

Now $r' \leq r$, because the row-sums of the defining matrix of $P^{\leq}/I$ (which is a submatrix of $A^{\leq}$) do not exceed the row-sums of $A^{\leq}$. Since $w \in P^{\leq}(A^{\leq})$, the row-sums of $A^{\leq}$ are at most r.

But then, if we replace in $1/r\underline{1}$ some coordinates by $1/r'$ some others by 1 the vector w'' we get majorates $1/r\underline{1} \in P^{\geq}(A^{\geq})$ whence it is also in $P^{\geq}(A^{\geq})$. Since $w' \in P^{\geq}/I$ is equivalent to the belonging to $P^{\geq}$ of such a vector w'', the claim is proved.

To finish the proof of Corollary 6 one can show that after deleting from $A^{\geq}$ an equality from '$\mathcal{A}_w$', the system remains minimal noninteger.

Using appropriate oracles, Corollary 3 provides a polynomial certificate. (For the right assumptions about providing the data and certifying the parameters of a partitionable clutter we refere to Seymour [12]. We need an additional oracle for the set-covering part.)

The polynomial certificates can be proved from the Claim using the fact that for partitionable clutters and perfect graphs the parameters can be certified in polynomial time.

For the non-full-integrality of the intersection of perfect and ideal polyhedra a simple polynomial certificate is provided by Corollary 5.

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